and it becomes, on division of numerator and denominator by D_x,

$$P=\frac{1}{1-\frac{(1+r)^{\frac{1}{2}}M_{x|n}}{D_x}}\cdot\frac{N_{x+n}}{D_x};$$

or, writing Q for $\frac{(1+r)^{\frac{1}{2}}M_{x|n}}{D_x}$, which is the value of a return of £1,

$$P=\frac{1}{1-Q}\cdot\frac{N_{x+n}}{D_x}.$$

This is Mr. Younger's expression, which, being thus identical with mine, ought, if properly applied—if the proper values of the elements composing it be made use of—to give the same numerical result. Mr. Younger, by an analytical process, which I think I am warranted in calling unnecessarily refined, determines an expression for Q which enables him to assign ·42556 as its value in the case in hand, which value differs but little from that given by my table—namely, ·425534. It is in the remaining element of the expression then—the annuity value—that the principal source of the discrepancy must be sought. Accordingly we find that Mr. Younger here uses the ordinary deferred annuity value; and in so doing—ignoring one of the contingencies on which, during the first 10 years, the value of the annuity depends—he, singular to say, commits an error precisely similar in character to that which he points out as vitiating Mr. Stephenson's investigation. The effect is, that the value thus assigned to P is one that will provide an annuity not only for those who neither die nor withdraw during the 10 years, but also for such of the latter class as shall survive the term!

Using the annuity value derived from my table, and Mr. Younger's value of Q (in which there is probably some arithmetical error), his formula gives for P, 5·60944.* Using also my value of Q, it gives, of course, 5·60920.

I must apologise for having occupied so much space. I hope, however, it may be found that something has been done towards the elucidation of the various points of interest that have arisen.

I am, Sir,
Yours most obediently,
London, 11*th May*, 1866. P. GRAY.

ON MR. YOUNGER'S LETTER, AND ON THE GENERAL SOLUTION OF PROBLEMS INVOLVING DISTINCT CONTINGENCIES.

To the Editor.

Sir,—After the non-success of my endeavour to convince Mr. Stephenson of the failure of his attempt to solve a new problem in life contingencies, I declared my intention of retiring from the contest; as I felt satisfied that a continuance of the discussion with an opponent who (in the face of the evidence I had adduced) still adhered to his notion that he had succeeded in determining the *value* of the "option of withdrawal," was not likely to lead to any useful result. The subject, however, having been taken up by Mr. Younger, in an able letter published in your last Number, wherein that gentleman (after endeavouring to point out the source of Mr. Stephenson's

* See Mr. Younger's letter, p. 118.—Ed. *J. I. A.*

singular error) proceeds to reduce the problem within the bounds of possibility—so far at least as regards its *theoretical* solution—by assuming the existence of a definite law of withdrawal, I now venture to request the favour of some further space for the discussion of the subject on its new footing.

Whatever opinion may be entertained of the practical value of the inquiry (on which point, before concluding, I shall have a few words to say), there can be no question that the investigation of the problem, on the basis upon which Mr. Y. has placed it, is a perfectly legitimate subject for the pages of the *Journal*, even if it be regarded merely in the light of a mathematical exercise. I propose, therefore, in the first place, to show wherein Mr. Younger has erred in the solution which he has given; and in the second, to point out what appears to me the true mode of dealing generally with problems of this and of a cognate character.

That Mr. Younger's solution *is* erroneous is apparent from the result of his numerical example. For he finds that, upon the hypothesis of each year's withdrawals being one-twentieth of the number existing at the beginning of the year, the premium for an annuity on a life aged 50, deferred 10 years, receives an addition, consequent upon the introduction of the option of withdrawal, of no less than 50 per cent.! Now I am prepared to maintain, that, in consequence of the "value" of a policy of this description being (as I had occasion to point out in a former letter) always *in excess* of the amount of the premium paid, it must inevitably follow that the limitation of the sum to be received on withdrawal to the amount of the premium *only*, must, as a matter of calculation, have the effect of *reducing* the premium instead of *raising* it. And examining Mr. Younger's equation of condition, $Q.P_x+\dfrac{N_{x+n}}{D_x}=P_x$, where Q denotes the value of £1 to be paid at death or withdrawal, we may see at once that it contains a very material error, inasmuch as it assumes that the value of the *annuity* is affected by the risk of *death* only, in the period during which it is deferred. The equation, in fact, expresses the condition necessary for determining the value of a deferred annuity with the return of the premium in the event of death, and also with the option of receiving a *gratuity* of P_x (unconditionally) any time before the annuity commences—assuming, of course, that the number who in each year will avail themselves of this option will be limited in accordance with the supposed "law of withdrawal."

