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A COMBINATORIAL CHARACTERIZATION OF THE LAGRANGIAN GRASSMANNIAN $\mathsf{LG}(3, 6)(\mathbb{K})$

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Abstract. We provide a combinatorial characterization of $\mathsf{LG}(3, 6)(\mathbb{K})$ using an axiom set which is the natural continuation of the Mazzocca–Melone set we used to characterize Severi varieties over arbitrary fields (Schillewaert and Van Maldeghem, Severi varieties over arbitrary fields, *Preprint*). This fits within a large project aiming at constructing and characterizing the varieties related to the Freudenthal–Tits magic square.

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1. Introduction. Classical varieties such as Veronese varieties, Segre varieties and Grassmann varieties are intensively studied in algebraic geometry, but are also important in combinatorial geometry, in particular in the area where groups and geometries meet and where the Tits buildings play a central role. However, in combinatorial geometry, the underlying field, if any, is arbitrary, and in this case a variety of tools from algebraic geometry can no longer be used. In 1984, Mazzocca & Melone suggested an axiom system for the Veronesean varieties over finite fields that was based on the very basic properties of these varieties as smooth complex varieties, but which can be phrased over any field (and they restricted to finite fields). The main property of such varieties responsible for allowing such an approach is the fact that they are the union of maximal quadratic varieties whose corresponding subspaces pairwise meet on the variety. This makes it possible to define the dimension of the variety via a condition on the tangent spaces to these quadrics over an arbitrary field, the variety is just a set of points, whereas tangent spaces to quadrics are defined over any field. The success of such an approach is illustrated in [**9**], where the authors generalize Zak's classification of complex Severi varieties [**14**] to their analogues over an arbitrary field, just using a straightforward extension of the axioms of Mazzocca & Melone. Another example is the recent characterization of the Veronese representation of projective planes over non-associative alternative division rings (Cayley-Dickson algebras) by Krauss [**6**]. Also, his axioms are based on the Mazzocca–Melone approach.

The Mazzocca–Melone approach, however, was, up to now, only applied when it concerned geometries with point-line diameter 2, and then the first axiom says that every pair of points is contained in a quadric. The types of the geometries thus characterized mainly belong to the second row of the so-called Freudenthal–Tits magic square. The latter is an arrangement of 16 Dynkin diagrams in a four-by-four square symmetric along the main diagonal. Chosen a field, the ith column is parametrized by a (split or non-split, depending on the point of view) quadratic alternative algebra of dimension 2^{i-1}, whereas the jth row is parametrized by a Jordan algebra over a quadratic alternative algebra of dimension 2^{j-1}. Each cell thus corresponds with an ordered pair of algebras, in a non-symmetric way, and a general construction method of Tits [**11**] associates to this pair a Lie algebra of the type indicated by the magic square. To each Lie algebra in the square can be associated a variety, which turns out to be a point-line geometry of a building with corresponding Dynkin type. The geometries of the second row all have diameter at most two (these comprise projective planes, products of two projective planes, line Grassmannians of projective 5-space, and the exceptional $\mathsf{E}_{6,1}$-geometry). In the present paper, we start to apply this approach to geometries of larger diameter. The first natural choice is the Lagrangian Grassmannian $\mathsf{LG}(3,6)(\mathbb{K})$, which is the geometry of totally singular planes of a symplectic space in five-dimensional projective space $\mathbb{P}^5(\mathbb{K})$. Not coincidently, it corresponds to the first cell of the third row of the Freudenthal–Tits magic square. The first cell of the third row is intimately related to the first cell of the second row, which contains the ordinary quadratic Veronesean representations of projective planes. To prove our main result, we strenghten the Mazzocca–Melone approach to these geometries: we basically show that the third axiom can be deleted, if one assumes the right bound on the dimension of the ambient space (the third axiom expresses the dimension of the variety by means of the tangents). This assumption cannot further be weakened as there exist counterexamples for higher dimensions.

NOTATION. In this paper, we will use the following notation: the subspace spanned by a set S of points will be denoted by $\langle S\rangle$. The finite field of q elements will be denoted by $\mathbb{F}_q$. The n-dimensional affine (projective) space over the skew field $\mathbb{K}$ will be denoted by $\mathbb{A}^n(\mathbb{K})$ (by $\mathbb{P}^n(\mathbb{K})$).

2. Statement of the main results.

Let us first recall the Mazzocca–Melone axioms for the quadratic Veronesean of the standard projective plane $\mathbb{P}^2(\mathbb{K})$ over any field $\mathbb{K}$. First note that an *oval* O in any projective plane is a set of points no three collinear and such that through every point $o \in O$ exactly one line does not intersect the set in two points. Examples are conics, if $\mathbb{K}$ is commutative.

Let X be a spanning point set of $\mathbb{P}^N(\mathbb{K})$, $N \in \mathbb{N} \cup \{\infty\}$, with $\mathbb{K}$ any skew field, and let Ξ be a collection of 2-spaces of $\mathbb{P}^N(\mathbb{K})$ containing at least two elements and such that for any $\xi \in \Xi$ the intersection $\xi \cap X =: X(\xi)$ is an oval in ξ (and then, for $x \in X(\xi)$, we denote the tangent line at x to $X(\xi)$ by $T_x(X(\xi))$, or sometimes simply by $T_x(\xi)$). Then, (X, Ξ) is called a *Veronesean cap* if (VC1), (VC2) and (VC3) below hold. It is called a *pre-Veronesean cap* if (VC1) and (VC2) hold.

(VC1) Any pair of points x and y of X is contained in an element of Ξ, denoted by $[x, y]$ (its uniqueness follows straight from (VC2)).

(VC2) If $\xi_1, \xi_2 \in \Xi$, with $\xi_1 \neq \xi_2$, then $\xi_1 \cap \xi_2 \subset X$.

(VC3) If $x \in X$, then all tangent lines $T_x(\xi)$, $x \in \xi \in \Xi$, are contained in a plane.

It is proved in [**8**] that such a Veronesean cap is always the Veronesean representation of the standard projective plane over $\mathbb{K}$, and $\mathbb{K}$ is a field. Recall that the Veronesean representation of $\mathbb{P}^2(\mathbb{K})$ is the image $\mathcal{V}_2(\mathbb{K})$ of $\mathbb{K}^3 \setminus \{(0,0,0)\}$ under the Veronesean map $(x, y, z) \mapsto (x^2, y^2, z^2, yz, zx, xy)$, where the latter is conceived as a point of $\mathbb{P}^5(\mathbb{K})$. Writing $(x^2, y^2, z^2, yz, zx, xy)$ as $(x, y, z)^T(x, y, z)$ (where T means "transposed") it is obvious that the points of the Veronesean representation of $\mathsf{PG}(2, \mathbb{K})$ can be seen as the points corresponding to the rank 1 symmetric (3×3)-matrices in the projective space corresponding to the vector space of all symmetric (3×3)-matrices over $\mathbb{K}$. In the proof, Axiom (VC3) seems to play an important, if not crucial, role. However, we will show below that, if $|\mathbb{K}| > 2$ and $N \leq 5$, then, we can delete Axiom (VC3)! This is our first Main Result.

MAIN RESULT 1. *If (X, Ξ) is a pre-Veronesean cap in $\mathbb{P}^N(\mathbb{K})$, $N \leq 5$, with $\mathbb{K}$ any skew field distinct from $\mathbb{F}_2$, then $\mathbb{K}$ is commutative, (X, Ξ) is a Veronesean cap, and hence, X is projectively equivalent with $\mathcal{V}_2(\mathbb{K})$, the Veronesean representation of the standard projective plane over $\mathbb{K}$.*

We also classify the pre-Veronesean caps if $\mathbb{K} \cong \mathbb{F}_2$, see Proposition 4.4; there is essentially one more example besides the Veronesean cap. Furthermore, we also provide a further weakening of the axioms by allowing X to contain lines. For the motivation and precise statements, see Subsection 4.2.

Now we turn to the Lagrangian Grassmannian $\mathsf{LG}(3, 6)(\mathbb{K})$. As a point set, this is the set of points of $\mathbb{P}^{19}(\mathbb{K})$ on the plane Grassmannian of $\mathbb{P}^5(\mathbb{K})$, restricted to the planes totally isotropic with respect to a non-degenerate alternating bilinear form, which forces this point set into a 13-dimensional subspace $\mathbb{P}^{13}(\mathbb{K})$. As natural point-line geometry (lines are those from $\mathbb{P}^{13}(\mathbb{K})$ completely contained in $\mathsf{LG}(3, 6)(\mathbb{K})$), $\mathsf{LG}(3, 6)(\mathbb{K})$ has diameter 3, but we want to leave the diameter open in the axioms (even infinite diameter will in principle be possible). Also, in the real case, $\mathsf{LG}(3, 6)(\mathbb{R})$ has dimension 6; in the finite case, $\mathsf{LG}(3, 6)(\mathbb{F}_q)$ has $(q^3 + 1)(q^2 + 1)(q + 1)$ points, confirming the six-dimensionality. All this leads to the following definition (noting that quadrics only exist in projective spaces over fields, hence there is no point in starting from a skew field).

