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QUASI θ-SPACES AND PAIRWISE θ-PERFECT IRREDUCIBLE MAPPINGS

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Abstract

In this paper we extend the notion of perfect, θ-continuous, irreducible and θ-perfect mappings to bitopological spaces. The main result is the following: the (small) image of an (i, j)-canonical open sets is an (i, j)-canonical open set under a pairwise θ-closed irreducible surjective mapping. Also we extend the notion of θ-proximity spaces to quasi θ-proximity spaces and point out the interrelation between it and separated quasi-proximity spaces by means of a pairwise θ-perfect irreducible mappings.

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1. Introduction

The notion of bitopological spaces was introduced by Kelly [10]. In this paper we investigate a less restrictive definition of pairwise perfect maps than that given by M. C. Datta [2] and study some of its properties. Then we introduce and study the concepts of pairwise θ-continuous, pairwise irreducible and pairwise θ-perfect mappings. Furthermore, we introduce the notion of a quasi θ-proximity space and prove the following.

(1) The (small) image of an (i, j)-canonical open set is an (i, j)-canonical open set under a pairwise θ-closed irreducible mapping.
(2) Every separated quasi-proximity space is a quasi θ-proximity space.
(3) A bitopological space admits a maximal quasi θ-proximity if the space is pairwise Hausdorff.

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(4) If f is a pairwise θ-perfect irreducible mapping from a pairwise Tychonoff space (X, τ_1, τ_2) onto a pairwise Hausdorff space (Y, Δ_1, Δ_2) and if δ is a compatible separated quasi-proximity on (X, τ_1, τ_2), then there exists a quasi θ-proximity θ on (Y, Δ_1, Δ_2), associated with f and δ.

Finally, we like to remark in the context of the present paper that, by $i, j, i, \neq j$, we mean that i is either 1 or 2 for instance if $i = 1$ then $j = 2$. Also we will use $P-$ to denote pairwise and "bts" to denote bitopological space.

2. Preliminaries

Let (X, τ_1, τ_2) be a bts and A a subset of X. The closure and interior of A with respect to τ_i are denoted by τ_i-cl(A) and τ_i-int(A), respectively. The family of all τ_i-closed sets will be denoted by τ_i'. When the appropriate topology is clear from the context, O_A (respectively O_x) denotes an open set containing A (respectively an open neighbourhood of x).

DEFINITION 2.1 [10, 14]. A bts (X, τ_1, τ_2) is called

(1) $PT_1 \Leftrightarrow (\forall x \in X)(\forall i \in \{1, 2\})(\{x\} = \tau_i\text{-cl}\{x\})$

(2) PT_2 or P-Hausdorff $\Leftrightarrow (\forall x, y \in X,\ x \neq y)(\exists O_x \in \tau_i)(\exists O_y \in \tau_j)(O_x \cap O_y = \varnothing)$

(3) $PT_{2\frac{1}{2}}$ or P-Urysohn $\Leftrightarrow (\forall x, y \in X,\ x \neq y)(\exists O_x \in \tau_i)(\exists O_y \in \tau_j)(\tau_j\text{-cl}(O_x) \cap \tau_i\text{-cl}(O_y) = \varnothing)$

(4) PR_2 or P-regular $\Leftrightarrow (\forall x \in X)(\forall Ox \in \tau_i)(\exists O_x^* \in \tau_i)(\tau_j\text{-cl}(O_x^*) \subseteq O_x)$

(5) $PR_{2\frac{1}{2}}$ or P-completely regular $\Leftrightarrow (\forall x \in X)(\forall F \in \tau_i',\ x \notin F)$ ($\exists$ a mapping $f\colon X \to [0, 1]$)(f is τ_i-lower semicontinuous, and f is τ_j-upper semicontinuous and $f(x) = 0$ and $f(F) = 1$), where $[0, 1]$ is the closed unit interval

(6) $PT_{3\frac{1}{2}}$ or P-Tychonoff if and only if it is $PR_{2\frac{1}{2}}$ and PT_1.

DEFINITION 2.2 [10]. A mapping $f\colon (X, \tau_1, \tau_2) \to (Y, \Delta_1, \Delta_2)$ is called *P-continuous* (respectively *P-open*, *P-closed*) if the induced mappings $f\colon (X, \tau_i) \to (Y, \Delta_i)$, $i = 1, 2$, are continuous (respectively open, closed).

DEFINITION 2.3 [8]. A cover $\mathscr{U}$ of a bts (X, τ_1, τ_2) is called *a $\tau_1\tau_2$-open cover* if $\mathscr{U} \subseteq \tau_1 \cup \tau_2$. If in addition $\mathscr{U}$ contains at least one nonempty member of τ_1 and at least one nonempty member of τ_2, then $\mathscr{U}$ is called *a P-open cover.*

Although there are several different notions of P-compactness in the

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literature [1, 8, 11], we use the definition given in [8]. An equivalent concept of P-compactness has been introduced by Y. M. Kim [11].

DEFINITION 2.4 [8]. A bts (X, τ_1, τ_2) is called *P-compact* if every P-open cover of X has a finite subcover.

We make use of the following results from [8].