But independently of the oversight here adverted to, I find fault with Mr. Younger's solution as being by no means the simplest which the problem admits of. For if we consider the actual effect of the introduction of the contingency of withdrawal, we shall see, first, that its operation is confined to the period intervening between the date of the contract and the commencement of the annuity; and, secondly, that during the period in question its effect is identical with that of the risk of *death*. We have, therefore, in dealing with the problem, merely to form a table of the numbers remaining from year to year in a body or community subject to the combined action of death and withdrawal from age x (supposed to be the age at entry) to age $x+n$, at which the annuity is to be entered upon; and then to continue it from the latter age upon the supposition that the members then remaining are affected by the risk of death only. The usual D and N columns being formed from a table so constructed, the ordinary formula for a deferred annuity with the return of the premium at death, viz., $\frac{N_{x+n}}{(N_{x-1}-N_{x+n-1})(1-v)+D_{x+n}}$, should give the value of the proposed benefit.

Seeing, then, that the whole difficulty of the problem lies in the construction of a table of the values of λ_x, or the numbers remaining, at each successive year of age, among a body of members subject to death and withdrawal, I shall confine myself (so far as regards this part of my subject) to the consideration of the best means of effecting this object. And first I observe that, in the numerical solution of his problem, Mr. Younger assumes that the number withdrawing in each year bears a constant proportion to the number entering upon the year—an assumption which involves the supposition that the inclination or disposition to withdraw increases with the risk of death, for otherwise the proportion of yearly withdrawals would become less as the average quantity of existence enjoyed during each year diminishes. I shall, on the other hand, assume the law of withdrawal to be entirely independent of the law of mortality—a supposition which, besides being somewhat more consistent with reason, will, I think, be still more fully justified by the simplicity of the results to which it will lead us.

With the view of generalizing the investigation, and of adapting it to the assumption last mentioned, let us (in the first instance) discard the ideas of death and withdrawal, and suppose merely that the body or community under observation, is subject to two independent contingencies, neither of which can occur more than once to the same individual, but the happening of either of which will in no way affect the occurrence of the other. Let p denote the probability that, in reference to any particular individual, a year will elapse without the occurrence of the first contingency (which we will designate by c_1), and p' the same probability as regards c_2, the second contingency. We shall then have the following five possible cases:—

Case.	Event.	The probability of which is
1	That neither c_1 nor c_2 happens.	pp'.
2	That c_1 happens and c_2 not.	$(1-p)p'$.
3	That c_2 happens and c_1 not.	$(1-p')p$.
4	That c_1 and c_2 both happen: c_1 first.	$\frac{1}{2}(1-p)(1-p')$.*
5	" " c_2 first.	$\frac{1}{2}(1-p)(1-p')$.*

By combining these elementary cases, we obtain the following three, which are all we shall require for the purposes to which our conclusions will be applied:—

Cases.	Event.	The probability of which is
1	That neither c_1 nor c_2 happens.	pp'.
2 and 4	That c_1 happens, c_2 not having previously occurred.	$\frac{1}{2}(1-p)(1+p')$.
3 and 5	That c_2 happens, c_1 not having previously occurred.	$\frac{1}{2}(1-p')(1+p)$.

* According to the hypothesis usually resorted to in life contingencies, viz., that if an event is certain to happen in a given year, it is as likely to occur in any one part of that year as in any other part.

Let the contingency c_1 *now* represent death, and c_2 withdrawal; let the age of the individual be x, and let p_x and p'_x denote the respective probabilities of surviving, and of non-withdrawal, for one year—the former determined upon the supposition that there are no withdrawals, and the latter upon the supposition that there are no deaths. We shall then have, by substitution in the last table, for the probability that the individual

(1) remains to the end of the year (*i.e.*, that he neither dies nor withdraws) . . . $p_x . p'_x$;
(2) disappears by death within the year . . $\frac{1}{2}(1-p_x)(1+p'_x)$;
(3) disappears by withdrawal within the year . $\frac{1}{2}(1-p'_x)(1+p_x)$.