Let X be a spanning point set of $\mathbb{P}^N(\mathbb{K})$, $N \in \mathbb{N} \cup \{\infty\}$, with $\mathbb{K}$ any field, and let Ξ be a collection of at least two 4-spaces of $\mathbb{P}^N(\mathbb{K})$ (called the *quadratic spaces*) such that, for any $\xi \in \Xi$, the intersection $\xi \cap X =: X(\xi)$ is a non-singular parabolic quadric $\mathsf{Q}(4, \mathbb{K})$ (which we will call a *symp*, inspired by the theory of parapolar spaces, see [**10**]) in ξ. For $x \in X(\xi)$, we denote the tangent space at x to $X(\xi)$ by $T_x(X(\xi))$ or sometimes simply by $T_x(\xi)$. A line of $\mathbb{P}^N(\mathbb{K})$ all of whose points are contained in X is called a *singular* line, and the set of singular lines is denoted by $\mathcal{S}$. Also,we denote by $\mathcal{G}(X)$ the geometry $(X, \mathcal{S})$ of points and singular lines, and with $\Gamma(X)$ we denote the point graph of $\mathcal{G}(X)$ (two points being adjacent if they are collinear in $\mathcal{G}(X)$). We call (X, Ξ) a *Lagrangian set* if (LS1), (LS2) and (LS3) below hold.

- (LS1) $\mathcal{G}(X)$ is connected and any pair of points x and y of X such that the distance between x and y in $\Gamma(X)$ is at most two is contained in at least one element of Ξ, denoted by $[x, y]$, if unique.
- (LS2) If $\xi_1, \xi_2 \in \Xi$, with $\xi_1 \neq \xi_2$, then $\xi_1 \cap \xi_2 \subset X$.
- (LS3) If $x \in X$, then all 3-spaces $T_x(\xi)$, $x \in \xi \in \Xi$, generate a subspace T_x of $\mathbb{P}^N(\mathbb{K})$ of dimension at most six.

Our second Main Result says that $\mathsf{LG}(3, 6)(\mathbb{K})$ is the only Lagrangian set. More precisely:

MAIN RESULT 2. *If (X, Ξ) is a Lagrangian set in $\mathbb{P}^N(\mathbb{K})$, $N \in \mathbb{N} \cup \{\infty\}$, then $N = 13$ and X is projectively equivalent to the Lagrangian Grassmannian* $\mathsf{LG}(3, 6)(\mathbb{K})$.

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The rest of the paper is devoted to proving Main Results 1 and 2. In the next section, we show that $\mathsf{LG}(3, 6)(\mathbb{K})$ is a Lagrangian set. Then, in Section 4 we show Main Result 1. In Section 5, we show Main Result 2. This proof consists of two major parts. In the first part, we show that the diameter of $\mathcal{G}(X)$ cannot be equal to 2. In the second part, we show that this implies that the diameter is equal to 3 and that X is projectively equivalent to the Lagrangian Grassmannian $\mathsf{LG}(3, 6)(\mathbb{K})$.

3. The Lagrangian Grassmannian $\mathsf{LG}(3, 6)(\mathbb{K})$. In this section, we give an explicit description of $\mathsf{LG}(3, 6)(\mathbb{K})$ and show that it is a Langrangian set. As already mentioned, $\mathsf{LG}(3, 6)(\mathbb{K})$ is the plane Grassmannian of $\mathbb{P}^5(\mathbb{K})$ restricted to the planes totally isotropic with respect to a non-degenerate alternating form. As a geometry, consequently, it is isomorphic to the *dual polar space* denoted by $\mathsf{DW}(5, \mathbb{K})$; the points are the planes of the symplectic polar space $\mathsf{W}(5, \mathbb{K})$ and the lines correspond to the sets of planes of $\mathsf{W}(5, \mathbb{K})$ containing a common line of $\mathsf{W}(5, \mathbb{K})$. In this setting, a symp is the set of points corresponding to the planes of the polar space $\mathsf{W}(5, \mathbb{K})$ containing a common point. It is naturally isomorphic to an orthogonal polar space of rank 2, the so-called *orthogonal generalized quadrangle* $\mathsf{Q}(4, \mathbb{K})$ (which is a *parabolic quadric*; note that every parabolic quadric over a field of characteristic 2 admits a *nucleus*, which is a point through which no secant line passes). The following construction is taken from [**2**] (see also [**3**]).

We define certain types of points in $\mathbb{P}^{13}(\mathbb{K})$.

Type I. A point denoted by $[\infty]$ has coordinates

$$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1).$$

Type II. For $k \in \mathbb{K}$, a point denoted by $[k]$ has coordinates

$$(0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, k).$$

Type III. For $k, x \in \mathbb{K}$, a point denoted by $[x; k]$ has coordinates

$$(0, 0, 0, 0, 0, 0, 0, 0, x^2, 1, -x, 0, 0, k).$$

Type IV. For $k_1, k_2, x \in \mathbb{K}$, a point denoted by $[x; k_1, k_2]$ has coordinates

$$(0, 1, 0, 0, 0, 0, 0, 0, k_1, k_2, x, 0, 0, k_1k_2 - x^2).$$

Type V. For $k, x_1, x_2 \in \mathbb{K}$, a point denoted by $[x_1, x_2; k]$ has coordinates

$$(0, 0, 0, 0, 0, 0, 0, 1, x_1^2, x_2^2, -x_1x_2, x_2, x_1, k).$$

Type VI. For k_1, k_2, x_1, x_2, a point denoted by $[x_1, x_2; k_1, k_2]$ has coordinates

$$(0, x_2^2, 0, 1, 0, x_2, 0, k_1, k_2, k_1x_2^2, -x_1x_2, k_1x_2, x_1, k_1k_2 - x_1^2).$$

Type VII. For $k_1, k_2, x_1, x_2, x_3 \in \mathbb{K}$, a point denoted by $[x_1, x_2, x_3; k_1, k_2]$ has coordinates

$$\big(0, x_3^2, 1, x_1^2, -x_1, -x_3x_1, x_3, k_1, k_2x_1^2 + k_1x_3^2 + x_2(x_1x_3) + (x_3x_1)x_2, k_2,$$
$$-x_3x_2 - k_1x_1, x_2, x_2x_1 + k_1x_3, k_1k_2 - x_2^2\big).$$

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Type VIII. For $k_1, k_2, k_3, x_1, x_2, x_3 \in \mathbb{K}$, a point denoted by $[x_1, x_2, x_3; k_1, k_2, k_3]$ has coordinates

$$\big(1, k_1, k_2, k_3, x_1, x_2, x_3, k_2k_3 - x_1^2, k_3k_1 - x_2^2, k_1k_2 - x_3^2, k_1x_1 - x_3x_2, k_2x_2 - x_3x_1, k_3x_3 - x_2x_1, k_1k_2k_3 + 2x_1x_2x_3 - k_1x_1^2 - k_2x_2^2 - k_3x_3^2\big).$$

The set X of all these points, together with the lines of $\mathbb{P}^{13}(\mathbb{K})$ contained in it is the dual polar space $\mathsf{DW}(5, \mathbb{K})$ and defines the Lagrangian Grassmannian variety $\mathsf{LG}(3, 6)(\mathbb{K})$. An example of a symp is given by all points of Type I, II, III and IV. These points all lie in the subspace U defined by $X_1 = X_3 = X_4 = \cdots = X_8 = X_{12} = X_{13} = 0$ and their (other) coordinates satisfy the equation $X_2X_{14} = X_9X_{10} - X_{11}^2$. Conversely, every point in U whose coordinates satisfy this equation lies on $\mathsf{LG}(3, 6)(\mathbb{K})$. Now it is shown in Corollary 1.2 (ii) of [**5**] that, if $|\mathbb{K}| > 2$, this is the absolutely universal embedding of $\mathsf{DW}(5, \mathbb{K})$, i.e., every other (full) embedding arises as a quotient (i.e., a projection from a suitable subspace) from this one. If $|\mathbb{K}| = 2$, then the universal embedding happens in a 14-dimensional projective space $\mathbb{P}^{14}(\mathbb{F}_2)$ [**1**, **7**], and the Lagrangian Grassmannian is a projection of it. Also, for arbitrary $\mathbb{K}$, the absolutely universal embedding (and for $\mathbb{K} \cong \mathbb{F}_2$ also the Lagrangian Grassmannian) is *homogeneous*, i.e., the group of collineations of the ambient projective space stabilizing the embedding induces the full group of automorphisms of the dual polar space. In particular, this group is transitive on the family of pairs of symps that intersect non-trivially, and also on the family of pairs of symps that have empty intersection; this group is also transitive on the set of points of the embedded dual polar space.

First, we want to check that the intersection of a quadratic space with the point set X is a symp. We can take, by the transitivity properties mentioned in the previous paragraph, the symp Σ_1 consisting of the points of Types I, II, III and IV. Put $n_U = \{1, 3, 4, \ldots, 8, 12, 13\}$ and note that $U = \langle \Sigma_1 \rangle$ is determined by the equations $X_i = 0$, for all $i \in n_U$. Since, points of Type V, VI, VII and VIII have a non-zero coordinate in position 8, 4, 3 and 1, respectively, and all these numbers belong to n_U, we deduce that $X \cap U = \Sigma_1$, which completes the proof.

We now verify the axioms (LS1), (LS2) and (LS3).

Axiom (LS1) follows from the fact that $\mathsf{DW}(5, \mathbb{K})$ is a strong parapolar space, see Example 2 of Section 13.4.2 in [**10**].

For (LS2), we introduce the following two symps:

- Σ_2 consists of the points of Type I, II, V (with $x_2 = 0$) and VI (with $x_2 = 0$). This symp spans the subspace U_2 with equations $X_1 = X_2 = X_3 = X_5 = X_6 = X_7 = X_{10} = X_{11} = X_{12} = 0$, which is indeed four-dimensional. Clearly, $U \cap U_1$ is the line spanned by $(0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0)$ and $(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$, which is a line contained in both symps (namely, the line consisting of all points of Types I and II).
- Σ_3 consists of the points of Type V (with $x_1 = x_2 = k = 0$), VI (with $x_1 = x_2 = k_1 = 0$), VII (with $x_2 = x_3 = k_2 = 0$) and VIII (with $x_2 = x_3 = k_1 = 0$). This symp spans the subspace U_3 with equations $X_2 = X_6 = X_7 = X_9 = X_{10} = \cdots = X_{14} = 0$, which is clearly disjoint from U.

