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PURE-INJECTIVITY FROM A DIFFERENT PERSPECTIVE

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Abstract. The study of pure-injectivity is accessed from an alternative point of view. A module M is called *pure-subinjective* relative to a module N if for every pure extension K of N, every homomorphism $N \to M$ can be extended to a homomorphism $K \to M$. The *pure-subinjectivity domain* of the module M is defined to be the class of modules N such that M is N-pure-subinjective. Basic properties of the notion of pure-subinjectivity are investigated. We obtain characterizations for various types of rings and modules, including absolutely pure (or, FP-injective) modules, von Neumann regular rings and (pure-) semisimple rings in terms of pure-subinjectivity domains. We also consider cotorsion modules, endomorphism rings of certain modules, and, for a module N, (pure) quotients of N-pure-subinjective modules.

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1. Introduction and preliminaries. The study of injectivity has frequently been approached from the perspective of relative notions. For a module M, its *injectivity domain*, denoted by $\mathcal{I}^{-1}(M)$, consists of all modules N such that M is injective relative to N (or N-injective). In [**8**], it is proposed that one view the class of all injectivity domains of modules over a ring R as an ordered structure $\mathcal{P}(R)$ (the *injective profile* of the ring R) and investigate the interactions between properties of that injective profile and those of the ring itself. In recent papers, authors have explored an alternative perspective: Instead of using the injectivity domain of a module M as a mean to gauge the extent of its injectivity, [**3**] proposes to consider the so-called *subdomain of injectivity* or *subinjectivity domain* $\underline{\mathcal{I}}^{-1}(M) = \{N \mid M \text{ is } N\text{-subinjective}\}$. The expression M is N-*subinjective* means that if for every extension K of N and every homomorphism $f: N \to M$, there exists a homomorphism $g: K \to M$ such that $g|_N = f$. This idea yields naturally the notion of the subinjectivity profile $\underline{\mathcal{P}}(R)$ of a ring R.

In [**6**], the *pure-injectivity profile* of a ring is introduced as an analogue to the injectivity profile of [**8**]. The *pure-injectivity domain* of a module M, denoted by

$\mathcal{PI}^{-1}(M)$, consists of those modules N such that M is N-pure-injective. Several complications arise in the process. For example, it is pointed out in [**8**] that the injective profile of a ring is always in a one-to-one correspondence with a set; that may not be the case for the pure-injectivity profile. The aim of this paper is to investigate the viability of obtaining valuable information about a ring R from yet another perspective as we consider the *pure-subinjectivity profile* (also called *sub pure-injectivity profile*) inspired by the notion of relative subinjectivity from [**3**].

In the study of the various profiles of a ring, in contrast to the injective modules (or the pure-injective modules in the case of [**6**]), an effort has been made to understand also the diametrical opposite notion of modules which are injective (subinjective, pure-injective) only with respect to the smallest possible family of modules. Such modules are often named *poor* (with appropriate modifications as needed, e.g., injectively poor, subinjectively poor, pure-injectively poor, etc.). While a leading objective is to lay down the foundations for the study of *pure-subinjectively poor modules*, we are saving our report on the study of pure-subinjectively poor for a future paper; the research for which is currently in progress. Such modules are the analogue of the poor modules in [**1**], [**5**], and the indigent modules of [**3**].

Throughout this paper, R denotes an associative ring with identity and modules are unital right R-modules unless otherwise stated. In what follows $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$, $E(M)$, $PE(M)$ and $C(M)$ denote the natural numbers, integers, rational numbers, the ring of integers modulo n, the injective hull, the pure-injective hull and the cotorsion envelope of a module M, respectively.

Let P be a submodule of a right R-module M and $i\colon P \to M$ be the inclusion map. Then P is called a *pure submodule* of M if for any left R-module X, the natural induced map on tensor products $i \otimes 1_X\colon P \otimes X \to M \otimes X$ is injective. This is well-known to be equivalent to say that for any finite system of equations over P which is solvable in M, the system is also solvable in P. A submodule K of a module M is said to be *pure-essential* in M if K is pure in M and for any non-zero submodule N of M, either $K \cap N \neq 0$ or $(K \oplus N)/N$ is not pure in M/N. Let M and N be R-modules. Recall that M is called *N-pure-injective* if every homomorphism from a pure submodule of N to M can be extended to a homomorphism from N to M, and M is called *pure-injective* if it is N-pure-injective for every module N. A ring R is called *right pure-semisimple* if every right R-module is a direct sum of finitely generated modules.

The reader should consult standard references such as [**15**] and [**16**] for more information about the subject. For convenience, the following remarks summarize various well-known results so that they may be easily referenced in the paper.

REMARK 1.1 ([**15**, 53.6]). A ring R is right pure-semisimple if and only if every right R-module is pure-injective.

REMARK 1.2 ([**15**, 34.13(2)]). Every $\mathbb{Z}$-module K, with $nK = 0$ for some $n \in \mathbb{N}$, is pure-injective.

REMARK 1.3 ([**14**, Theorem 6]). If B is a pure-injective module, A is a module, and $f\colon A \to B$ is an embedding of A as a pure submodule in B, then f extends to a homomorphism $PE(A) \to B$, embedding $PE(A)$ as a pure submodule of B.

REMARK 1.4 ([**6**, Lemma 3.6]). Let A be a pure submodule of a module B. Then A/AJ can be embedded in B/BJ as a pure submodule for any ideal J of R.

REMARK 1.5 ([**15**, 34.6]). For any module M, the map

$$\varphi_M : M \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$$

defined by $\varphi_M(m)(\alpha) = \alpha(m)$ for any $m \in M$ and $\alpha \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a pure monomorphism.

In [**9**], a module M is said to be *absolutely pure* if it is pure in every module containing it as a submodule, equivalently, $\mathrm{Ext}^1_R(N, M) = 0$ for every finitely presented module N, i.e., M is FP-injective in the terminology of [**12**].