These expressions afford a very easy mode of constructing a table of (λ_x), the numbers remaining at the expiration of each successive year of age, as well as of the deaths and withdrawals at every age; and it will be seen that the formulæ are precisely the same as would be used for calculating the probabilities of joint existence and of survivorship between two lives of equal ages, but subject to different laws of mortality. It is also evident that with the functions in question tabulated for every age we shall be furnished with every requisite for calculating directly the values of all contingencies depending upon the probabilities of death and withdrawal, whether separately or in combination with each other.

Let us now suppose that the disposition to withdraw does not vary, and put $p'_x = k'$ a constant quantity. We then have $\frac{\lambda_{x+1}}{\lambda_x} = \frac{l_{x+1}}{l_x} \cdot k'$, and generally $\frac{\lambda_{x+n}}{\lambda_x} = \frac{l_{x+n}}{l_x} k'^n$, a condition which is satisfied by making $\lambda_x = l_x k'^x$.

I shall now apply the results we have obtained to the consideration of a question of some practical importance, viz., the determination of the rates of premium to be charged for assurances on the lives of persons exposed to risk of death from extraneous causes—that is, from causes independent of the ordinary law of mortality. The usual (perhaps I might almost say the invariable) way of dealing with these cases is to charge a fixed extra premium, irrespective of the age of the party or of the nature of the assurance, which extra premium is discontinued when the extra risk ceases to be incurred. Now the foregoing investigation shows that we have already at our command the means of determining the proper extra premium for every case in the ordinary D and N columns calculated at different rates of interest, for we have seen that the constant extra risk (represented by k') affects these functions precisely in the same way as the interest of money.

I shall not attempt to determine the difference between the *usual* mode and the *correct* mode of dealing with these risks in all the various forms in which the cases may present themselves, but shall confine myself to the following specimens of two of the more usual ones, viz., single life assurances for one year and for the whole term of life. The formula for an assurance for one year on a life exposed to extra risk will be $v\left(1-\frac{\lambda_{x+1}}{\lambda_x}\right) = v\left(1-\frac{l_{x+1}}{l_x}k'\right) = v - \frac{l_{x+1}}{l_x}k'v = v - \frac{l_{x+1}}{l_x}v'$ (putting $k'v = v'$). If we denote $l_x v'^x$ by D'_x, this becomes $v - \frac{D'_{x+1}}{D'_x}$. Again, representing

$(D'_{x+1}+D'_{x+2}+\ldots)\div D'_x$ by a'_x, the formula for the annual premium for a whole life assurance becomes $\frac{1}{1+a'_x}-(1-v)$. By these formulæ the values in the second division of the following table have been computed, vk' or v' being taken$=1\cdot06^{-1}$.

Carlisle Mortality—Interest 4 per Cent.
Annual Premiums for an Assurance of £100, with a Loading of 30 per Cent.

Age.	I. Without extra Risk.		II. With extra Risk.		III. Difference.		Age.
	One Year.	Whole Life.	One Year.	Whole Life.	One Year.	Whole Life.	
	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	
20	0 17 8	1 14 3	3 4 5	3 15 3	2 6 9	2 1 0	20
40	1 12 7	3 1 9	3 19 1	5 0 0	2 6 6	1 18 3	40
60	4 3 10	7 3 10	6 9 4	8 19 6	2 5 6	1 15 8	60

We see by this table, that when the extra risk is not influenced by the age of the individual, the practice of providing for it by a fixed addition to the annual premium is a very clumsy and inefficient one. In the case of "whole life" assurances the extra premium required is greater (and not insignificantly so) at the lower than at the higher ages; while it is materially greater in "term" than in "whole life" assurances.