The transitivity properties of the automorphism group of the Lagrangian Grassmannian variety mentioned before conclude the proof of (LS2).

Finally, (LS3) follows by (1) of Theorem 1.3 of [**2**].

Let $x \in X$ be a point of the variety $\mathsf{LG}(3, 6)(\mathbb{K})$. Then, we denote by η_x the subspace of $\mathbb{P}^{13}(\mathbb{K})$ generated by all points of X collinear to x in $\mathsf{DW}(5, \mathbb{K})$, and by ζ_x the subspace of $\mathbb{P}^{13}(\mathbb{K})$ generated by all points of X contained in a common symp with x in $\mathsf{DW}(5, \mathbb{K})$. Obviously, we have $\eta_x \subseteq \zeta_x$, and it follows by (1) of Theorem 1.3 of [**2**] that $\dim \eta_x = 6$ and $\dim \zeta_x = 12$, for all $x \in X$.

LEMMA 3.1. *Let $x \in X$. Then, every seven-dimensional subspace U containing η_x and not contained in ζ_x contains a unique point $y \in X$ not in a common symp with x.*

Proof. By homogeneity, we may take $x = [\infty]$. It follows from the definition of X in [**2**] (see also [**3**]) that η_x is generated by the points of Types I, II, III and V, whereas ζ_x is generated by the points of Types I, II, III, IV, V, VI and VII. It is easy to see that ζ_x has equation $X_1 = 0$. Hence, an arbitrary point z outside $\langle \zeta_x \rangle$ can be written as $(1, \ell_1, \ell_2, \ell_3, y_1, y_2, y_3, \ldots)$. Then, one also calculates that η_x has equations $X_1 = X_2 = \cdots = X_7 = 0$. Now, there is a unique point y of Type VIII in X sharing the same initial seven coordinates with z, we see that $y \in \langle \eta_x, z \rangle$ and the lemma is proved. □

We now want to show that no non-trivial projection of the Lagrangian Grassmannian is a Lagrangian set. We first need a lemma.

LEMMA 3.2. *Let Y be a point set of $\mathbb{P}^5(\mathbb{K})$ isomorphic to $\mathcal{V}_2(\mathbb{K})$, and let p be a point in $\mathbb{P}^5(\mathbb{K})$ not belonging to Y. Then, there exists a point $y \in Y$ such that some plane containing a conic on Y shares a point with $\langle p, y \rangle$ distinct from y.*

Proof. We identify Y with the rank 1 symmetric 3×3-matrices over $\mathbb{K}$, up to a scalar non-zero multiple. Those of rank 2 correspond to points contained in a plane of one of the conics of $\mathcal{V}_2(\mathbb{K})$, and those of rank 3 correspond to points not contained in any such plane. Clearly, we may assume that p corresponds to a rank 3 symmetric matrix M. Let $y \in Y$ correspond to the matrix with a 1 in one place somewhere on the diagonal and 0 elsewhere. Then, clearly the line $\langle p, y \rangle$ contains a point t corresponding to a rank ≤ 2 matrix (in which case the assertion easily follows) if and only if the corresponding cofactor is non-zero. Hence, we may assume that all diagonal cofactors vanish. If the characteristic of $\mathbb{K}$ is 2, then, this implies that M is singular, a contradiction.

If the characteristic of $\mathbb{K}$ is not 2, then, we play the same game with the point $z \in Y$ corresponding with the rank 1 matrix all entries of which are 0 except for the entries in the north-west 2×2-square which are all 1. Since, M is non-singular, an easy calculation implies that $\langle p, z \rangle$ must contain a point distinct from z corresponding to a rank ≤ 2 matrix. □

PROPOSITION 3.3. *No non-trivial projection of the Lagrangian Grassmannian is a Lagrangian set.*

Proof. Since, (LS3) holds for the Lagrangian Grassmannian, it will hold for all its projections. Hence, we have to find a contradiction against (LS1) or (LS2), and it suffices to do so for the projection X' of $\mathsf{LG}(3, 6)(\mathbb{K})$ from an arbitrary point p.

Suppose first that p is contained in ζ_x, for each $x \in X$. By [**2**] we have $\operatorname{char}(\mathbb{K}) = 2$, p is contained in a unique quadratic space and it is the nucleus of the corresponding symp. Hence, the projection of this symp reduces to a 3-space, violating (LS1).

Hence, we may assume that there is some $x \in X$ with $p \notin \zeta_x$. In that case the 7-space $\langle p, \eta_x \rangle$ contains by Lemma 3.1 a point y of X outside η_x and at distance 3 from x in $\Gamma(X)$. So, in the projection from p the subspace generated by η_x contains the projection of y. Hence, our assertion boils down to showing that η_x cannot contain a point of X' at distance 3 from x. Let, for a contradiction, q be such a point. Lemma 3.2

implies that there is some line $L \subseteq X'$ through x such that the plane $\langle q, L\rangle$ contains a line $L' \neq L$ through x which is also contained in a symp through x. Now, since some point u on L is at distance 2 from q in $\Gamma(X)$, (LS2) yields $\langle q, u\rangle \subseteq X'$ and so the distance from q to x is at most 2, a contradiction. □

4. Proof of main result 1.

4.1. Pre-Veronesean caps. Let X be a spanning point set of $\mathbb{P}^N(\mathbb{K})$, $N \leq 5$, with $\mathbb{K}$ any skew field, which for the moment we allow to be isomorphic to $\mathbb{F}_2$, and let Ξ be a collection of at least two 2-spaces of $\mathbb{P}^N(\mathbb{K})$ such that for any $\xi \in \Xi$ the intersection $\xi \cap X$ is an oval in ξ. Suppose that, (X, Ξ) satisfies (VC1) and (VC2) above, in other words, suppose (X, Ξ) is a pre-Veronesean cap.

With "oval", we will in this section always refer to the intersection of X with a member of Ξ. If $\mathbb{K} \cong \mathbb{F}_2$, then an oval has only three points x, y, z not on a common line. In this case, there is a unique line L in $\langle x, y, z\rangle$ disjoint from $\{x, y, z\}$, and the unique point in $\langle x, y, z\rangle$ not on that line and not on the oval will be denoted by $x + y + z$; it is usually called the *nucleus* of the oval. The points of L will be denoted by $x + y, y + z, z + x$, where $\{x, y, x + y\}$ is a line, etc.

LEMMA 4.1. *Under the above assumptions, let $\pi \in \Xi$ and let U be a subspace of $\mathbb{P}^N(\mathbb{K})$ complementary to π. Then, the projection of $X \setminus X(\pi)$ from π onto U is injective.*

Proof. If x_1, x_2 are two points of $X \setminus X(\pi)$ projected onto the same point, then $\langle \pi, x_1.x_2\rangle$ is a 3-space. Hence, $[x_1, x_2]$ intersects π in a point of $\langle x_1, x_2\rangle$, which belongs to X by (VC2), contradicting the fact that $X([x_1, x_2])$ is an oval. □

LEMMA 4.2. *Under the above assumptions, we have $N = 5$.*

Proof. Suppose for a contradiction that $N \leq 4$.

First, suppose $\mathbb{K} \cong \mathbb{F}_2$. In this case, X contains an odd number of points. Indeed, if there are ℓ ovals containing a point $x \in X$, then, (VC1) implies that there are $2\ell + 1$ points in X. Since, there are at least two ovals, we have a point x and an oval $C = \{x_1, x_2, x_3\} \not\ni x$. The ovals $X([x, x_i])$, $i \in \{1, 2, 3\}$ are distinct, hence, $|X| \geq 7$. On the other hand, Lemma 4.1 implies that there are at most six points in X (three in the plane π and three projected onto the at most one-dimensional subspace U), a contradiction.

Now suppose $\mathbb{K} \ncong \mathbb{F}_2$. Clearly, (VC2) implies that $N \geq 4$, since $|\Xi| \geq 2$. Now suppose $N = 4$. Consider two intersecting ovals C, C_1, then the intersection x_1 of $\langle C\rangle$ and $\langle C_1\rangle$ belongs to X by (VC2). Let $x_2 \in C \setminus \{x_1\}$. Let C_2 be an oval containing x_2 and some point $y \in C_1 \setminus \{x_1\}$. We project $(C_1 \cup C_2) \setminus \{x_1, x_2\}$ from $\langle C\rangle$ onto a line L skew to C. It is clear that the images of both these sets comprise all points of the line L, except one, say p_1 and p_2, respectively (then p_i, $i = 1, 2$ corresponds to the tangent line in x_i at C_i). Since $|\mathbb{K}| > 2$, there is a point z on L in the image of both $C_1 \setminus \{x_1, y\}$ and $C_2 \setminus \{x_2, y\}$, contradicting Lemma 4.1. □

LEMMA 4.3. *Every pair of ovals intersects non-trivially.*

Proof. Suppose, by way of contradiction, that two ovals C and D do not meet. We consider the projection of $X \setminus C$ from $\langle C\rangle$ onto $\langle D\rangle$, which is injective by Lemma 4.1.