REMARK 1.6 ([**10**, Theorem 2.8, Example 3.4]). For any module M, the following are equivalent:
(1) M is absolutely pure.
(2) M is a pure submodule of an injective module.
(3) $PE(M) = E(M)$.

In Section 2, we introduce relative pure-subinjectivity for modules and pure-subinjectivity domains. Comparing pure-subinjectivity domains with (pure-)injectivity domains and subinjectivity domains yields characterizations of various classes of modules and rings such as pure-injective modules, absolutely pure modules, von Neumann regular rings and (pure-)semisimple rings.

In Section 3, we consider cotorsion modules and, for any module N, the (pure) quotients of N-pure-subinjective modules. We also determine when an R-module is R-pure-subinjective. The notion of pure-subinjectivity is used to provide conditions equivalent to being a right pure hereditary ring. We also analyse the quotient module $PE(N)/N$ of an arbitrary module N in terms of flatness.

Section 4 consists of further results on the notion of pure-subinjectivity domain of a module. We study the endomorphism rings of various types of modules. Using pure-subinjectivity, we provide an alternative proof of the known result that endomorphism rings of pure-injective modules are also right pure-injective.

2. The pure-subinjectivity domain of a module. In this section, we introduce the notion of the pure-subinjectivity domain of a module and investigate its basic properties. The (sub)injective domains and (pure-)subinjective domains are compared, also some characterizations of absolutely pure modules, (pure-)injective modules, von Neumann regular rings, right pure-semisimple rings and semisimple rings are obtained in terms of pure-subinjectivity.

DEFINITION 2.1. Let M and N be R-modules. M is called N-*pure-subinjective* if for every pure extension K of N, every homomorphism from N to M can be extended to a homomorphism from K to M. The *pure-subinjectivity domain* of M (denoted $\underline{\mathcal{PI}}^{-1}(M)$) consists of those modules N such that M is N-pure-subinjective.

We start by offering alternative characterizations of the pure-subinjectivity domain $\underline{\mathcal{PI}}^{-1}(M)$ of a module M.

THEOREM 2.2. *Let M and N be R-modules. Then the following are equivalent:*
(1) $N \in \underline{\mathcal{PI}}^{-1}(M)$.
(2) *For every pure-essential extension K of N, every homomorphism from N to M can be extended to a homomorphism from K to M.*

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(3) *Every homomorphism from N to M can be extended to a homomorphism from $PE(N)$ to M.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Clear.
(3) $\Rightarrow$ (1) Let K be a pure extension of N and $f: N \to M$ a homomorphism. By (3), there exists a homomorphism $g: PE(N) \to M$ such that $g|_N = f$. By Remark 1.3, $PE(N)$ is a direct summand of $PE(K)$. Hence, g extended a homomorphism $g \oplus 0: PE(K) \to M$. Since $(g \oplus 0)|_K i = f$ where $i: N \to K$ is the inclusion map, the proof is complete. □

THEOREM 2.3. *An R-module M is pure-injective if and only if $\underline{\mathcal{PI}}^{-1}(M) = Mod\text{-}R$ if and only if $M \in \underline{\mathcal{PI}}^{-1}(M)$.*

Proof. Let M be a pure-injective R-module. For any R-module N, pure-injectivitiy of M implies that every homomorphism $N \to M$ can be extended to a homomorphism $PE(N) \to M$. Hence, $\underline{\mathcal{PI}}^{-1}(M) = \text{Mod-}R$.
Now assume that $M \in \underline{\mathcal{PI}}^{-1}(M)$. Since M is pure in $PE(M)$, the homomorphism $1_M: M \to M$ can be extended to a homomorphism $PE(M) \to M$. This implies that M is a direct summand of $PE(M)$. Therefore, M is pure-injective. □

THEOREM 2.4. *The intersection of pure-subinjectivity domains of all R-modules is the class of all pure-injective R-modules.*

Proof. Since a pure-injective module is a direct summand of its pure extensions, the class of all pure-injective modules is contained in the intersection of pure-subinjectivity domains of all modules. Let M be a module for which every module is M-pure-subinjective. Since $M \in \underline{\mathcal{PI}}^{-1}(M)$, by Theorem 2.3, M is pure-injective. □

Clearly, the subinjectivity domain $\underline{\mathcal{I}}^{-1}(M)$ of a module M is contained in $\underline{\mathcal{PI}}^{-1}(M)$. They need not be equal, as the following example shows.

EXAMPLE 2.5. Let M be the socle of the $\mathbb{Z}$-module $\mathbb{Z}/4\mathbb{Z}$. By Remark 1.2, M is pure-injective and so by Theorem 2.3, $M \in \underline{\mathcal{PI}}^{-1}(M)$. On the other hand, since M is not a direct summand of $\mathbb{Z}/4\mathbb{Z}$, it does not belong to $\underline{\mathcal{I}}^{-1}(M)$.

The example above basically illustrates the fact that for any $\mathbb{Z}$-module M, if there exists $0 \neq n \in \mathbb{Z}$ such that $nM = 0$, then M is pure-injective (see Remark 1.2) but not injective, and therefore $M \in \underline{\mathcal{PI}}^{-1}(M) \setminus \underline{\mathcal{I}}^{-1}(M)$.

It is well-known that the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of a module M is pure-injective, so it is N-pure-subinjective for every module N. In the next theorem, we determine the intersection of subinjectivity domains of character modules. In addition, we see that N-pure-subinjectivity and N-subinjectivity coincide for an absolutely pure module N; moreover, this condition is a characterization of N being an absolutely pure module.