RESULTS 2.5 [8]. (1) *P-compactness is P-continuous invariant.*

(2) *In a P-Hausdorff space, a τ_i-compact subset is τ_j-closed.*

(3) *If (X, τ_1, τ_2) is P-compact, then a proper τ_i-closed subset is τ_j-compact.*

DEFINITION 2.6. A subset A of a bts (X, τ_1, τ_2) is called *(i, j)-canonical open* (or *(i, j)-regular open*) if $A = \tau_i\text{-}\mathrm{int}(\tau_j\text{-}\mathrm{cl}(A))$. Specifically, $(\forall A \subseteq X)(\tau_i\text{-}\mathrm{int}(\tau_j\text{-}\mathrm{cl}(A))$ is always (i, j)-canonical open).

DEFINITION 2.7 [5]. If $f: X \to Y$ is a mapping from X into Y and $A \subseteq X$, then we define a mapping $f^{\#}: 2^X \to 2^Y$ by

$$f^{\#}(A) = \{y | y \in Y \text{ and } f^{-1}(\{y\}) \subseteq A\},$$

and $f^{\#}(A)$ is called the *small image of A under the mapping f.*

THEOREM 2.8 [5]. *The mapping $f^{\#}$ has the following properties*:

(1) $f^{\#}(A) \subseteq f(A)$;
(2) $f^{\#}(A) = \mathrm{co}(f(\mathrm{co}\, A))$, *where* co *denotes complementation*;
(3) $f^{\#}(A \cap B) = f^{\#}(A) \cap f^{\#}(B)$;
(4) $f^{-1} f^{\#}(A) \subseteq A$.

DEFINITION 2.9 [13]. A mapping $\delta: 2^X \times 2^X \to \{0, 1\}$ is called a *quasi-proximity on X* if it satisfies the following axioms:

(P_1) $\delta(A, B) = 0 \Rightarrow A \neq \varnothing$ and $B \neq \varnothing$;
(P_2) $\delta(A, B \cup C) = \delta(A, B) \cdot \delta(A, C)$ and,
$\delta(A \cup B, C) = \delta(A, C) \cdot \delta(B, C)$;
(P_3) $A \cap B \neq \varnothing \Rightarrow \delta(A, B) = 0$;
(P_4) $\delta(A, B) = 1 \Rightarrow (\exists U \subseteq X)(\delta(A, U) = \delta(\mathrm{co}\, U, B) = 1)$.

The pair (X, δ) is called a *quasi-proximity space.* A quasi-proximity δ is said to be *separated* if it satisfies the following axiom:

(P_5) $\delta(\{x\}, \{y\}) = 0 \Leftrightarrow x = y$.

If δ is a quasi-proximity, then δ^{-1}, defined by $\delta^{-1}(A, B) = \delta(B, A)$, is also a quasi-proximity and it is called the *conjugate of δ*.

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DEFINITION 2.10 [12]. If (X, δ) is a quasi-proximity space, then two topologies $\tau(\delta)$ and $\tau(\delta^{-1})$ are defined on X if for arbitrary $A \subseteq X$ we let

$$\tau(\delta)\text{-cl}(A) = \{x \in X : \delta(\{x\}, A) = 0\},$$

and

$$\tau(\delta^{-1})\text{-cl}(A) = \{x \in X : \delta(A, \{x\}) = 0\}.$$

DEFINITION 2.11. A quasi-proximity space (X, δ) is called *compatible with a bts* (X, τ_1, τ_2) if $\tau(\delta) = \tau_1$ and $\tau(\delta^{-1}) = \tau_2$.

LEMMA 2.12. *The axiom* (P_4) *implies the following axiom:*
(P_4^*) $\delta(A, B) = 1 \Rightarrow (\exists U = \tau(\delta^{-1})\text{-int}(\tau(\delta)\text{-cl}(U)))(\delta(A, U) = \delta(\text{co}(\tau(\delta))\text{-cl}(U)), B) = 1)$.

DEFINITION 2.13. A bts (X, τ_1, τ_2) is said to be *P-extremally disconnected* if the τ_i-closure of each τ_j-open sets is τ_j-open.

3. Pairwise perfect mappings

DEFINITION 3.1. A P-continuous, P-closed mapping f from a bts (X, τ_1, τ_2) into a bts (Y, Δ_1, Δ_2) is called *P-perfect* if it satisfies

$$(\forall y \in Y)(\forall i \in \{1, 2\})(f^{-1}(\{y\}) \text{ is } \tau_i\text{-compact subset in } X).$$

Our definition of P-perfect mappings differs from the definition given by Datta [2] in that we do not insist that point inverses by P-compact.

LEMMA 3.2. *Every P-continuous mapping from a P-compact-bts* (X, τ_1, τ_2) *into a* PT_2*-bts* (Y, Δ_1, Δ_2) *is P-perfect.*

PROOF. Let $A \in \tau_i' \backslash \{X, \varnothing\}$. Since (X, τ_1, τ_2) is P-compact, by 2.5(3), A is a τ_j-compact subset of X. Hence by 2.5(1), $f(A)$ is a τ_j-compact subset of Y. So by 2.5(2), $f(A) \in \Delta_i'$ and hence f is P-closed.

To prove (iii), consider $y \in Y$. Since (Y, Δ_1, Δ_2) is PT_2, then it is PT_1 and hence $\{y\} \in \Delta_j'$. By P-continuity of f it follows that $f^{-1}(\{y\}) \in \tau_j'$ and hence by 2.5(3), $f^{-1}(\{y\})$ is a τ_i-compact subset of X.