Again, we first suppose that $\mathbb{K} \cong \mathbb{F}_2$. Since $\langle D\rangle$ contains seven points, and since $|X|$ is odd, we have $|X| \in \{7, 9\}$. Hence, the geometry induced by the ovals on X is either a $2-(7, 3, 1)$ design or a $2-(9, 3, 1)$ design, which are both unique and isomorphic to

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$\mathbb{P}^2(\mathbb{F}_2)$ and $\mathbb{A}^2(\mathbb{F}_3)$, respectively. In the former case, every pair of ovals intersects; in the latter case, there exist three pairwise disjoint ovals C, D, E. Since $\langle C\rangle$ and $\langle D\rangle$ are also disjoint (by (VC2)), every point z_i of $E = \{z_1, z_2, z_3\}$ is contained in a unique line L_i that intersects both $\langle C\rangle$ and $\langle D\rangle$ non-trivially; put $C = \{x_1, x_2, x_3\}$ and $D = \{y_1, y_2, y_3\}$. Since the projection from $\langle C\rangle$ onto $\langle D\rangle$ is injective, we clearly have $L_i \cap \langle D\rangle \notin D$, for all $i \in \{1, 2, 3\}$. Similarly, $L_i \cap \langle C\rangle \notin C$, for all $i \in \{1, 2, 3\}$. At most one of L_1, L_2, L_3 contains $x_1 + x_2 + x_3$ (by Lemma 4.1 projecting from $\langle D\rangle$), and likewise at most one $y_1 + y_2 + y_3$. Hence, there is at least one line, say L_1 containing a point $x_2 + x_3$ and a point $y_2 + y_3$ (without loss of generality). The oval $X([x_1, z_1])$ contains at most one of $\{y_2, y_3\}$, hence we find an oval F containing, without loss of generality, the points z_1, y_2, x_3. Then, F is contained in the 3-space $\langle x_2, x_3, y_2, y_3\rangle$ and so $\langle F\rangle$ and $\langle x_2, y_3\rangle$ meet non-trivially, implying by (VC2) that $X([x_2, y_3])$ contains three collinear points, a contradiction.

So we may assume that $\mathbb{K} \ncong \mathbb{F}_2$. Let $x \in D$ be arbitrary. By the injectivity of the projection, and since $|\mathbb{K}| > 2$, the projections of the planes generated by the conics containing x and a point varying on C are distinct lines through x. Consequently, there is a conic E such that the projection E' is not contained in the tangent line to D at x. By injectivity, if t is the projection of the tangent line to E at $E \cap C$, then $E' \cup \{t\}$ is a full projective line, and $t \in D$. Let $u \in E \setminus (C \cup \{x\})$ be arbitrary. Since the projection is injective, the projection of $C_u := X([t, u])$ does not coincide with $\langle x, t\rangle$, and so the projection C'_u of C_u is an oval through t.

Now, let v be an arbitrary point of C and let $C_v = X([t, v])$. Let C'_v be the projection of C_v. Then, by injectivity, C'_v is not contained in $\langle x, t\rangle$. For finite $\mathbb{K} \ncong \mathbb{F}_2$, this is a contradiction, as there are precisely $|\mathbb{K}| + 1$ choices for v and exactly as many lines in $\langle D\rangle$ through p. So we may assume that $\mathbb{K}$ is infinite. But then we consider two choices for u, say u_1 and u_2, and we can choose v such that C'_v is neither contained in the tangent line to C'_{u_1} at t, nor in the tangent line to C'_{u_2} at t. By injectivity of the projection, C'_v is contained in a line minus two points (the latter are points in $C'_{u_1} \cup C'_{u_2}$, which are distinct, again by injectivity of the projection), a contradiction.

The proof of the lemma is complete. □

We can now finish the proof of Main Result 1. Since two distinct ovals always meet, the geometry of points of X and ovals is a projective plane. Now Theorem 2.3 of **[8]** completes the proof.

We now briefly study the case $|\mathbb{K}| = 2$. In this case, we have seven points in $\mathbb{P}^5(\mathbb{F}_2)$ and the geometry of ovals determines a projective plane of order 2 (a so-called *Fano plane*). Consider arbitrarily five points of X. In a Fano plane, every set of five points is the union of two lines, hence, by (VC2), the corresponding ovals generate a 4-space. Hence, every set of five points in X generates a 4-space. If every set of six points of X generates a 5-space, then X consists of a skeleton, and this is isomorphic to $\mathcal{V}_2(\mathbb{F}_2)$. Hence, we may assume that there is a 6-subset of X forming a skeleton in some 4-subspace U of $\mathbb{P}^5(\mathbb{F}_2)$. Since the seventh point must lie outside U, and every point in the Fano plane plays the same role, this gives rise to a projectively unique situation, and the resulting point set will be called a *disturbed Veronesean cap*. Hence, we have the following result.

PROPOSITION 4.4. *Let X be a pre-Veronesean cap in $\mathbb{P}^N(\mathbb{F}_2)$, $N \leq 5$, then it is either a Veronesean cap, and hence, X is projectively equivalent with $\mathcal{V}_2(\mathbb{F}_2)$, or it is a disturbed Veronesean cap.*

4.2. Singular pre-Veronesean caps. Motivated by the proof of Main Result 2, we will now extend Main Result 1 in case the ambient space has dimension at most 5. We will weaken the hypotheses to again end up with a Veronesean cap in case $\mathbb{K} \ncong \mathbb{F}_2$. For $\mathbb{K} \cong \mathbb{F}_2$, some more possibilities will turn up. The idea is to also allow degenerate conics, i.e., lines. However, we will only need to deal with the situation where the degenerate conic is a line with multiplicity 2 (and not a point, or a pair of distinct lines). These lines will be called *singular lines*, and, although a set containing lines is not a cap in the technical sense, we will call the new objects *singular pre-Veronesean caps*. This is harmless, as we will show that a singular pre-Veronesean cap is a Veronesean cap after all, at least when the underlying field is not the smallest field. In the latter case, a few more possibilities occur, see below.

Let X be a point set of $\mathbb{P}^5(\mathbb{K})$, with $\mathbb{K}$ any skew field, and let Ξ be a collection of 2-spaces (called the *quadratic planes*) of $\mathbb{P}^5(\mathbb{K})$ containing at least two elements and such that for any $\xi \in \Xi$ the intersection $\xi \cap X =: X(\xi)$ is an oval in ξ. Then, (X, Ξ) a called a *singular pre-Veronesean cap* if (VC1$'$) and (VC2) below hold.

(VC1$'$) If $x, y \in X$, then either all points of $\langle x, y\rangle$ belong to X, or there exists a unique member $[x, y]$ of Ξ containing both x and y.

(VC2) If $\xi_1, \xi_2 \in \Xi$, with $\xi_1 \neq \xi_2$, then $\xi_1 \cap \xi_2 \subset X$.

Clearly, every Veronesean cap is a singular pre-Veronesean cap. The converse is not true for $\mathbb{K} \cong \mathbb{F}_2$, and there are some counter examples.

Example 1 (The projected Veronesean cap). If we project one conic of the Veronesean cap $\mathcal{V}_2(\mathbb{F}_2)$ from its nucleus onto a secant, then we obtain a singular pre-Veronesean cap, as one checks easily. If $\{e_1, \ldots, e_6\}$ is a basis of $\mathbb{P}^5(\mathbb{F}_2)$, then such a set is projectively equivalent with $\{e_1, e_2, e_3, e_4, e_5, e_6, e_4 + e_5\}$. The corresponding set of quadratic planes contains six elements, namely those corresponding with the conics $\{e_1, e_2, e_4\}$, $\{e_2, e_3, e_5\}$, $\{e_3, e_4, e_6\}$, $\{e_1, e_5, e_6\}$, $\{e_2.e_6, e_4 + e_5\}$ and $\{e_1, e_3, e_4 + e_5\}$.

Example 2 (The biaffine singular cap). Let $\{e_1, \ldots, e_6\}$ again be a basis for $\mathbb{P}^5(\mathbb{F}_2)$ and let Ξ be the set of planes generated by the triples of points corresponding to the lines of a biaffine plane of order 3 (i.e., an affine plane with three points per line and one parallel class of lines removed, giving rise to parallel classes of points) with point set $X := \{e_1, e_2, e_1 + e_2, e_3, e_4, e_3 + e_4, e_5, e_6, e_5 + e_6\}$, where the triples $\{e_1, e_2, e_1 + e_2\}$, $\{e_3, e_4, e_3 + e_4\}$ and $\{e_5, e_6, e_5 + e_6\}$ are point parallel classes. Then, (X, Ξ) is a singular pre-Veronesean cap.

In a series of lemmas, we will show the following classification.

Proposition 4.5. *Every singular pre-Veronesean cap in $\mathbb{P}^5(\mathbb{K})$ is a Veronesean cap, except if $\mathbb{K} \cong \mathbb{F}_2$, in which case it could also be isomorphic to either a disturbed Veronesean cap, or a projected Veronesean cap, or a biaffine singular cap.*

So let (X, Ξ) be a singular pre-Veronesean cap, which we may assume to contain at least one singular line by Main Result 1. If all points of a certain subspace are contained in X, then we call that subspace *singular*. In the sequel, an *oval* is the intersection of X with a member of Ξ. We start with proving a lemma similar to Lemma 4.1 now using (VC1$'$) and (VC2) instead of (VC1) and (VC2).