THEOREM 2.6. *The following are equivalent for a module N.*

(1) *N is an absolutely pure module.*
(2) *Every module M is N-pure-subinjective if and only if it is N-subinjective.*
(3) *$Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is N-subinjective for every module M.*
(4) *$Hom_{\mathbb{Z}}(Hom_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ is N-subinjective.*
(5) *$PE(N)$ is N-subinjective.*

Proof. (1) $\Rightarrow$ (2) Since N is absolutely pure, by Remark 1.6, $PE(N) = E(N)$. The rest is clear.

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(2) $\Rightarrow$ (3) $\Rightarrow$ (4) It is obvious from the fact that the character module of any module is pure-injective and Theorem 2.3.
(4) $\Rightarrow$ (1) Let N^{**} denote the module $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$. For the homomorphism φ_N mentioned in Remark 1.5, by (4), there exists $f : E(N) \to N^{**}$ such that $fi_N = \varphi_N$ where $i_N : N \to E(N)$ is the inclusion. Since i_N is an essential monomorphism, f is a monomorphism. Since $\varphi_N(N)$ is a pure submodule of N^{**} and $\varphi_N(N) = f(N) \subseteq f(E(N)) \subseteq N^{**}$, $f(N)$ is a pure submodule of $f(E(N))$. Thus, N is pure in $E(N)$. Therefore, N is absolutely pure by Remark 1.6.
(2) $\Rightarrow$ (5) By Theorem 2.3, $\underline{\mathcal{PI}}^{-1}(PE(N)) =$ Mod-R. Hence, $PE(N)$ is N-pure-subinjective. By (2), $PE(N)$ is also N-subinjective.
(5) $\Rightarrow$ (1) Let $i_1 : N \to E(N)$ and $i_2 : N \to PE(N)$ be the inclusions. By (5), there exists a homomorphism $f : E(N) \to PE(N)$ such that $fi_1 = i_2$. Since i_1 is an essential monomorphism, f is a monomorphism. It follows that N is pure in $E(N)$. So Remark 1.6 completes the proof. □

It is natural at this stage to consider when the various injectivity and subinjectivity domains (regular and pure) may coincide for certain modules or for all modules. The next five results deal with related conditions and all point out that they will coincide only in the most of the trivial cases. We start by showing that the only time when pure-injectivity domains and pure-subinjectivity domains are, respectively, the same as their non-pure counterparts for all modules is the trivial case when the ring is von Neumann regular (where, by trivial, we are referring to the fact that, for those rings, all submodules are pure and pure-injectives are injective).

COROLLARY 2.7. *The following are equivalent for a ring R:*
(1) *R is von Neumann regular.*
(2) *For every R-module M, $\underline{\mathcal{PI}}^{-1}(M) = \underline{\mathcal{I}}^{-1}(M)$.*
(3) *For every R-module M, $\underline{\mathcal{PI}}^{-1}(M) \subseteq \underline{\mathcal{I}}^{-1}(M)$.*
(4) *For every R-module M, $\mathcal{PI}^{-1}(M) = \mathcal{I}^{-1}(M)$.*
(5) *For every R-module M, $\mathcal{PI}^{-1}(M) \subseteq \mathcal{I}^{-1}(M)$.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4) They are easy by the fact that every extension of a module over a von Neumann regular ring is a pure extension.
(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious.
(3) $\Rightarrow$ (1) Let M be a pure-injective R-module. Then by Theorem 2.3, M belongs to $\underline{\mathcal{PI}}^{-1}(M)$. By hypothesis, $M \in \underline{\mathcal{I}}^{-1}(M)$ and hence M is M-subinjective, and so it is injective. Thus, R is von-Neumann regular by [**16**, Theorem 3.3.2].
(5) $\Rightarrow$ (1) Let M be a pure-injective R-module. Then Mod-$R = \mathcal{PI}^{-1}(M)$ and by hypothesis, Mod-$R = \mathcal{I}^{-1}(M)$. Hence, M is injective and so R is von Neumann regular by [**16**, Theorem 3.3.2]. □

THEOREM 2.8. *The following are equivalent for a module M:*
(1) *M is pure-injective.*
(2) *$\underline{\mathcal{PI}}^{-1}(M)$ is closed under pure submodules.*
(3) *$\underline{\mathcal{PI}}^{-1}(M) = \mathcal{PI}^{-1}(M)$.*
(4) *$\underline{\mathcal{PI}}^{-1}(M) \subseteq \mathcal{PI}^{-1}(M)$.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are clear since $\underline{\mathcal{PI}}^{-1}(M) =$ Mod-$R = \mathcal{PI}^{-1}(M)$.
(2) $\Rightarrow$ (1) Since $PE(M) \in \underline{\mathcal{PI}}^{-1}(M)$, by (2), M is also in $\underline{\mathcal{PI}}^{-1}(M)$. Then Theorem 2.3 completes the proof.
(3) $\Rightarrow$ (4) Obvious.

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(4) $\Rightarrow$ (1) By hypothesis, $PE(M)$ belongs to $\mathcal{PI}^{-1}(M)$. This implies that M is a direct summand of $PE(M)$, and so M is pure-injective. $\square$

COROLLARY 2.9. *The following are equivalent for a ring R:*

(1) *R is right pure-semisimple.*
(2) *For every R-module M, $\underline{\mathcal{PI}}^{-1}(M) \subseteq \mathcal{PI}^{-1}(M)$.*
(3) *For every R-module M, $\underline{\mathcal{PI}}^{-1}(M) = \mathcal{PI}^{-1}(M)$.*