THEOREM 3.3. *Let* (X, τ_1, τ_2) *be a* PR_2*-bts and let* A *be* τ_i*-compact. Then* $(\forall B \in \tau_i')(A \cap B = \varnothing \Rightarrow (\exists O_A \in \tau_i)(\exists O_B \in \tau_j)(O_A \cap O_B = \varnothing))$.

PROOF. Since (X, τ_1, τ_2) is a PR_2-bts, it follows that $(\forall x \in A)(\exists O_x \in \tau_i)(\exists O_B^{(x)} \in \tau_j)(O_x \cap O_B^{(x)} = \varnothing)$. Clearly $(O_x)_{x \in A}$ is an τ_i-open cover of

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A, so there exists a finite subcover $(O_{x_s})_{s=1}^n$ of A. One readily verifies that $O_A = \bigcup_{s=1}^n O_{x_s}$ and $O_B = \bigcup_{s=1}^n O_B^{(x_s)}$ have the required property.

THEOREM 3.4. *If* (X, τ_1, τ_2) *is a* PT_2*-bts,* $x \in X$ *and* B *is* τ_i*-compact such that* $x \notin B$, *then* $(\exists O_x \in \tau_j)(\exists O_B \in \tau_i)(O_x \cap O_B = \varnothing)$. *Moreover, if* A *is* τ_j*-compact and* B *is* τ_i*-compact such that* $A \cap B = \varnothing$, *then* $(\exists O_A \in \tau_j)(\exists O_B \in \tau_i)(O_A \cap O_B = \varnothing)$.

PROOF. Theorem 3.4 can be proved similarly to Theorem 3.3.

THEOREM 3.5. *The axioms* PT_2, PR_2 *and* PR_3 *are invariant under a P-perfect surjective mapping.*

PROOF. Let f be an P-perfect mapping from a PT_2-bts (X, τ_1, τ_2) onto an arbitrary bts (Y, Δ_1, Δ_2). Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Then we have $f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\}) = \varnothing$. Moreover, since f is P-perfect, $f^{-1}(\{y_1\})$ and $f^{-1}(\{y_2\})$ are τ_i-compact. Hence by Theorem 3.4, we have $(\exists O_{f^{-1}(\{y_1\})} \in \tau_i)(\exists O_{f^{-1}(\{y_2\})} \in \tau_j)(O_{f^{-1}(\{y_1\})} \cap O_{f^{-1}(\{y_2\})} = \varnothing)$. Putting $U = \mathrm{co}(f(\mathrm{co}(O_{f^{-1}(\{y_1\})})))$ and $V = \mathrm{co}(f(\mathrm{co}(O_{f^{-1}(\{y_2\})})))$, we obtain the following.

(i) $y_1 \in U$ and $y_2 \in V$. Indeed, from $f^{-1}(\{y_1\}) \subseteq O_{f^{-1}(\{y_1\})}$ we obtain $f^{-1}(\{y_1\}) \cap \mathrm{co}(O_{f^{-1}(\{y_1\})}) = \varnothing$. Then $ff^{-1}(\{y_1\}) \cap f(\mathrm{co}(O_{f^{-1}(\{y_1\})})) = \varnothing$ and hence, since f is surjective, $y_1 \notin f(\mathrm{co}(O_{f^{-1}(\{y_1\})}))$ or equivalently, $y_1 \in U$.

(ii) $U \in \Delta_i$ and $V \in \Delta_j$, since f is P-closed.

(iii) $U \cap V = \varnothing$.

Thus (Y, Δ_1, Δ_2) is a PT_2-bts.

The invariance of the axioms PR_2 and PR_3 is proved in a similar way.

THEOREM 3.6. *P-compactness is inverse invariant under P-perfect mapping.*

PROOF. Theorem 3.6 can be proved similarly to [2, Lemma 5.2].

4. Pairwise θ-continuous mappings

DEFINITION 4.1. A mapping f from a bts (X, τ_1, τ_2) into a bts (Y, Δ_1, Δ_2) is said to be $P \cdot \theta$-continuous if

$$(\forall x \in X)(\forall O_{f(x)} \in \Delta_i)(\exists O_x \in \tau_i)(f(\tau_j\text{-}\mathrm{cl}(O_x)) \subseteq \Delta_j\text{-}\mathrm{cl}(O_{f(x)})).$$

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It is obvious that a P-continuity is a $P\cdot\theta$-continuity. The converse is not true in general as the following example shows.

EXAMPLE 4.2. Let $X=\{a,b\}$ and $\tau_1=\{X,\varnothing,\{a\}\}$, $\tau_2=\{X,\varnothing,\{b\}\}$, $\Delta_1=\{X,\varnothing,\{a\},\{b\}\}$, $\Delta_2=\{X,\varnothing\}$. Let $f\colon (X,\tau_1,\tau_2)\to(X,\Delta_1,\Delta_2)$ be the identity mapping. Then f is $P\cdot\theta$-continuous but not P-continuous, since for $b\in X$ and for each $O_{f(b)}\in\Delta_1$, there does not exist any $O_b\in\tau_1$ such that $f(O_b)\subseteq O_{f(b)}$.