Lemma 4.6. *Let $\pi \in \Xi$ and let U be a complementary subspace to π in U_p. Then, the projection from π onto U is injective when restricted to the points of $X \setminus \pi$ which are not on a singular line that intersects π.*

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Proof. Suppose two points $x, y \in X_p \setminus \pi$ have the same image. Then, $\langle x, y\rangle$ intersects π and so, if $\langle x, y\rangle$ is not singular, then the conic through x, y contains three collinear points, a contradiction. □

We now rule out singular subspaces of dimension at least 2. We denote by k the dimension of $\langle X\rangle$. First, we note that distinct maximal singular subspaces must be disjoint.

LEMMA 4.7. *Two singular subspaces U and V sharing a point z generate a singular subspace.*

Proof. Since every point in the span $\langle U, V\rangle$ is contained in the span of two lines containing z, one in U and one in V, it suffices to assume that both U and V are lines. Let p be arbitrary in $\langle U, V\rangle \setminus \{z\}$ and assume $p \notin X$. Choose points $x_1, x_2 \in U \setminus \{z\}$ and $y_1, y_2 \in V \setminus \{z\}$ such that $p = \langle x_1, y_1\rangle \cap \langle x_2, y_2\rangle$. Since $p \notin X$, Axiom (VC1′) implies that $[x_1, y_1]$ and $[x_2, y_2]$ are well defined. But Axiom (VC2) implies $p \in X$, a contradiction. Hence, $p \in X$ and so $\langle U, V\rangle$ is a singular plane. □

LEMMA 4.8. *There are no singular planes in X.*

Proof. Let U be a singular subspace of dimension $\ell \geq 2$ in X and assume that ℓ is maximal with this property. Since X contains at least one plane π such that $\pi \cap X_p$ is a conic, we see that $\ell \leq k - 2 \leq 3$. If $\ell = 3$, then $k = 5$, and we can consider a point $x \in X$ outside U. For any $u \in U$, the line $\langle u, x\rangle$ is non-singular, as Lemma 4.7 would otherwise lead to a singular subspace of dimension 4. Pick two distinct points $u, v \in U$. So we have an oval $C \subseteq X$ through x and u, and for each point y of $C \setminus \{u\}$, we have an oval C_y containing v and y. Let y_1, y_2 be two distinct points of $C \setminus \{u\}$. An arbitrary 4-space W through U not containing the tangent lines at v to the ovals C_{y_1} and C_{y_2}, respectively, intersects C_{y_i} in a point z_i, $i = 1, 2$. The line $\langle z_1, z_2\rangle$ intersects U and so is singular, a contradiction to Lemma 4.7.

Next suppose $\ell = 2$. If $k = 4$, then we argue similarly as above and obtain a contradiction. So we may assume that $k = 5$. Let π be a plane in $\mathbb{P}^5(\mathbb{K})$ skew to U. If $\pi \in \Xi$, then, by Lemma 4.6, the projection from π onto U is injective, even on $X \setminus \pi$, which leads to $X_p = U \cup X_p(\pi)$, a contradiction as is easily seen. Hence, all ovals intersect U nontrivially. Now, the projection of $X \setminus U$ from U onto π is also injective, as the line joining two points with same image must meet U and hence is singular, a contradiction to Lemma 4.7. If $\mathbb{K} \cong \mathbb{F}_2$, then considering all conics joining a point off U with a point of U, we obtain $7 + 1$ points of X off U, contradicting the injectivity. So suppose $\mathbb{K} \ncong \mathbb{F}_2$. Now let C_1, C_2 be two conics intersecting U in the same point u. The projection onto π of $C_1 \setminus \{u\}$ and $C_2 \setminus \{u\}$ are two affine lines (an affine line is the point set of a line, except for one point) $L_1 \setminus \{c_1\}, L_2 \setminus \{c_2\}$, respectively, where L_1, L_2 are lines of π and c_i is a point of L_i, $i = 1, 2$. Suppose $c_1 \neq c_2$. By injectivity, we may assume $c_1 \in L_2 \setminus \{c_2\}$. Take an arbitrary point c_1' of $L_1 \setminus \{c_1\}$. The conic defined by the inverse images of c_1, c_1' in X intersects U and projects into L_1, contradicting the injectivity (since that conic is certainly different from both C_1, C_2).

Hence all conics in X through the same point u of U project onto affine lines of π sharing the same point p_u. For different u, the points p_u are also different as otherwise, by injectivity of the projection, we find two conics through a common point of $X \setminus U$ intersecting in all points but the ones in U, a contradiction. This now implies that two different conics containing a (possibly different) point of U meet in a unique point of X. We now choose a line $L \subseteq U$ and project $X \setminus L$ from L onto some skew 3-space

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Σ. Let u_i, $i = 1, 2, 3$, be three distinct points on L. The conics through these points project onto three families of lines such that lines from different families intersect in a unique point. Considering two families, we see that these lie either on a hyperbolic quadric, and the third family cannot exist ($|\mathbb{K}| > 2$), or in a plane. In the latter case, we easily see that all points of $X_p \setminus U$ are contained in a 4-space together with L, a contradiction considering a conic through some point of $U \setminus L$ (and once again using $|\mathbb{K}| > 2$). $\square$

So we now know that X does not contain planes. Before we start a detailed analysis when there are singular lines, we note two easy properties.

LEMMA 4.9. *No point x of any singular line is contained in the span of two other singular lines. Also, no oval C misses at least two singular lines L_1, L_2.*

Proof. Note that by Lemma 4.7 singular lines do not intersect each other. But the transversal through x – i.e., the line through x intersecting the two singular lines – must be a singular line by (LS1), a contradiction. For the second assertion, let $x \in \langle C \rangle \cap \langle L_1, L_2 \rangle$ (our assumption implies $x \notin L_1 \cup L_2$) and consider the unique transversal to L_1, L_2 containing x. Then, Axiom (LS2) implies that $x \in X$ and the transversal is singular, a contradiction. $\square$

We first treat the case where X spans a 4-space. From now one, we will frequently have to make a distinction between $|\mathbb{K}| = 2$ en the rest (a few times also $|\mathbb{K}| = 3$ requires special arguments). Note that $|X|$ is odd if $|\mathbb{K}| = 2$ (if there are ℓ ovals through a point $x \in X$, then there are either $2\ell + 1$ points—if there is no singular line through x—or $2\ell + 3$—otherwise). Also, if $|\mathbb{K}| = 2$, the geometry induced by the singular lines and the ovals on X is a 2-design, which is the Fano plane if $|X| = 7$, and the affine plane of order 3 if $|X| = 9$.

LEMMA 4.10. *We have $k = \dim\langle X \rangle = 5$.*

Proof. Since there are at least two ovals, Axiom (VC2) implies $k \geq 4$. Hence for a contradiction, we assume $k = 4$. We claim that there are at most two singular lines. Indeed, suppose L_1, L_2, L_3 are three different singular lines. Notice that they are disjoint by Lemma 4.7. The 3-space $\langle L_1, L_2 \rangle$ intersects L_3 in at least a point, contradicting Lemma 4.9. The claim is proved.

Now suppose that there are precisely two singular lines L_1, L_2. Let $x_i \in L_i$, $i = 1, 2$ and consider the projection of $X \setminus [x_1, x_2]$ from $[x_1, x_2]$ onto some skew line L. Clearly, $\langle L_1, L_2, [x_1, x_2] \rangle$ is 4-dimensional, so we can choose L to contain a point y_i of L_i, $i = 1, 2$.

- If $|\mathbb{K}| = 2$, then the injectivity of the projection on $X \setminus ([x_1, x_2] \cup L_1 \cup L_2)$ implies $6 \leq |X| \leq 8$. Hence $|X| = 7$, contradicting the fact that in a Fano plane every two lines meet (and L_1 and L_2 are disjoint).
- If $|\mathbb{K}| > 2$, we consider three conics through y_1 intersecting $L_2 \setminus \{x_2\}$ non-trivially. These project onto three affine lines in L containing y_1, y_2. Hence, there is at least one point $L \setminus \{y_1, y_2\}$ covered twice. This contradicts the injectivity of the projection on $X \setminus ([x_1, x_2] \cup L_1 \cup L_2)$.

Now suppose that there is a unique singular line L. If some conic C is disjoint from L, then projecting $X \setminus C$ from $\langle C \rangle$ onto L implies that $X = C \cup L$, an easy contradiction. Hence, every conic intersects L and the geometry of conics and L is a projective plane (as in a 4-space every pair of planes intersects). We project $X \setminus C$ from $\langle C \rangle$ onto some disjoint line M, which we may assume to contain a point $x \in L \setminus C$.

- If $|\mathbb{K}| > 2$, then we may consider three conics through y, which all project onto some affine line in M containing x; this again implies that two distinct points have the same image giving rise to an extra singular line, a contradiction.
- If $|\mathbb{K}| = 2$, then $|X_p| = 7$ and we can coordinatize as follows: the points on L are $e_1, e_2, e_1 + e_2$. Let x_1, x_2, x_3 three arbitrary other points of X. The planes $[x_1, x_2]$ and $[x_2, x_3]$ generate the 4-space, hence we may assume that they are e_3, e_4, e_5, with $\{e_1, e_2, e_3, e_4, e_5\}$ a basis. There are two projectively inequivalent choices for the last point, namely $e_1 + e_3 + e_4 + e_5$ and $e_3 + e_4 + e_5$. In the former case, the plane $[e_1, e_1 + e_3 + e_4 + e_5]$, which we may assume to contain without loss of generality e_3, contains $e_4 + e_5$, a contradiction. In the latter case, we may assume that the conic planes through e_1 are $\langle e_1, e_3, e_4 \rangle$ and $\langle e_1, e_5, e_3 + e_4 + e_5 \rangle$, which both contain $e_3 + e_4$, a contradiction. □

So from now on, we may assume that $k = 5$. We first treat the case where there are at least three singular lines.