THEOREM 2.10. *The following are equivalent for module M:*

(1) *M is injective.*
(2) $\underline{\mathcal{PI}}^{-1}(M) = \mathcal{I}^{-1}(M)$.
(3) $\underline{\mathcal{PI}}^{-1}(M) \subseteq \mathcal{I}^{-1}(M)$.
(4) $\underline{\mathcal{I}}^{-1}(M) = \mathcal{I}^{-1}(M)$.
(5) $\underline{\mathcal{I}}^{-1}(M) \subseteq \mathcal{I}^{-1}(M)$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) $\Rightarrow$ (5) are clear.
(3) $\Rightarrow$ (1) Since $E(M)$ is pure-injective, $E(M) \in \underline{\mathcal{PI}}^{-1}(M)$. By hypothesis, $E(M) \in \mathcal{I}^{-1}(M)$, and hence M is injective.
(5) $\Rightarrow$ (1) Since $E(M)$ is injective, $E(M) \in \underline{\mathcal{I}}^{-1}(M)$ and so by hypothesis, $E(M) \in \mathcal{I}^{-1}(M)$. Therefore, M is injective. $\square$

COROLLARY 2.11. *The following are equivalent for a ring R.*

(1) *R is semisimple.*
(2) *For every R-module M, $\underline{\mathcal{PI}}^{-1}(M) = \mathcal{I}^{-1}(M)$.*
(3) *For every R-module M, $\underline{\mathcal{PI}}^{-1}(M) \subseteq \mathcal{I}^{-1}(M)$.*
(4) *For every R-module M, $\underline{\mathcal{I}}^{-1}(M) = \mathcal{I}^{-1}(M)$.*
(5) *For every R-module M, $\underline{\mathcal{I}}^{-1}(M) \subseteq \mathcal{I}^{-1}(M)$.*
(6) *For every R-module M, $\mathcal{PI}^{-1}(M) = \underline{\mathcal{I}}^{-1}(M)$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1) are clear by Theorem 2.10.
(1) $\Rightarrow$ (6) Obvious.
(6) $\Rightarrow$ (1) Let M be an R-module. Then Mod-$R = \mathcal{PI}^{-1}(PE(M))$ and by hypothesis, Mod-$R = \underline{\mathcal{I}}^{-1}(PE(M))$. Hence, $PE(M)$ is injective and so $PE(M) \in \underline{\mathcal{I}}^{-1}(M) \subseteq \mathcal{PI}^{-1}(M)$. Thus, M is $PE(M)$-pure-injective. Theorem 2.2 implies that $M \in \underline{\mathcal{PI}}^{-1}(M)$, and by Theorem 2.3 , M is pure-injective, i.e., $\mathcal{PI}^{-1}(M) =$ Mod-R. By (6), M is injective. Since every R-module M is injective, R is semisimple. $\square$

REMARK 2.12. Let R be a ring. If $R \in \underline{\mathcal{PI}}^{-1}(M)$ for a module M, then $\underline{\mathcal{PI}}^{-1}(M)$ need not be equal to Mod-R. Consider a right pure-injective ring R which is not right pure-semisimple (for example, an infinite direct product of the ring $\mathbb{Z}/2\mathbb{Z}$ is right self-injective as a ring but not right pure-semisimple because it is von Neumann regular). Then $R \in \underline{\mathcal{PI}}^{-1}(M)$ for every module M by Theorem 2.4. On the other hand, by Remark 1.1, there exists a module M which is not pure-injective.

Recall that a module M is called *cotorsion* if $\mathrm{Ext}^1_R(F, M) = 0$ for every flat module F. It is known that every pure-injective module is cotorsion.

THEOREM 2.13. *Let M and N be R-modules. Consider the following conditions:*

(1) $N \in \underline{\mathcal{PI}}^{-1}(M)$.
(2) *There exists a pure-injective extension K of N such that every homomorphism from N to M can be extended to a homomorphism from K to M.*

(3) *There exists a cotorsion extension K of N such that every homomorphism from N to M can be extended to a homomorphism from K to M.*
(4) *For every extension K of N with K/N flat, every homomorphism from N to M can be extended to a homomorphism from K to M.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4). *Also, if $PE(N)/N$ is flat, then* (4) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) Obvious by Theorem 2.2.
(2) $\Rightarrow$ (1) Let L be a pure extension of N and $f\colon N \to M$ a homomorphism and $i\colon N \to L$ the inclusion. By hypothesis, there exists a pure-injective module K with $N \leq K$ and a homomorphism $g\colon K \to M$ such that $gi_N = f$ where $i_N\colon N \to K$ is the inclusion. Also, there exists a homomorphism $h\colon L \to K$ with $hi = i_N$ because of the pure-injectivity of K. Hence, $ghi = f$, and so f is extended to gh.
(2) $\Rightarrow$ (3) Since every pure-injective module is cotorsion, the proof is clear.
(3) $\Rightarrow$ (4) Let L be an extension of N with L/N flat and $f\colon N \to M$ a homomorphism. Hence, $0 \to N \to L \to L/N \to 0$ is a pure-exact sequence. If we apply the functor $\mathrm{Hom}_R(-, K)$ to the pure-exact sequence where K is a cotorsion module in (3), then we obtain the exact sequence

$$\cdots \to \mathrm{Hom}_R(L, K) \to \mathrm{Hom}_R(N, K) \to \mathrm{Ext}^1_R(L/N, K) = 0.$$

Since $\mathrm{Hom}_R(L, K) \to \mathrm{Hom}_R(N, K)$ is surjective, by a similar discussion in the proof of the implication (2) $\Rightarrow$ (1), f is extended to a homomorphism $L \to M$.
(4) $\Rightarrow$ (3) By Wakamatsu's Lemma (see [**16**, Lemma 2.1.2]), $C(N)/N$ is flat. The rest is clear by (4).
(4) $\Rightarrow$ (1) Let $f\colon N \to M$ be a homomorphism. Since $PE(N)/N$ is flat, f is extended to a homomorphism $PE(N) \to M$. This implies that $N \in \underline{\mathcal{PI}}^{-1}(M)$ due to Theorem 2.2. □