THEOREM 4.3. *If f is a $P\cdot\theta$-continuous mapping from an arbitrary bts (X,τ_1,τ_2) into a PR_2-bts (Y,Δ_1,Δ_2), then f is P-continuous.*

PROOF. Since (Y,Δ_1,Δ_2) is PR_2, we find that $(\forall x\in X)(\forall O_{f(x)}\in\Delta_i)\cdot(\exists O^*_{f(x)}\in\Delta_i)(O^*_{f(x)}\subseteq\Delta_j\text{-cl}(O^*_{f(x)}\subseteq O_{f(x)})$. By $P\cdot\theta$-continuity of f, $(\exists O_x\in\tau_i)(f(\tau_j\text{-cl}(O_x))\subseteq\Delta_j\text{-cl}(O^*_{f(x)}))$. Hence we have

$$f(O_x)\subseteq f(\tau_j\text{-cl}(O_x))\subseteq\Delta_j\text{-cl}(O^*_{f(x)})\subseteq O_{f(x)}.$$

THEOREM 4.4. *The composition of two $P\cdot\theta$-continuous mappings is $P\cdot\theta$-continuous.*

PROOF. This is straightforward.

THEOREM 4.5. *The P-Urysohn axiom is inverse invariant under a $P\cdot\theta$-continuous injective mapping.*

PROOF. Let f be a $P\cdot\theta$-continuous injective mapping from a bts (X,τ_1,τ_2) into a $PT_{2\frac{1}{2}}$-bts (Y,Δ_1,Δ_2). Let $x_1,x_2\in X$ such that $x_1\neq x_2$. Hence $f(x_1)\neq f(x_2)$. Since (Y,Δ_1,Δ_2) is $PT_{2\frac{1}{2}}$-bts, we obtain $(\exists O_{f(x_1)}\in\Delta_i)\cdot(\exists O_{f(x_2)}\in\Delta_j)(\Delta_j\text{-cl}(O_{f(x_1)})\cap\Delta_i\text{-cl}(O_{f(x_2)})=\varnothing)$. By $P\cdot\theta$-continuity of f, we obtain $(\exists O_{x_1}\in\tau_i)(\exists O_{x_2}\in\tau_j)(f(\tau_j\text{-cl}(O_{x_1}))\subseteq\Delta_j\text{-cl}(O_{f(x_1)})$ and $f(\tau_i\text{-cl}(O_{x_2}))\subseteq\Delta_i\text{-cl}(O_{f(x_2)}))$. Hence $f(\tau_j\text{-cl}(O_{x_1}))\cap f(\tau_i\text{-cl}(O_{x_2}))=\varnothing$ and so $\tau_j\text{-cl}(O_{x_1})\cap\tau_i\text{-cl}(O_{x_2})=\varnothing$. Thus (X,τ_1,τ_2) is $PT_{2\frac{1}{2}}$-bts.

THEOREM 4.6. *Let f be a $P\cdot\theta$-continuous and P-closed mapping from a bts (X,τ_1,τ_2) onto a bts (Y,Δ_1,Δ_2). Then $(\forall U\in\Delta_i)$ we have*

$$f^{-1}(\Delta_j\text{-cl}(U))=\tau_j\text{-cl}(f^{-1}(U)).$$

PROOF. Let $x\notin\tau_j\text{-cl}(f^{-1}(U))$. Then $f(x)\notin f(\tau_j\text{-cl}(f^{-1}(U))$ and hence $f(x)\notin\Delta_j\text{-cl}(U)$ since f is P-closed and onto. So $x\notin f^{-1}(\Delta_j\text{-cl}(U))$. Thus $f^{-1}(\Delta_j\text{-cl}(U))\subseteq\tau_j\text{-cl}(f^{-1}(U))$.

To prove the converse inclusion, let $x \notin f^{-1}(\Delta_j\text{-cl}(U))$. Then $f(x) \notin \Delta_j\text{-cl}(U)$. From f being onto we obtain that $ff^{-1}(\Delta_j\text{-cl}(U)) = \Delta_j\text{-cl}(U)$ and hence $(\exists O_{f(x)} \in \Delta_j)(O_{f(x)} \cap U = \varnothing)$. From $U \in \Delta_i$, we find that $\Delta_i\text{-cl}(O_{f(x)}) \cap U = \varnothing$. By $P \cdot \theta$-continuity of f, $(\exists O_x \in \tau_j) \cdot (f(\tau_i\text{-cl}(O_x)) \subseteq \Delta_i\text{-cl}(O_{f(x)}))$ and hence $f(\tau_i\text{-cl}(O_x)) \cap U = \varnothing$ which implies that $\tau_i\text{-cl}(O_x) \cap f^{-1}(U) = \varnothing$ and so $x \notin \tau_j\text{-cl}(f^{-1}(U))$. Thus, $\tau_j\text{-cl}(f^{-1}(U)) \subseteq f^{-1}(\Delta_j\text{-cl}(U))$.

5. Pairwise θ-perfect irreducible mappings

DEFINITION 5.1. A mapping f from a bts (X, τ_1, τ_2) onto a bts (Y, Δ_1, Δ_2) is called *P-irreducible* if $(\forall F = F_1 \cup F_2,\ F_1 \in \tau_1' \setminus \{X\}$ and $F_2 \in \tau_2' \setminus \{X\}) \cdot (f(F) \neq Y)$.

We omit the proofs of Lemma 5.2 and Theorem 5.3, which are straightforward.