LEMMA 4.11. *There are at most three singular lines, and in case there are three of them,* $|\mathbb{K}| = 2$ *and* (X, Ξ) *is a biaffine singular cap.*

Proof. Suppose for a contradiction that there are at least four singular lines L_1, L_2, L_3, L_4. Then the 3-spaces $\langle L_1, L_2 \rangle$ and $\langle L_3, L_4 \rangle$ have a line K in common, with $K \cap X = \emptyset$. Each point $a \in K$ belongs to a transversal to L_1, L_2, and to a transversal to L_3, L_4. Axiom (LS2) now implies that these two transversals span a quadratic plane. We conclude that every point $a \in K$ is contained in a unique such quadratic plane, and so each point $x \in L_1 \cup L_2 \cup L_3 \cup L_4$ is contained in a unique oval C_x which intersects each L_i, $i \in \{1, 2, 3, 4\}$, non-trivially. This already implies $|\mathbb{K}| > 2$. Consider two such ovals C_x and C_y, with $x, y \in L_1$ and project $X \setminus C_x$ from C_x onto $\langle C_y \rangle$. Let a_i be the projection of L_i, $i = 2, 3, 4$.

- Suppose first that $|\mathbb{K}| > 3$. Let $u, v \in C_x \setminus \{x\}$ be arbitrary and consider the projections L_u and L_v of the ovals $X([u, y])$ and $X([v, y])$, respectively. If $L_u = L_v$, then at least three points on L_u are the image of at least two points of $X([u, y]) \cup X_p([v, y])$, and at most one of these is contained in C_y. So there are at least two singular lines intersecting C_x and projected off C_y. It follows that C_y misses at least two singular lines, a contradiction. Hence $L_u \neq L_v$, and since there are at least four choices for $u \in C_x \setminus \{x\}$, there is a choice such that L_u misses $\{a_2, a_3, a_4\}$, and hence $X([u, y])$ misses at least two of $\{L_2, L_3, L_4\}$ (it intersects at most one of these in the plane $\langle C_x \rangle$).
- Now let $|\mathbb{K}| = 3$. Then, an oval through x and a point of $C_y \setminus \{y\}$ gives rise to a point $z \in X$ not contained in $L_1 \cup L_2 \cup L_3 \cup L_4$. Every oval through z must have precisely three points in common with $L_1 \cup L_2 \cup L_3 \cup L_4$, which has 16 points, a contradiction as 16 is not divisible by 3.

Now assume that there are exactly three singular lines L_1, L_2, L_3. Suppose for a contradiction that some point $x \in X$ is not contained in $L_1 \cup L_2 \cup L_3$. Then, $\langle L_1, L_2 \rangle$ shares a point y with $\langle x, L_3 \rangle$. As above, this implies that x and the unique transversal to L_1, L_2 through y span a quadratic plane. We conclude that every point outside L_1, L_2, L_3 is contained in an oval intersecting each of L_1, L_2, L_3 (and so $|\mathbb{K}| > 2$). Let C be such an oval and project $X \setminus C$ from $\langle C \rangle$ onto some disjoint plane π. The projection of L_i is some point a_i, $i = 1, 2, 3$. Let $z \in X$ be a point not contained in $L_1 \cup L_2 \cup L_3 \cup C$, which we may assume to be contained in π.

- Suppose $|\mathbb{K}| > 3$. The conics through z and a point of C project into distinct lines of π through z, because, if not, then by Lemma 4.6, there are at least three points on such projection with inverse image consisting of at least three points, contradicting the fact that a_1, a_2, a_3 are the only such points, and they are not contained in one line. Hence, at least one such line misses a_1, a_2 and a_3, and so can only meet one of L_1, L_2, L_3 (namely, in a point of C), a contradiction to the second assertion of Lemma 4.9.
- Suppose $|\mathbb{K}| = 3$. Since x is contained in exactly one conic meeting each of L_1, L_2, L_3, and every other conic through x meets exactly two of L_1, L_2, L_3, an easy count implies that there are 1+9/2 conics through x, a contradiction.

Consequently, $|\mathbb{K}| = 2$ and we have exactly nine points and nine ovals, as is easily checked. These ovals form a biaffine plane; if we add the singular lines, we have an affine plane of order 3. The uniqueness of this structure in easily proved. □

LEMMA 4.12. *The set X cannot contain exactly two singular lines.*

Proof. Suppose for a contradiction that there are exactly two singular lines L_1, L_2.

- If $|\mathbb{K}| > 3$, then choose a conic C containing a point $x_i \in L_i$ and project $X \setminus C$ from C onto some disjoint plane π. Let a_i be the projection of L_i, $i = 1, 2$. Since $k = 5$, there is some point $x_3 \in X$, which we may assume to be in π, such that $\langle a_1, a_2, x_3\rangle = \pi$. As in the previous proof, no two conics through x_3 and a point of $C \setminus \{x_1, x_2\}$ project into the same line. Hence, we can find such an oval whose projection misses a_1 and a_2 and hence which does not contain a point of $L_1 \cup L_2$, a contradiction to Lemma 4.9.
- Now suppose $|\mathbb{K}| = 3$. If some point $x \in X_p \setminus (L_1 \cup L_2)$ is only contained in ovals which meet $L_1 \cup L_2$ in two points, then all points are contained in $\langle L_1, L_2, x\rangle$, contradicting $k = 5$.
  Hence, each point is contained in at least one oval intersecting $L_1 \cup L_2$ in exactly one point. Suppose the point $x \in X \setminus (L_1 \cup L_2)$ is contained in t ovals intersecting $L_1 \cup L_2$ in exactly two points; then it is contained $8 - 2t$ ovals intersecting $L_1 \cup L_2$ in exactly one point. Hence, $|X| = 1 + 3(8 - t)$ and it follows that t is constant. If $y \in L_1 \cup L_2$, then exactly four ovals through y intersect $L_1 \cup L_2$ in two points, leaving $9 - 3t$ points. Hence, there are $3 - t$ ovals through y intersecting $L_1 \cup L_2$ in just y. So in total there are $24 - 8t$ ovals intersecting $L_1 \cup L_2$ in just one point. On the other hand, there are $17 - 3t$ points in $X \setminus (L_1 \cup L_2)$, each in $8 - 2t$ ovals intersecting $L_1 \cup L_2$ in just one point. Hence, there are $\frac{(17-3t)(8-2t)}{3}$ ovals intersecting $L_1 \cup L_2$ in exactly one point. Equating the two expressions obtained for this number, we obtain $136 - 58t + 6t^2 = 72 - 24t$, implying $32 - 17t + 3t^2 = 0$, a contradiction.
- Now suppose $|\mathbb{K}| = 2$. A similar count as in the case $|\mathbb{K}| = 3$ implies (with similar definition for t) that $\frac{(7-2t)(6-2t)}{2} = 18 - 6t$, so $t = \frac{1}{2}$, a contradiction. □

The next lemma concludes the proof of Proposition 4.5.

LEMMA 4.13. *If the set X contains a unique singular line L, then $|\mathbb{K}| = 2$ and (X, Ξ) is a projected Veronesean cap.*

Proof.

- Suppose first $|\mathbb{K}| > 2$. We claim that every two ovals that intersect L, intersect mutually. Indeed, let C, D be two ovals intersecting L in x, y, respectively. Suppose

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C and D are disjoint. The projection from $\langle C\rangle$ onto $\langle D\rangle$ of $X \setminus (C \cup L)$ is injective. Hence, there are at least two ovals E_1, E_2 through y meeting C and projected onto affine lines A_1, A_2, respectively, whose projective extensions M_1, M_2, respectively, are not tangent to D at x. Injectivity implies that $M_i \setminus A_i \in D$, $i = 1, 2$. If there were a second oval D' through y disjoint from C, then its projection would be an oval through y tangent to both M_1, M_2, a contradiction. Hence, the points of $X \setminus (C \cup L)$ are projected onto the union U of a set of affine lines through y and the oval D. Now consider an oval through a point of $L \setminus \{x, y\}$ and some point of $C \setminus \{x\}$. Its projection is an affine line T through y, contained in U. Since no affine line can be contained in D, T is contained in the projective extension M of some projection A of an oval through y and some point of C. If $|\mathbb{K}| > 3$, then $|(A \cap T) \setminus \{y\}| \geq 2$, contradicting injectivity. If $|\mathbb{K}| = 3$, then there are 16 points in total, hence five ovals through a point z of $X \setminus L$. Consequently, there is an oval E through z disjoint from L. The projection of $X \setminus E$ from $\langle E\rangle$ onto a plane π skew to $\langle E\rangle$ containing L is injective. The four ovals through x intersecting E project into four distinct lines of π. But these lines should also differ from L, contradicting the fact that we have only four lines through x in π.

Hence, all conics that intersect L meet mutually. Projection from L onto some disjoint 3-space yields a system of $|\mathbb{K}| + 1$ families of $|\mathbb{K}|$ lines generating 3-space such that each pair of lines from different families intersect non-trivially. This is only possible for $|\mathbb{K}| = 2$.