THEOREM 2.14. *Let R be a ring, $\{M_i\}_{i\in I}$ a class of R-modules for any index set I and N an R-module. Then $N \in \underline{\mathcal{PI}}^{-1}(\prod_{i\in I} M_i)$ if and only if $N \in \underline{\mathcal{PI}}^{-1}(M_i)$ for all $i \in I$.*

Proof. Let $N \in \underline{\mathcal{PI}}^{-1}(\prod_{i\in I} M_i)$, $i \in I$ and $f\colon N \to M_i$ be a homomorphism. Then there exists a homomorphism $g\colon PE(N) \to \prod_{i\in I} M_i$ such that $gi_N = i_{M_i}f$ where $i_N\colon N \to PE(N)$ and $i_{M_i}\colon M_i \to \prod_{i\in I} M_i$ are the inclusions. Let π_{M_i} denote the natural projection $\prod_{i\in I} M_i \to M_i$. Since $\pi_{M_i} g i_N = f$, f is extended to $\pi_{M_i} g$. Therefore, $N \in \underline{\mathcal{PI}}^{-1}(M_i)$. Conversely, let $N \in \underline{\mathcal{PI}}^{-1}(M_i)$ for all $i \in I$ and $f\colon N \to \prod_{i\in I} M_i$ be a homomorphism. Hence for each $i \in I$, there exists $g_i\colon PE(N) \to M_i$ with $g_i i_N = \pi_{M_i} f$. Now define $g\colon PE(N) \to \prod_{i\in I} M_i$ by $g\colon x \to (g_i(x))$. Since $gi_N = f$, g extends f. Thus, $N \in \underline{\mathcal{PI}}^{-1}(\prod_{i\in I} M_i)$. □

COROLLARY 2.15. *Let N be a module. Then the following hold:*
(1) *Every finite direct sum of N-pure-subinjective modules is N-pure-subinjective.*
(2) *Every direct summand of an N-pure-subinjective module is also N-pure-subinjective.*

THEOREM 2.16. *Let M, N_1 and N_2 be R-modules. Then $N_1 \oplus N_2 \in \underline{\mathcal{PI}}^{-1}(M)$ if and only if $N_i \in \underline{\mathcal{PI}}^{-1}(M)$ for $i = 1, 2$.*

Proof. Let $N_1 \oplus N_2 \in \underline{\mathcal{PI}}^{-1}(M)$ and $f\colon N_1 \to M$ a homomorphism. Since N_1 is pure in $N_1 \oplus N_2$, by Remark 1.3, $PE(N_1)$ is a direct summand of $PE(N_1 \oplus N_2)$. Let $\pi_{N_1}\colon N_1 \oplus N_2 \to N_1$ and $i_{N_1\oplus N_2}\colon N_1 \oplus N_2 \to PE(N_1 \oplus N_2)$ be the natural projection and inclusion, respectively. Then there exists a homomorphism $g\colon PE(N_1 \oplus$

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$N_2) \to M$ such that $gi_{N_1 \oplus N_2} = f\pi_{N_1}$. It is easy to check that $gi_{PE(N_1)}i'_{N_1} = f$, where $i_{PE(N_1)}\colon PE(N_1) \to PE(N_1 \oplus N_2)$ and $i'_{N_1}\colon N_1 \to PE(N_1)$ are the inclusions. Thus, $N_1 \in \underline{\mathcal{PI}}^{-1}(M)$. Similarly, $N_2 \in \underline{\mathcal{PI}}^{-1}(M)$, as desired. Now let $f\colon N_1 \oplus N_2 \to M$ be a homomorphism, $i_{N_1 \oplus N_2}$ and i_{N_j} denote the inclusions $N_1 \oplus N_2 \to PE(N_1 \oplus N_2)$ and $N_j \to N_1 \oplus N_2$, respectively, for each $j = 1, 2$. Since $N_j \in \underline{\mathcal{PI}}^{-1}(M)$, there exist $g_j\colon PE(N_j) \to M$ with $g_j i'_{N_j} = f i_{N_j}$ for $j = 1, 2$. Since $PE(N_1)$ and $PE(N_2)$ are direct summands of $PE(N_1 \oplus N_2)$, consider the homomorphism $g_1\pi_{PE(N_1)} + g_2\pi_{PE(N_2)}\colon PE(N_1 \oplus N_2) \to M$, where $\pi_{PE(N_1)}$ and $\pi_{PE(N_2)}$ are natural projections. Due to $(g_1\pi_{PE(N_1)} + g_2\pi_{PE(N_2)})i_{N_1 \oplus N_2} = f$, M is $(N_1 \oplus N_2)$-pure-subinjective. □

As a consequence of Theorem 2.16 and Corollary 2.15, we have the next result.

COROLLARY 2.17. *Let $\{M_i\}_{i \in I}$ and $\{N_i\}_{i \in I}$ be classes of R-modules for an index set $I = \{1, \dots, n\}$ with n a positive integer. Then $\bigoplus_{i \in I} M_i$ is $\bigoplus_{i \in I} N_i$-pure-subinjective if and only if M_i is N_j-pure-subinjective for all $i, j \in I$.*

The following examples show that Theorem 2.14 and Theorem 2.16 do not hold for infinite direct sums.

EXAMPLES 2.18.

(1) Consider the modules $M_i = \mathbb{Z}_{p_i}$ and $N = \bigoplus_{i \in \mathbb{N}} M_i$ where p_i is a prime integer for all $i \in \mathbb{N}$. Since every M_i is pure-injective, $N \in \underline{\mathcal{PI}}^{-1}(M_i)$ and $M_i \in \underline{\mathcal{PI}}^{-1}(N)$ for all $i \in \mathbb{N}$. But N is a pure submodule of $\prod_{i \in \mathbb{N}} M_i$ and N is not a direct summand of it; hence, N is not pure-injective, and so $N \notin \underline{\mathcal{PI}}^{-1}(N) = \underline{\mathcal{PI}}^{-1}(\bigoplus_{i \in \mathbb{N}} M_i)$.
(2) If R is a right pure-injective ring which is not Σ-pure-injective over itself, then $R \in \underline{\mathcal{PI}}^{-1}(R^{(I)})$ and $R^{(I)} \in \underline{\mathcal{PI}}^{-1}(R)$ but $R^{(I)} \notin \underline{\mathcal{PI}}^{-1}(R^{(I)})$ for some infinite index set I.