LEMMA 5.2. *A mapping f from a bts (X, τ_1, τ_2) onto a bts (Y, Δ_1, Δ_2) satisfies: f is P-irreducible if and only if $(\forall U = U_1 \cap U_2,\ U_1 \in \tau_1 \setminus \{\varnothing\}$ and $U_2 \in \tau_2 \setminus \{\varnothing\})(f^{\#}(U) \neq \varnothing)$.*

THEOREM 5.3. *Let f be a P-closed mapping from a bts (X, τ_1, τ_2) into a bts (Y, Δ_1, Δ_2) and $U \in \tau_i$, $i = 1, 2$. Then*

(1) $f^{\#}(U) \in \Delta_i$

(2) $f^{\#}(U) \subseteq \Delta_i\text{-int}(f(U))$.

DEFINITION 5.4. A $P \cdot \theta$-continuous map is called *$P \cdot \theta$-closed irreducible* if it is both P-closed and P-irreducible.

LEMMA 5.5. *If f is a $P \cdot \theta$-closed irreducible mapping from a bts (X, τ_1, τ_2) onto a bts (Y, Δ_1, Δ_2), then $(\forall U \in \tau_i \setminus \{\varnothing\})$ we have*

$$\tau_i\text{-int}(f^{-1}(\Delta_j\text{-cl}(f^{\#}(U)))) \subseteq \tau_j\text{-cl}(U) \subseteq f^{-1}(\Delta_j\text{-cl}(f^{\#}(U))).$$

PROOF. Let $x \notin \tau_j\text{-cl}(U)$. Then we obtain successively

$f(x) \notin f(\tau_j\text{-cl}(U))$ (monotonicity of direct image)

$f(x) \notin \Delta_j\text{-cl}(f(U))$ (f is P-closed)

$f(x) \notin \Delta_j\text{-cl}(f^{\#}(U))$ (property (1) of Theorem 2.8)

$x \notin f^{-1}(\Delta_j\text{-cl}(f^{\#}(U)))$ (monotonicity of inverse image)

$x \notin \tau_i\text{-int}(f^{-1}(\Delta_j\text{-cl}(f^{\#}(U))))$.

Thus, $\tau_i\text{-int}(f^{-1}(\Delta_j\text{-cl}(f^{\#}(U)))) \subseteq \tau_j\text{-cl}(U)$.

Now, it is required to prove that $\tau_j\text{-cl}(U) \subseteq f^{-1}(\Delta_j\text{-cl}(f^{\#}(U)))$. Let $x \in \tau_j\text{-cl}(U)$. Then we obtain successively:

$(\forall O_x \in \tau_j)(O_x \cap U \neq \varnothing)$

$(\forall O_x \in \tau_j)(f^{\#}(O_x \cap U) \neq \varnothing)$ (Lemma 5.2)

$(\forall O_x \in \tau_j)(f^{\#}(O_x) \cap f^{\#}(U) \neq \varnothing)$ (property (3) of Theorem 2.8)

$(\forall O_x \in \tau_j)(f(O_x) \cap f^{\#}(U)) \neq \varnothing)$ (property (1) of Theorem 2.8)

Now, since f is $P \cdot \theta$-continuous,

$$(\forall O_{f(x)} \in \Delta_j)(f(O_x) \subseteq f(\tau_i\text{-cl}(O_x)) \subseteq \Delta_i\text{-cl}(O_{f(x)})).$$

Hence, $\Delta_i\text{-cl}(O_{f(x)}) \cap f^{\#}(U) \neq \varnothing$ and since $f^{\#}(U) \in \Delta_i$, we have $O_{f(x)} \cap f^{\#}(U) \neq \varnothing$ and so $f(x) \in \Delta_j\text{-cl}(f^{\#}(U))$ which implies that $x \in f^{-1}(\Delta_j\text{-cl}(f^{\#}(U)))$. Thus, $\tau_j\text{-cl}(U) \subseteq f^{-1}(\Delta_j\text{-cl}(f^{\#}(U)))$.

Now we are ready to prove the main theorem in this section.

THEOREM 5.6. *The (small) image of an (i, j)-canonical open set is an (i, j)-canonical open set under a $P \cdot \theta$-closed irreducible surjective mapping.*