- Hence, let $\mathbb{K} \cong \mathbb{F}_2$. If $|X| = 7$, then the four points of X off L are projectively unique; indeed, if $L = \{e_1, e_2, e_1 + e_2\}$, then the other points are e_3, e_4, e_5, e_6, where $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ is a basis. The conics and L form a Fano plane. If $|X| > 7$, then there is an oval disjoint from L, and hence projection from such an oval onto a plane containing L is injective, implying $|X| \leq 10$. Since $|X|$ is odd, we have $|X| = 9$ and the ovals and L form an affine plane of order 3. Hence, there are two disjoint ovals that are also disjoint from L. If $\{e_1, \ldots, e_6\}$ is a basis as above, we may assume that the two ovals are $C_1 = \{e_1, e_2, e_3\}$ and $C_2 = \{e_4, e_5, e_6\}$. Since projection from $\langle C_1\rangle$ onto $\langle C_2\rangle$ is injective on $X \setminus C_1$ and vice versa, and there is only one line disjoint from C_i in $\langle C_i\rangle$, $i = 1, 2$, we deduce that L is contained in $\langle e_1 + e_2, e_2 + e_3, e_3 + e_1, e_4 + e_5, e_5 + e_6, e_6 + e_4\rangle$. Hence without loss of generality, we may take $L = \{e_1 + e_2 + e_4 + e_5, e_2 + e_3 + e_5 + e_6, e_3 + e_1 + e_6 + e_4\}$. The plane $[e_1, e_4]$ does not contain $e_1 + e_2 + e_4 + e_5$ as it would also contain $e_2 + e_5$, which does not belong to X_p, but is also contained in $[e_2, e_5]$. Likewise $[e_1, e_4]$ does not contain $e_3 + e_1 + e_6 + e_4$. Hence, it must contain $e_2 + e_3 + e_5 + e_6$. Likewise, $[e_2, e_5]$ contains $e_3 + e_1 + e_6 + e_4$. But then $[e_1, e_4]$ and $[e_2, e_5]$ share the point $e_1 + e_2 + \cdots + e_6$, which does not belong to X, contradicting (VC2). □

5. Proof of main result 2.

5.1. General properties. In this section, (X, Ξ) is a Lagrangian set in $\mathbb{P}^N(\mathbb{K})$, with $\mathbb{K}$ a field and N possibly infinite. We denote by $\mathcal{G}(X)$ the corresponding geometry of points and singular lines, and by $\Gamma(X)$ we denote the point graph of $\mathcal{G}(X)$ (which is the graph with point set X and adjacency is collinearity). The diameter of $\Gamma(X)$ is by definition the diameter of (X, Ξ). The distance between two points $x, y \in X$ in $\Gamma(X)$ is denoted by $\delta(x, y)$. Two points of X on a singular line will be called X-collinear. The

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elements of Ξ are called the *quadratic spaces*. Subspaces of $\mathbb{P}^N(\mathbb{K})$ consisting entirely of points of X are called *singular*.

The following is exactly the Quadrangle Lemma of [**9**], proved there for similar objects, although having diameter 2. We give a proof for completeness' sake.

LEMMA 5.1 (The quadrangle lemma). *Let L_1, L_2, L_3, L_4 be four (not necessarily pairwise distinct) singular lines such that L_i and L_{i+1} share a (not necessarily unique) point p_i, $i = 1, 2, 3, 4$ mod 4, and suppose that p_1 and p_3 are not X-collinear. Then, L_1, L_2, L_3, L_4 are contained in a unique common symp.*

Proof. Since $\langle p_1, p_3\rangle$ is not singular, we can pick a point $p \in \langle p_1, p_3\rangle$ which does not belong to X. Since p_1 and p_3 are X-collinear with p_2, we have $\delta(p_1, p_3) = 2$. Hence, by (LS1), there is a unique quadratic space ξ containing p_1 and p_3. We choose two arbitrary distinct lines M_1, M_2 through p inside the plane $\langle L_1, L_2\rangle$ not containing p_2. Denote $M_i \cap L_j = \{p_{ij}\}$, $\{i, j\} \subseteq \{1, 2\}$, then $\delta(p_{i1}, p_{i2}) = 2$, $i = 1, 2$. By (LS1), there is a quadratic space ξ_i containing p_{i1} and p_{i2}, $i = 1, 2$. If $\xi_1 \neq \xi_2$, then (LS2) implies that p, which is contained in $\xi_1 \cap \xi_2$, belongs to X, a contradiction. Henc,e $\xi_1 = \xi_2 = \xi$ and contains L_1, L_2. We conclude ξ contains L_1, L_2, and similarly also L_3, L_4. □

Now let $p \in X$ be arbitrary. Let U_p be a hyperplane of T_x not containing p and define X_p to be the set of points obtained by intersecting U_p with all singular lines of X through p. Let Ξ_p be the set of subspaces of U obtained by intersecting U with all tangent spaces at p to the symps of (X, Ξ) through p. The pair (X_p, Ξ_p) is called the *residue* of (X, Ξ) in p. We denote the dimension of U_p by k. Note $k \leq 5$.

We have the following result.

LEMMA 5.2. *For every $p \in X$, the residue (X_p, Ξ_p) is a singular pre-Veronesean cap.*

Proof. Clearly, for any $\xi \in \Xi_p$, we have $X_p \cap \xi$ is a conic. Also, clearly (VC2) is inherited from (X, Ξ). Now suppose $x, y \in X_p$. Assume first that some point of $\langle x, y, p\rangle$ does not belong to X. Then, there are two points on $\langle x, p\rangle \cup \langle y, p\rangle$ which are not X-collinear and the Quadrangle Lemma implies that a unique quadratic space ξ contains $\langle x, p\rangle \cup \langle y, p\rangle = X \cap \langle x, y, p\rangle$. In this case, $T_p(\xi) \cap X_p$ is a conic. Assume now that all points of $\langle x, y, p\rangle$ belong to X. Then, all points of $\langle x, y\rangle$ belong to X_p. This shows (VC1′).

Since Ξ contains at least two elements, it follows from the connectivity and (LS1) that there is at least one symp ξ through p. Now let $x \in X \setminus \xi$. Let $(p, p_1, p_2, p_3, \ldots, x)$ be a minimal path connecting p and x in $\Gamma(X)$. If $p_2 \notin \xi$, then $X([p, p_2])$ is a second symp through p. So suppose $p_2 \in \xi$. Then, $p_2 \neq x$ and p_3 exists. But now we find a point y outside ξ in $X([p_1, p_3])$ collinear with p_1, and so $X([p, y])$ is a symp distinct from ξ and containing p. Hence, $|\Xi_p| \geq 2$ and the lemma is proved. □

The previous lemma motivates the following terminology. For $p \in X$, if (X_p, Ξ_p) is a Veronesean cap, we call p a *straight point*. All points are straight as soon as $\mathbb{K} \ncong \mathbb{F}_2$. If $\mathbb{K} \cong \mathbb{F}_2$, then we also have *almost straight points* (when (X_p, Ξ_p) is a projected Veronesean), 1-*singular points* (when (X_p, Ξ_p) contains exactly one singular line) and 3-*singular points* (when (X_p, Ξ_p) contains exactly three singular lines).

5.2. Lagrangian sets of diameter 2. We now suppose that $\Gamma(X)$ has diameter 2 and prove the following lemma.

LEMMA 5.3. *If* (X, Ξ) *has diameter* 2, *and* $x \in X$ *is not a* 3-*singular point, then every point at distance* 2 *from* x *in* $\Gamma(X)$ *is a* 3-*singular point.*

Proof. Let $y \in X$ be at distance 2 from x and let L be any singular line of X through y not contained in $[x, y]$. Choose a point $z \in L \setminus \{y\}$. Since x is not 3-singular, every pair of ovals in (X_x, Ξ_x) intersects and hence the symps $X([x, y])$ and $X([x, z])$ intersect in a line M containing x. Consequently, looking in $X([x, z])$, there is a point x' of M collinear with z. Clearly, $x' \neq y$, hence the Quadrangle Lemma implies that y and x' are X-collinear, and hence $\langle z, y, x'\rangle$ is a singular plane.

Hence, every point of (X_y, Ξ_y) not on the oval corresponding to $[x, y]$ is contained in a singular line. This implies, by Proposition 4.5, that y is a 3-singular point. □

We are now ready to prove the nonexistence of Lagrangian sets of diameter 2.

PROPOSITION 5.4. *Lagrangian sets of diameter* 2 *do not exist.*

Proof. Since, if $\mathbb{K} \ncong \mathbb{F}_2$, every point of X is straight, the assertion follows in that case directly from Lemma 5.3.

Now suppose $\mathbb{K} \cong \mathbb{F}_2$. Suppose first that some point p is not 3-singular. Then, we can select two singular lines L_1, L_2 through p which are not contained in a singular plane. It follows that, if $x_i \in L_i$, $i = 1, 2$, are points distinct from p, none of the points x_1, x_2 is 3-singular. But by our choice, we have $\delta(x_1, x_2) = 2$. This contradicts Lemma 5.3.

Hence, we may assume that all points of X are 3-singular. Select an arbitrary point p; there are nine symps through p giving rise to exactly 72 points of X at distance 2 from p. There are nine singular lines through p giving rise to exactly 18 points X-collinear with p. Together with p, this amounts to $91 = |X|$ points. A double count of the pairs $(x, \xi) \in X \times \Xi$ with $x \in \xi$ yields $91 \times 9 = |\Xi| \times 15$, a contradiction. □

REMARK 5.5. In fact, in the proof of Proposition 5.4 we did not use Axiom (LS3) explicitly anymore; the facts that in every residue every pair of conics intersects non-trivially and that there are no singular planes, or that $|\mathbb{K}| = 2$ and that either every residue has seven points with at most one singular line, or nine points with exactly three singular lines, suffice.

5.3. Lagrangian sets of diameter at least 3. From now on, we may assume that the diameter of $\Gamma(X)$ is either unbounded or at least 3. We first aim at showing that the diameter is always equal to 3. Along the way, this will also prove that there are no singular planes.