A few words now, in conclusion, in reference to the practical utility of Mr. Younger's assumed "law of withdrawal." We have seen that, in the particular case to which it has been applied, the effect of its introduction into the calculation must be to *reduce* the premium; and, therefore, I think nothing more need be said against its application to problems of this class. Nor, in my opinion, would it be a whit more reasonable if applied to cases where the "value of the policy" is less than the amount of premium paid upon it (which is always the case where the interest on the premium paid is insufficient to cover the risk incurred), and where, consequently, the option of withdrawal would have a positive value. For in that case those policy-holders who refrain from exercising the option (which they are supposed to have paid for) are at a disadvantage as compared with those who avail themselves of it; and it would in reality be to the interest of the former to withdraw for the purpose of effecting new assurances with the premiums refunded to them. To illustrate this point, let us suppose that an ordinary life assurance is effected (by an annual premium) with the condition that, *during the first year only*, the option of withdrawal shall be allowed. Let us suppose, in the first instance, that no extra charge is made for this option; then, starting with the assumption that the value of the policy at the end of the first year is less than the premium paid, or that

$$1-\frac{1+a_{x+1}}{1+a_x}<\frac{1}{1+a_x}-(1-v); \quad [1]$$

that is,

$$(1+a_{x+1})(\pi_{x+1}-\pi_x)<\pi_x, \quad [2]$$

(π_x denoting the pure annual premium), we see from [2] that this *inequation*

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holds good if for *pure* we substitute *loaded* premiums, whether the loading be in the shape of a constant percentage, a constant addition, or a combination of the two. For in the first case we should have

$$(1+a_{x+1})(c\pi_{x+1}-c\pi_x) < c\pi_x, \qquad [3]$$

which evidently follows from [2]. In the second

$$(1+a_{x+1})(\overline{\pi_{x+1}+c}-\overline{\pi_x+c}) < \overline{\pi_x+c}, \qquad [4]$$

$$\text{or } (1+a_{x+1})(\pi_{x+1}-\pi_x) < \overline{\pi_x+c},$$

which also follows evidently from [2]. And the third case is already proved, as it follows from (3) and (4) by precisely the same processes.

It appears, then, that whatever be the nature of the loading, $\pi'_x > (1+a_{x+1})(\pi'_{n+1}-\pi'_x)$, where π'_x denotes the loaded premium, follows from $\frac{1}{1+a_x}-(1-v) > 1-\frac{1+a_{x+1}}{1+a_x}$; and, indeed, a little attention to the preceding process will show that the inequality is *increased* by the loading. Now $(1+a_{x+1})(\pi'_{x+1}-\pi'_x)$ is the equivalent which an Office would require to be paid for granting a new assurance at age $x+1$ at the rate of premium for age x; and as this has been shown to be less than π'_x, it appears that when no extra charge is made for the option of withdrawal, the policy-holder would derive an advantage by withdrawing and effecting a new assurance in the way indicated. But if π'_x is greater than $(1+a_{x+1})(\pi'_{x+1}-\pi'_x)$, still more so is π'_x+e, e being any positive quantity. So that the imposition of an extra charge (e), to cover the risk of withdrawal, merely has the effect of increasing the inducement to withdraw—viz., the saving which the policy-holder would effect by availing himself of the option allowed him for the purpose of effecting a new assurance.

For convenience of illustration I have restricted the option of withdrawal to one year; but I submit that my example is sufficient to show that the system—if a practical application of it should ever be attempted—would rest on a very sandy foundation. As observed in my letter which appeared in July last, the option of withdrawal (being a contingency depending upon the *will* of the individual) is not a benefit susceptible of valuation, and it can be safely allowed only where the value of the policy is equal to or greater than the premium paid—or, in other words, either where there has been no risk incurred, or where the *interest* on the premium paid is alone a sufficient compensation for the risk. In such cases no extra charge is necessary; but whether the option should not, in certain instances, be subject to a restriction as regards the health of the policyholder, is a question altogether foreign to the points I have touched upon.

I remain, Sir,

Your very obedient servant,

London, 24th May, 1866. W. M. MAKEHAM.

ON THE TABLES OF DEFERRED ANNUITIES AS PUBLISHED BY THE GOVERNMENT.

To the Editor.

DEAR SIR,—Since I wrote to you last year on the question of options, &c., two letters have appeared in your *Journal* on the same subject; one by Mr. Makeham and the other by Mr. Younger—both writers differing