PROOF. Let f be a $P\cdot\theta$-closed irreducible mapping from a bts (X, τ_1, τ_2) onto a bts (Y, Δ_1, Δ_2) and $U \subseteq X$ be a (i, j)-canonical open set ($U = \tau_i\text{-int}(\tau_j\text{-cl}(U))$). We have to prove that $\Delta_i\text{-int}(\Delta_j\text{-cl}(f^{\#}(U))) = f^{\#}(U)$. Let $y \in \Delta_i\text{-int}(\Delta_j\text{-cl}(f^{\#}(U)))$. Then $(\exists O_y \in \Delta_i)(O_y \subseteq \Delta_j\text{-cl}(f^{\#}(U)))$ and hence $(\Delta_j\text{-cl}(O_y) \subseteq \Delta_j\text{--cl}(f^{\#}(U)))$. Since f is $P \cdot \theta$-continuous, we obtain $(\exists O_{f^{-1}(\{y\})} \in \tau_i)(f(O_{f^{-1}(\{y\})}) \subseteq f(\tau_j\text{-cl}(O_{f^{-1}(\{y\})})) \subseteq \Delta_j\text{-cl}(O_y))$, where $O_{f^{-1}(\{y\})} = \bigcup_{x \in f^{-1}(\{y\})} O_x$. Hence $f(O_{f^{-1}(\{y\})}) \subseteq \Delta_j\text{-cl}(f^{\#}(U))$ and so $O_{f^{-1}(\{y\})} \subseteq f^{-1}(\Delta_j\text{-cl}(f^{\#}(U)))$. Then $O_{f^{-1}(\{y\})} \subseteq \tau_i\text{-int}(f^{-1}(\Delta_j\text{-cl}(f^{\#}(U))))$. From Lemma 5.5, we have $O_{f^{-1}(\{y\})} \subseteq \tau_j\text{-cl}(U)$ and so $O_{f^{-1}(\{y\})} \subseteq \tau_i\text{-int}(\tau_j\text{-cl}(U)) = U$. Hence $f^{-1}(\{y\}) \subseteq U$ which implies that $y \in f^{\#}(U)$. Thus, $\Delta_i\text{-int}(\Delta_j\text{-cl} \times (f^{\#}(U))) \subseteq f^{\#}(U)$. The converse inclusion $f^{\#}(U) \subseteq \Delta_i\text{-int}(\Delta_j\text{-cl}(f^{\#}(U)))$ follows directly from Theorem 5.3(1).

DEFINITION 5.7. A $P \cdot \theta$-continuous, p-closed mapping f from a bts (X, τ_1, τ_2) onto a bts (Y, Δ_1, Δ_2) is called *$P \cdot \theta$-perfect* if it satisfies the following condition: $(\forall y \in Y)(\forall i \in \{1, 2\})(f^{-1}(\{y\})$ is τ_i-compact subset in $X)$. If f is also P-irreducible then it is called *$P \cdot \theta$-perfect irreducible.*

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It is direct consequence of Definitions 4.1 and 5.7 and Theorem 4.3 that every P-perfect map is $P \cdot \theta$-perfect and every $P \cdot \theta$-perfect mapping from an arbitrary bts into a PR_2-bts is P-perfect.

6. Quasi θ-proximity spaces

In this section the concept of θ-proximity spaces [4] is extended to bitopological spaces.

DEFINITION 6.1. A *quasi θ-proximity space* is a pair (X, θ), where X denotes a PT_2-bts and θ a mapping from $2^X \times 2^X$ onto $\{0, 1\}$ satisfying the following axioms:

(θ_1) $\theta(A, B) = 0 \Rightarrow A \neq \varnothing$ and $B \neq \varnothing$;

(θ_2) $\theta(A, B \cup C) = \theta(A, B) \cdot \theta(A, C)$ and,
$\theta(A \cup B, C) = \theta(A, C) \cdot \theta(B, C)$;

(θ_3) $\theta(\{x\}, A) = 0 \Rightarrow (\forall O_x \in \tau_1)(\forall O_A \in \tau_2)(O_x \cap O_A \neq \varnothing)$, and
$\theta(A, \{x\}) = 0 \Rightarrow (\forall O_x \in \tau_2)(\forall O_A \in \tau_1)(O_x \cap O_A \neq \varnothing)$;

(θ_4) $\theta(A, B) = 1 \Rightarrow (\exists E \subseteq X)(E$ is $(2, 1)$-canonical open and $\theta(A, E)$
$= \theta(\text{co}(\tau_1\text{-cl}(E)), B) = 1)$;

(θ_5) $\theta(\{x\}, \{y\}) = 0 \Leftrightarrow x = y$.

LEMMA 6.2. *The quasi θ-proximity space $(X\theta)$ has the following properties.*

(1) *If $\theta(A, B) = 0$ and $A \subseteq A_1$, $B \subseteq B_1$, then $\theta(A_1, B_1) = 0$.*

(2) $A \cap B \neq \varnothing \Rightarrow \theta(A, B) = 0$.

(3) $\theta(A, B) = 1 \Rightarrow (\exists O_A \in \tau_1)(\exists O_B \in \tau_2)(O_A \cap O_B = \varnothing)$.

(4) $\theta(A, B) = 1 \Rightarrow \theta(\text{int}(\text{cl}(A)), \text{int}(\text{cl}(B))) = 1$.

PROOF. Statement (1) follows from (θ_2), statement (2) follows from (θ_2) and (θ_5), statement (3) follows directly from (2), (θ_3) and (θ_4) and statement (4) follows directly from (2) and (θ_4).

THEOREM 6.3. *Every separated quasi-proximity space is quasi θ-proximity space.*

PROOF. Since the axioms (P_1), (P_2), (P_4^*) and (P_5) are (θ_1), (θ_2), (θ_4) and (θ_5) respectively, then it suffices to verify the axiom (θ_3). Let $\theta(\{x\}, A) = 0$. Then by Definition 2.10, we have $x \in \tau(\delta)\text{-cl}(A)$ and hence

$(\forall O_x \in \tau(\delta))(O_x \cap A \neq \varnothing)$ which implies that $(\forall O_A \in \tau(\delta^{-1}))(O_x \cap O_A \neq \varnothing)$. The proof of the second part of the axiom (θ_3) is proved in a similar way.

THEOREM 6.4. *On a P-extremally disconnected space, every quasi θ-proximity space is a separated quasi-proximity space.*

PROOF. Theorem 6.4 follows directly from Definitions 2.9, 2.13 and 6.1.