THEOREM 2.19. *Let M and N be R-modules and K a submodule of N with $Hom_R(K, M) = 0$. If M is (N/K)-pure-subinjective, then M is N-pure-subinjective.*

Proof. Let $f\colon N \to M$ be a homomorphism. Since $f i_K \in \mathrm{Hom}_R(K, M) = 0$ where $i_K\colon K \to N$ is the inclusion, $K \subseteq \mathrm{Ker} f$. By the Factor Theorem, there exists $\bar{f}\colon N/K \to M$ with $\bar{f}\pi_N = f$ where $\pi_N\colon N \to N/K$ is the natural projection. Also, N/K is pure in $PE(N)/K$. By hypothesis, there exists $h\colon PE(N)/K \to M$ with $hi_{N/K} = \bar{f}$ where $i_{N/K}\colon N/K \to PE(N)/K$ is the inclusion. Since $h\pi_{PE(N)}i_N = f$, $N \in \underline{\mathcal{PI}}^{-1}(M)$. □

3. When pure relative subinjectivity is inherited by (pure) quotients and extensions. In this section, we deal with the (pure) quotients of N-pure-subinjective modules for a module N and cotorsion modules, and investigate when the class of N-pure-subinjective modules is closed under extensions. We also obtain some characterizations of the module $PE(N)/N$ being flat for a module N in terms of the cotorsion envelope of N. The following theorem is a generalization of [**16**, Theorem 3.5.1].

THEOREM 3.1. *The following are equivalent for a module N:*

(1) *$PE(N)/N$ is flat.*
(2) *Every cotorsion module is N-pure-subinjective.*
(3) *$C(N)$ is N-pure-subinjective.*
(4) *$C(N)$ is pure-injective (i.e., $C(N) = PE(N)$).*
(5) *For every exact sequence $0 \to P' \to P \to P'' \to 0$ with P' pure-injective, if*
  (a) *P'' pure-injective, or*
  (b) *$P'' = PE(N)$, or*
  (c) *$P'' = C(N)$,*
  then P is N-pure-subinjective.
(6) *For every exact sequence $0 \to P' \to P \to P'' \to 0$ with P' pure-injective and P'' N-pure-subinjective, P is N-pure-subinjective.*

Proof. (1) $\Rightarrow$ (2) Let M be a cotorsion module. By applying the functor $\mathrm{Hom}_R(-, M)$ to the exact sequence $0 \to N \to PE(N) \to PE(N)/N \to 0$, we obtain

$$\cdots \to \mathrm{Hom}_R(PE(N), M) \to \mathrm{Hom}_R(N, M) \to \mathrm{Ext}^1_R(PE(N)/N, M) \cdots$$

Since $PE(N)/N$ is flat, $\mathrm{Ext}^1_R(PE(N)/N, M) = 0$. $\mathrm{Hom}_R(PE(N), M) \to \mathrm{Hom}_R(N, M)$ being surjective implies $N \in \underline{\mathcal{PI}}^{-1}(M)$.
(2) $\Rightarrow$ (3) Obvious.
(3) $\Rightarrow$ (4) Let $i_1 : N \to C(N)$ and $i: N \to PE(N)$ be the inclusions. By (3), $C(N)$ is N-pure-subinjective; hence, by Theorem 2.2, there exists a homomorphism $\phi: PE(N) \to C(N)$ such that $\phi i = i_1$. Since every pure-injective module is cotorsion and $C(N)$ is cotorsion envelope of N, there exists a homomorphism $\alpha: C(N) \to PE(N)$ such that $\alpha i_1 = i$. Thus, $\alpha\phi i = i$ and $\phi\alpha i_1 = i_1$; hence, by the definitions of injective hull and cotorsion envelope (see [**16**, Definition 1.2.1]), $\alpha\phi$ and $\phi\alpha$ are automorphisms and hence α and ϕ are isomorphisms.
(4) $\Rightarrow$ (2) Let M be a cotorsion module. Since $C(N)$ is the cotorsion envelope of N, every homomorphism from N to M can be extended to a homomorphism from $C(N)$ to M. Thus by Theorem 2.2, M is N-pure-subinjective.
(2) $\Rightarrow$ (5) Let $0 \to P' \to P \to P'' \to 0$ be an exact sequence with P' and P'' cotorsion R-modules. Since the class of cotorsion modules is closed under extension, P is also cotorsion and hence by (2), P is N-pure-subinjective.
(5)(c) $\Rightarrow$ (3) Let $P = C(N) \oplus P'$. By hypothesis, P is N-pure-subinjective hence by Corollary 2.15, $C(N)$ is N-pure-subinjective.
(6) $\Rightarrow$ (5)(a) $\Rightarrow$ (5)(b) Obvious.
(5)(b) $\Rightarrow$ (6) Let $0 \to P' \to P \xrightarrow{\alpha} P'' \to 0$ be an exact sequence with P' pure-injective and P'' N-pure-subinjective. By Theorem 2.2, for every homomorphism $f: N \to P$, there exists a homomorphism $\varphi: PE(N) \to P''$ such that $\alpha f = \varphi i$ where $i: N \to PE(N)$ is the inclusion. Now consider the following pullback diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P' & \longrightarrow & L & \xrightarrow{\psi} & PE(N) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & P' & \longrightarrow & P & \xrightarrow[\alpha]{} & P'' & \longrightarrow & 0
\end{array}$$

By pullback diagram property, there exists a homomorphism $\phi: N \to L$ such that $g\phi = f$ and $\psi\phi = i$. By hypothesis, L is N-pure-subinjective, so by Theorem 2.2, there

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exists a homomorphism $\lambda\colon PE(N) \to L$ such that $\lambda i = \phi$. Thus, $g\lambda i = g\phi = f$ and so by Theorem 2.2, P is N-pure-subinjective.