LEMMA 5.6. *If* π *is a singular plane, then every point* $p \in X$ *not contained in* π *is* X-*collinear with exactly one point of* π.

Proof. Suppose for a contradiction that no point of π is X-collinear with p. Then, by connectivity, we may assume that there is a point $x \in \pi$ with $\delta(p, x) = 2$. Note that, by Proposition 4.5, every symp through x intersects π in a line. Applied to $X([p, x])$, this yields a line $L \in \pi \cap [p, x]$. Inside the symp $X([p, x])$, there is a point $y \in L$ that is X-collinear with p, contradicting our hypothesis. Hence, there is always at least one point of π collinear with $p \in X \setminus \pi$.

If at least two points $x_1, x_2 \in \pi$ are collinear with p, then $\pi' = \langle x_1, x_2, p\rangle$ is singular and Lemma 4.7 applied to the residue at x_1 leads to a singular plane in that residue, a contradiction to Lemma 4.8. □

LEMMA 5.7. *There are no singular planes in X.*

Proof. This follows from Proposition 4.5 if $\mathbb{K} \not\cong \mathbb{F}_2$; so suppose $\mathbb{K} \cong \mathbb{F}_2$. For a contradiction, suppose there is a singular plane, and hence X contains some 1-singular point or 3-singular point. If all points are either 1-singular or 3-singular, then every point is contained in a singular plane and consequently, using Lemma 5.6, the diameter of $\Gamma(X)$ equals 2, a contradiction.

Hence, we may assume that some point p is (almost) straight. Noting that (almost) straight points are never X-collinear with 3-singular points, connectivity leads to the existence of a 1-singular point q. Since q is contained in a singular plane, it is at distance ≤ 2 from any other point of X, and hence every other point is contained in a symp together with q. Since there are six symps through q, there are exactly 48 points at distance 2 from q; since there are seven singular lines through q, there are 14 points X-collinear with q. Hence $|X| = 63$. But a similar count yields already $1 + 14 + 56 = 71$ points at distance 0, 1, 2 from p, a contradiction. □

LEMMA 5.8. *The graph $\Gamma(X)$ has diameter* 3.

Proof. Suppose for a contradiction that x_1, x_2, x_3, x_4, x_5 are five points of X with $\delta(x_i, x_j) = |i-j|$, $i, j \in \{1, 2, 3, 4, 5\}$. The symps $X([x_1, x_3])$ and $X([x_3, x_5])$ intersect in a line L. It follows that there are points z_1, z_5 on L which are X-collinear to x_1, x_5, respectively. This leads to a path (x_1, z_1, z_5, x_5) or (x_1, z_1, x_5) of length 3 or 2 joining x_1 to x_5 (depending on whether $z_1 \neq z_5$ or $z_1 = z_5$), a contradiction. □

We can now determine the isomorphism class of the geometry of points and singular lines of X.

LEMMA 5.9. *If $\mathcal{L}$ denotes the set of singular lines of X, then $(X, \mathcal{L})$ is the dual polar space associated to the building of absolute and relative type C_3 over the field $\mathbb{K}$; in other words, X can be viewed as the set of totally isotropic planes with respect to a symplectic polarity in $\mathbb{P}^5(\mathbb{K})$, and the singular lines correspond to the planes intersecting in a common totally isotropic line with respect to that polarity.*

Proof. Define a geometry $\mathcal{G}$ over the type set $\{1, 2, 3\}$ where the points of X are the elements of type 3, the singular lines in X are the elements of type 2, and the symps in X are the elements of type 1. Incidence is symmetrized containment. From the previous, it follows that this is a geometry of type C_3. Moreover, properties (LL) and (O) of [**12**], p. 543, required for the geometry to correspond to a building, are in our setting equivalent to the requirement that if two lines are both contained in two distinct quads S_1 and S_2, then they coincide, which trivially holds. Hence, the geometry corresponds to a building, and since the residue of the elements of type 1 are precisely the symps, hence orthogonal quadrangles $\mathsf{Q}(4, \mathbb{K})$, we see that $\mathcal{G}$ is the geometry of the totally isotropic subspaces of a symplectic polarity in $\mathbb{P}^5(\mathbb{K})$. Consequently, $(X, \mathcal{L})$ is the corresponding dual polar space $\mathsf{DW}(5, \mathbb{K})$. □

If $|\mathbb{K}| > 2$, then by [**4**] and [**5**], $\mathsf{LG}(3, 6)(\mathbb{K})$ is the absolute universal embedding of $(X, \mathcal{L})$, and Proposition 3.3 completes the proof of Main Result 2.

Finally, suppose $|\mathbb{K}| = 2$.

Let Y be the point set of the universal embedding of $\mathsf{DW}(5, \mathbb{F}_2)$. By [**13**], Y spans a 14-dimensional space $\mathbb{P}^{14}(\mathbb{F}_2)$ and the stabilizer of Y in $\mathsf{PGL}_{15}(2)$ induces the full

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group of automorphisms of $\mathsf{DW}(5, \mathbb{F}_2)$; in particular, it is transitive on the points of Y. For $y \in Y$, denote by η_y the subspace of $\mathbb{P}^{14}(\mathbb{F}_2)$ generated by all points of Y collinear with y in $\mathsf{DW}(5, \mathbb{F}_2)$. We now show that $\mathbb{P}^{14}(F_2)$ is generated by $\langle \eta_y, \eta_z \rangle$, for $y, z \in Y$ at distance 3 from each other.

Let x be an arbitrary point of Y. If x is at distance at most 2 from one of y and z, then we claim it is contained in $\langle \eta_y, \eta_z \rangle$. Indeed, suppose x is at distance 2 from y. Let S be the symp through x and y; then there is a unique point $t \in S$ collinear with z. Since t cannot be collinear with y, S is generated by t and the points of S in η_y. Hence, $S \subseteq \langle \eta_y, \eta_z \rangle$ and the claim follows.

So we may assume that x has distance 3 from both y and z. In the polar space $\mathsf{W}(5, \mathbb{F}_2)$, the points x, y, z correspond to mutually disjoint planes π_x, π_y, π_z. We claim that there is a plane π intersecting π_x in a line and $\pi_y \cup \pi_z$ in a single point. Indeed, clearly no plane intersecting π_x in a line can meet $\pi_y \cup \pi_z$ in more than two points. Suppose now, for a contradiction, that each plane π_L which intersects π_x in a line L and π_y in a point y_L (there are precisely seven such planes) intersects π_z in a point z_L. If $y_L = y_M$, for L, M lines of π_x, then y_L is collinear with all points of π_x, a contradiction. Since the planes π_x, π_y, π_z are disjoint, one deduces that the mapping $\pi_y \to \pi_z : y_L \mapsto z_L$ induces a collineation, and so also the mapping $\pi_y \to \pi_x : y_L \mapsto L \cap \langle y_L, z_L \rangle$ is a collineation. Now the projection mapping $\pi_x \to \pi_y : L \mapsto y_L$ is a duality; hence the mapping $\pi_x \to \pi_x : L \mapsto L \cap \langle y_L, z_L \rangle$ is a duality, every point of which is incident with its image. It is easy to see that this is a contradiction. This proves our claim. So there are planes α_y and α_z intersecting π_x in a common line L, intersecting π_y and π_z, respectively, in some point, and disjoint from π_z and π_y, respectively.

Now, this implies that the line L' in Y corresponding to the line L of $\mathsf{W}(5, \mathbb{F}_2)$ contains the point x, and the points at distance 2 from y and z on L' are distinct. Hence, $x \in \langle \eta_y, \eta_z \rangle$ by our first claim, and we have shown $\langle \eta_y, \eta_z \rangle$ is the whole space. By transitivity of the automorphism group on Y, we either have $\dim \eta_z = \dim \eta_y = 6$, or $\dim \eta_z = \dim \eta_y = 7$. In the former case, $\dim\langle \eta_y, \eta_z \rangle \leq 13$, a contradiction. Hence $\dim \eta_y = 7$, for all $y \in Y$. Since $\mathsf{LG}(3, 6)(\mathbb{F}_2)$ is isomorphic to the projection of Y from a point $c \notin Y$, Axiom (LS3) yields that c is contained in η_y, for all $y \in Y$. Choosing coordinates in η_y appropriately, we may assume $y = (1, 0, 0, 0, 0, 0, 0, 0, 0)$, and the other points of $Y \cap \eta_y$ are $(0, \dots, 0, 1, 0, \dots, 0)$ and $(1, 0, \dots, 0, 1, 0, \dots, 0)$ (the 1 is twice in the ith position), $i = 2, \dots, 8$. The point c consequently has coordinates either $(1, 1, \dots, 1)$ or $(0, 1, \dots, 1)$. Without loss of generality, we may assume the former.

Now suppose for a contradiction that X does not arise from Y by projection from c. Then, it must arise from Y by projection from a subspace C that intersects η_y in a unique point y_C, for every $y \in Y$. Since (the projection of) y is either a straight or an almost straight point, the point y_C either has coordinates $(0, 1, \dots, 1)$ (in case of a straight point), or we may assume without loss of generality that y_C has coordinates $(0, 0, 1, \dots, 1)$. In both cases, the projection of c coincides with the projection of a point of $Y \cap \eta_y$, namely, y and $(1, 1, 0, \dots, 0)$, respectively. Since this holds for all $y \in Y$, it implies that the projection from C is not injective on Y, a contradiction.

Hence, X arises from Y by projection from c, and we obtain $\mathsf{LG}(3, 6)(\mathbb{F}_2)$. The proof of Main Result 2 is complete.

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