THEOREM 6.5. *If (X, τ_1, τ_2) is a PR_2-bts, then the axiom (θ_3) is equivalent to the following axiom*:

(θ_3^*) $\theta(\{x\}, A) = 0 \Rightarrow x \in \tau_1\text{-cl}(A)$ *and*
$\theta(A, \{x\}) = 0 \Rightarrow x \in \tau_2\text{-cl}(A)$.

PROOF. It is clear that $(\theta_3^*) \Rightarrow (\theta_3)$. To prove the converse, let $\theta(\{x, \}, A) = 0$ and suppose $x \notin \tau_1\text{-cl}(A)$. Since (X, τ_1, τ_2) is PR_2, then $(\exists O_x \in \tau_1)(\exists O_A \in \tau_2)(O_x \cap O_A = \varnothing)$, which contradicts the first part of the axiom (θ_3). The second part of the axiom (θ_3^*) is proved in a similar way.

DEFINITION 6.6. Let θ_1 and θ_2 be two quasi θ-proximities on X. Then we say that

$$\theta_1 \leq \theta_2 \Leftrightarrow (\forall A, B \subseteq X)(\theta_1(A, B) \leq \theta_2(A, B)).$$

THEOREM 6.7. *Let (X, τ_1, τ_2) be a PT_2-bts. Then, the mapping $\theta: 2^X \times 2^X \to \{0, 1\}$ defined by*

$$(\forall A, B \subseteq X)(\theta(A, B) = 1 \Leftrightarrow (\exists O_A \in \tau_1)(\exists O_B \in \tau_2)(O_A \cap O_B = \varnothing)),$$

is the maximal quasi θ-proximity on X.

PROOF. The verification of the axioms (θ_s), $s \in \{1, 2, 3, 5\}$, being straightforward, we only need to prove (θ_4). Let A, B be two subsets of X such that $\theta(A, B) = 1$. Then $(\exists O_A \in \tau_1)(\exists O_B \in \tau_2)(O_A \cap O_B = \varnothing)$. Putting $E = \tau_2\text{-int}(\tau_1\text{-cl}(O_B))$, we have that E is a $(2, 1)$-canonical open set satisfying $O_A \cap E = \varnothing$, which implies that $\theta(A, E) = 1$. On the other hand $\text{co}(\tau_1\text{-cl}(E)) \cap O_B = \varnothing$ holds and hence $\theta(\text{co}(\tau_1\text{-cl}(E)), B) = 1$.

Now, we shall that, θ is the maximal quasi θ-proximity on X. Let θ_1 be another quasi θ-proximity on X and $\theta < \theta_1$. Let $\theta(A, B) = 0$ and suppose $\theta_1(A, B) = 1$. Then, $\theta(A, B) = 0 \Rightarrow (\forall O_A \in \tau_1)(\forall O_B \in \tau_2)(O_A \cap O_B \neq \varnothing)$. But, by Lemma 6.2(3), $\theta_1(A, B) = 1 \Rightarrow (\exists O_A \in \tau_1)(\exists O_B \in \tau_2)(O_A \cap O_B = \varnothing)$, which gives a contradiction.

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THEOREM 6.8. *Let δ be a compatible quasi-proximity on a $PT_{3\frac{1}{2}}$-bts (X, τ_1, τ_2). If A is a τ_1-compact and B is τ_1-closed, then $A \cap B = \varnothing \Rightarrow \delta(A, B) = 1$.*

PROOF. For each $x \in A$, $x \notin B = \tau(\delta)\text{-cl}(B)$ which implies that $\delta(\{x\}, B) = 1$. By axiom (P_4) we find $(\exists U \subseteq X)(\delta(\{x\}, U) = \delta(\text{co}\, U, B) = 1)$. Then $x \notin \tau_1\text{-cl}(U)$ and hence $x \in \text{co}(\tau_1\text{-cl}(U)) = O_x$ (say). Hence, we have $\delta(O_x, B) = 1$. Clearly $\{O_x : x \in A\}$ is an τ_1-open cover of the τ_1-compact set A, and so $A \subseteq \bigcup_{i=1}^n O_{x_i}$. Now by axiom (P_2), we have $\delta(\bigcup_{i=1}^n O_{x_i}, B) = 1$ and hence $\delta(A, B) = 1$.

THEOREM 6.9. *Let f be a $P \cdot \theta$-perfect irreducible mapping from a $PT_{3\frac{1}{2}}$-bts (X, τ_1, τ_2) onto a PT_2-bts (Y, Δ_1, Δ_2) and δ be a compatible separated quasi-proximity on X. A map $\theta: 2^Y \times 2^Y \to \{0, 1\}$ defined by*

$$(\forall A, B \subseteq Y)(\theta(A, B) = 0 \Leftrightarrow \delta(f^{-1}(A), f^{-1}(B)) = 0)$$

is a quasi θ-proximity on Y.

PROOF. The verification of axioms (θ_1) and (θ_2) is straightforward.

(θ_3). Let $y \in Y$ and $A \subseteq Y$. Consider $O_y \in \Delta_1$ and $O_A \in \Delta_2$ such that $O_y \cap O_A = \varnothing$ and so $\Delta_2\text{-cl}(O_y) \cap A = \varnothing$. From f is $P \cdot \theta$-continuous, we obtain $(\exists O_{f^{-1}(\{y\})} \in \tau_1)(f(\tau_2\text{-cl}(O_{f^{-1}(\{y\})})) \subseteq \Delta_j\text{-cl}(O_y))$. Hence we have $f(\tau_2\text{-cl}(O_{f^{-1}(\{y\})})) \cap A = \varnothing$ and so $f^{-1}(\{y\}) \cap \tau_1\text{-cl}(f^{-1}(A)) = \varnothing$. By Theorem 6.8, we have $\delta(f^{-1}(\{y\}), f^{-1}(A)) = 1$ and hence $\theta(\{y\}, A) = 1$. The proof of the second part is proved in a similar way.

(θ_4). Consider $A, B \subseteq Y$ and $\theta(A, B) = 1$. Then $\delta(f^{-1}(A), f^{-1}(B)) = 1$, and so by Lemma 2.12, $(\exists E \subseteq X, E$ is a $(2, 1)$-canonical open and $\delta(f^{-1}(A), E) = \delta(\text{co}(\tau_1\text{-cl}(E)), f^{-1}(B)) = 1)$, where $\tau_1 = \tau(\delta)$ and $\tau_2 = \tau(\delta^{-1})$. Putting $V = f^{\#}(E)$, we find by Theorem 5.6 that V is a $(2, 1)$-canonical open set in Y with $f^{-1}(V) \subseteq E$. It follows from Lemma 5.5 that $\tau_1\text{-cl}(E) \subseteq f^{-1}(\Delta_1\text{-cl}(f^{\#}(E)))$ and so by Theorem 4.6 we have $f^{-1}(\text{co}(\Delta_1\text{-cl}(V))) = \text{co}(f^{-1}(\Delta_1\text{-cl}(V)) = \text{co}(f^{-1}(\Delta_1\text{-cl}(f^{\#}(E)))) \subseteq \text{co}(\tau_1\text{-cl}(E))$. Then $\delta(f^{-1}(A), f^{-1}(V)) = \delta(f^{-1}(\text{co}(\Delta_1\text{-cl}(V))), f^{-1}(B)) = 1$ and hence $\theta(A, V) = \theta(\text{co}(\Delta_1\text{-cl}(V), B) = 1$.

(θ_5). Consider $y_1, y_2 \in Y$ such that $\theta(\{y_1\}, \{y_2\}) = 1$. Then

$$\delta(f^{-1}(\{y_1\}), f^{-1}(\{y_2\})) = 1$$

and hence $f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\}) = \varnothing$ which implies that $y_1 \neq y_2$. Conversely, let $y_1 \neq y_2$. Then we have $f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\}) = \varnothing$. Since f

is $P \cdot \theta$-perfect, then both $f^{-1}(\{y_1\})$ and $f^{-1}(\{y_2\})$ are τ_i-compact subsets in X. By 2.5(2), we find that $\tau_2\text{-cl}(f^{-1}(\{y_1\})) \cap \tau_1\text{-cl}(f^{-1}(\{y_2\})) = \varnothing$ and hence by Theorem 6.8 we have $\delta(f^{-1}(\{y_1\}), f^{-1}(\{y_2\})) = 1$ which implies that $\theta(\{y_1\}, \{y_2\}) = 1$.

References

[1] I. E. Cooke and I. L. Reilly, 'On bitopological compactness', *J. London Math. Soc.* **2** (9) (1975), 518–522.
[2] M. C. Datta, 'Projective bitopological spaces II', *J. Austral. Math. Soc.* **14** (1972), 119–128.
[3] R. Engelking, *General Topology*, Warszawa, 1977.
[4] V. V. Fedorcuk, ' θ-spaces and perfect irreducible mappings of topological spaces,' *Soviet Math. Dokl.* **8** (3) (1967), 684–686.
[5] V. V. Fedorcuk, 'Perfect irreducible mappings and generalized proximities', *Soviet Math. Dokl.* **9** (3) (1968), 661–664.
[6] V. V. Fedorcuk, 'Uniform spaces and perfect irreducible mappings of topological spaces', *DAN SSSR* **192** (1970), 1228–1230.
[7] V. V. Fedorcuk, 'On H-closed extension of θ-proximity spaces', *Math. Sbornik* **89** (1972), 400–418.
[8] P. Fletcher, H. B. Hoyle and C. W. Patty, 'The comparison of topologies', *Duke Math. J.* **36** (1969), 325–331.
[9] A. Kandil, On dimension of θ-spaces, Ph.D. Thesis, Moscow University, 1977.
[10] J. C. Kelly, 'Bitopological spaces', *Proc. London Math. Soc.* **13** (1963), 71–89.
[11] Y. M. Kim, 'Pairwise compactness', *Pub. Math. Debrecen* **15** (1968), 87–90.
[12] E. P. Lane, 'Quasi-proximities and bitopological spaces', *Portugal. Math.* **28** (1969), 151–159.
[13] W. J. Pervin, 'Quasi-proximities for topological spaces', *Math. Ann.* **150** (1963) 325–326.
[14] K. Singal and A. R. Singal, 'Some separation axioms in bitopological spaces', *Ann. Soc. Sci. Bruxelles* **84** (1970), 207–230.
[15] J. Swart, 'Total disconnectedness in bitopological spaces and product bitopological spaces', *Indag. Math.* **33** (1971), 135–145.

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