(5)(b) $\Rightarrow$ (1) By [**16**, Lemma 3.4.1], it is enough to show that $\mathrm{Ext}^1_R(PE(N)/N, K) = 0$ for every pure-injective module K. Let K be a pure-injective module. For every exact sequence $0 \to K \to H \xrightarrow{f} PE(N)/N \to 0$, consider the pullback diagram of $f\colon H \to PE(N)/N$ and $\pi\colon PE(N) \to PE(N)/N$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & L & \xrightarrow{\alpha} & PE(N) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & K & \longrightarrow & H & \xrightarrow[f]{} & PE(N)/N & \longrightarrow & 0 \end{array}$$

and since the following diagram commutes:

$$\begin{array}{ccc} N & \xrightarrow{i} & PE(N) \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \pi} \\ H & \xrightarrow[f]{} & PE(N)/N \end{array}$$

so by the property of pullback diagram, there exists a homomorphism $\sigma\colon N \to L$ such that $\alpha\sigma = i$ and $g\sigma = 0$. By hypothesis, L is N-pure-subinjective, thus by Theorem 2.2, there exists a homomorphism $\phi\colon PE(N) \to L$ such that $\phi i = \sigma$ and so $\alpha\phi i = i$. By the property of pure-injective hulls, $\alpha\phi$ is an automorphism of $PE(N)$; hence, $i = (\alpha\phi)^{-1} i$ and $g\phi(\alpha\phi)^{-1} i = g\phi i = g\sigma = 0$. Therefore, $\pi = f g\phi(\alpha\phi)^{-1}$ and it implies that $i_{PE(N)/N} = f\psi$ where $\psi = \overline{g\phi(\alpha\phi)^{-1}}\colon PE(N)/N \to H$ and $i_{PE(N)/N}\colon PE(N)/N \to PE(N)/N$. Consequently, $\mathrm{Ext}^1_R(PE(N)/N, K) = 0$. $\square$

COROLLARY 3.2. *The following statements hold:*

(1) *For a pure submodule N of a module M, if $PE(M)$ is flat, then $C(N)$ is a pure-injective flat module.*
(2) *For a cotorsion module M, $PE(M)/M$ is flat if and only if $M = PE(M)$.*
(3) *Let $\mathcal{A}$ be a class of all cotorsion $\mathbb{Z}$-modules. Then for every $\mathbb{Z}$-module M in $\mathcal{A}$, $\bigoplus_{i\in\mathbb{N}} \mathbb{Z}_{p_i}$ and $\bigoplus_{i\in\mathbb{N}} \mathbb{Z}_{p^i}$ belong to $\underline{\mathcal{PI}}^{-1}(M)$, where p and p_i are prime integers for all $i \in \mathbb{N}$.*

Proof.

(1) Let N be a pure submodule of M. By Remark 1.3, $PE(N)$ is a direct summand of $PE(M)$, and so $PE(N)$ is flat by hypothesis. Consider the pure-exact sequence $0 \to N \to PE(N) \to PE(N)/N \to 0$. Flatness of $PE(N)$ implies that $PE(N)/N$ is flat by [**7**, Corollary 4.86]. Therefore, $C(N)$ is pure-injective due to Theorem 3.1.
(2) It is an easy consequence of Theorem 3.1.
(3) It is a consequence of Theorem 3.1 by the facts that $PE(\bigoplus_{i\in\mathbb{N}} \mathbb{Z}_{p_i}) = \prod_{i\in\mathbb{N}} \mathbb{Z}_{p_i}$ where p_i is a prime integer for all $i \in \mathbb{N}$ and $\prod_{i\in\mathbb{N}} \mathbb{Z}_{p_i} / \bigoplus_{i\in\mathbb{N}} \mathbb{Z}_{p_i}$ is divisible torsion-free, and so a flat $\mathbb{Z}$-module. Similar discussion also holds for $\bigoplus_{i\in\mathbb{N}} \mathbb{Z}_{p^i}$ where p is a prime integer.

$\square$

Recall that a ring R is called *left coherent* provided that every finitely generated left ideal is finitely presented.

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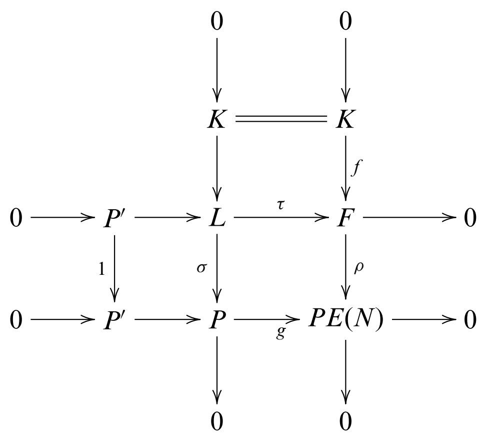
THEOREM 3.3. *Let R be a left coherent ring. Assume for any R-module N and every exact sequence $0 \to A \to B \to C \to 0$, A being pure-injective and B being N-pure-subinjective implies that C is N-pure-subinjective. Then $PE(N)/N$ is flat.*

Proof. Let N be an R-module such that for every exact sequence $0 \to A \to B \to C \to 0$, A being pure-injective and B being N-pure-subinjective implies that C is N-pure-subinjective. Consider an exact sequence $0 \to P' \to P \to PE(N) \to 0$ with P' pure-injective, we shall show that P is N-pure-subinjective, then by Theorem 3.1, $PE(N)/N$ is flat. By [**4**], $PE(N)$ has a flat cover F. Consider the following pullback diagram:



By [**16**, Lemma 3.2.4], F and K are pure-injective. Also, F being flat implies that the sequence $0 \to P' \to L \to F \to 0$ is pure-exact. Since P' is pure-injective, $L \cong F \oplus P'$, and so L is pure-injective. Hence by hypothesis, P is N-pure subinjective. Therefore, $PE(N)/N$ is flat. □

THEOREM 3.4. *The following statements are equivalent for a module N:*

(1) *The class of N-pure-subinjective modules is closed under extensions.*
(2) *For every exact sequence $0 \to P' \to P \to C(N) \to 0$ with P' N-pure-subinjective, P is N-pure-subinjective.*
(3) *For every exact sequence $0 \to P' \to P \to PE(N) \to 0$ with P' N-pure-subinjective, P is N-pure-subinjective.*

Proof. (1) ⇒ (2) Let $0 \to P' \to P \to C(N) \to 0$ be an exact sequence with P' N-pure-subinjective. We prove that $C(N)$ is N-pure-subinjective. Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence with A' pure-injective and A'' N-pure-subinjective. Then A' is also N-pure-subinjective. By hypothesis, A is N-pure-subinjective. Thus by Theorem 3.1(3) ⇔ (6), $C(N)$ is N-pure-subinjective and so again by hypothesis, P is N-pure-subinjective.
(2) ⇒ (3) Let $0 \to P' \to P \to C(N) \to 0$ be an exact sequence with P' pure-injective. Then P' is N-pure-subinjective. By (2), P is N-pure-subinjective. By Theorem 3.1(4) ⇔ (5)(c), $C(N)$ is pure-injective. Therefore, $C(N) = PE(N)$.
(3) ⇒ (1) It is similar to the proof of (5)(b) ⇒ (6) of Theorem 3.1. □

The next lemma is analogue of the characterization of projectivity in terms of injective modules in [**11**, Lemma 4.22].

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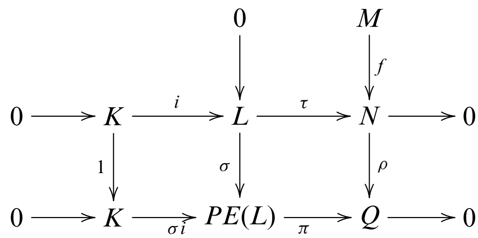
LEMMA 3.5. *A module M is pure-projective if and only if it has the projective property relative to every pure-exact sequence $0 \to A \to B \to C \to 0$ of modules with B pure-injective.*

Proof. The necessity is clear. For the sufficiency, let $0 \to K \to L \to N \to 0$ be a pure-exact sequence of modules and consider the following diagram:



where $\sigma : L \to PE(L)$ is the inclusion and $Q = PE(L)/\mathrm{Im}(\sigma i)$. Note that ρ exists by diagram chasing with $\pi\sigma = \rho\tau$. Since the first row is pure-exact and L is pure in $PE(L)$, the second row is pure-exact. By hypothesis, there exists $g : M \to PE(L)$ such that $\pi g = \rho f$. We claim that $\mathrm{Im} g \subseteq L$. Let $m \in M$. Since τ is surjective, there exists $x \in L$ such that $\tau(x) = f(m)$. Then $\pi\sigma(x) = \rho\tau(x) = \rho f(m) = \pi g(m)$. Hence, $\sigma(x) - g(m) \in \mathrm{Ker}\pi = \mathrm{Im}(\sigma i)$, so $g(m) = \sigma(x - i(k))$ for some $k \in K$. Since σ is the inclusion, $g(m) \in L$. Thus, $\mathrm{Im} g \subseteq L$, and so M is pure-projective. □

THEOREM 3.6. *Let N be a module and consider the following conditions:*

(1) *N is pure-projective.*
(2) *Every pure quotient of an N-pure-subinjective module is N-pure-subinjective.*
(3) *Every pure quotient of a cotorsion module is N-pure-subinjective.*
(4) *Every pure quotient of a pure-injective module is N-pure-subinjective.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (4) *and* (3) $\Rightarrow$ (4). *If $PE(N)$ is pure-projective, then* (4) $\Rightarrow$ (1). *Also,* (2) $\Rightarrow$ (3) *if $PE(N)/N$ is flat.*

Proof. (1) $\Rightarrow$ (2) Let M be an N-pure-subinjective module, K a pure submodule of M and $f : N \to M/K$ a homomorphism. Let $\pi : M \to M/K$ denote the natural projection. By the pure-projectivity of N, there exists a homomorphism $g : N \to M$ such that $f = \pi g$. Since M is N-pure-subinjective, $g = hi_N$ for some homomorphism $h : PE(N) \to M$ where $i_N : N \to PE(N)$ is the inclusion. Due to $\pi hi_N = f$, M/K is N-pure-subinjective.
(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4) are obvious.
(4) $\Rightarrow$ (1) Let M be a pure-injective module and consider a pure-exact sequence $0 \to K \to M \xrightarrow{\pi} M/K \to 0$ where K is a pure submodule of M. Let $f : N \to M/K$ be a homomorphism. By (4), $N \in \underline{\mathcal{PI}}^{-1}(M/K)$. Then $gi = f$ for some $g : PE(N) \to M/K$ where $i : N \to PE(N)$ is the inclusion. By the pure-projectivity of $PE(N)$, there exists $h : PE(N) \to M$ such that $\pi h = g$. Hence, we have $\pi hi = f$. Therefore, N is pure-projective by Lemma 3.5.
If $PE(N)/N$ is flat, then (2) $\Rightarrow$ (3) follows from Theorem 3.1(1)$\Leftrightarrow$(2). □

COROLLARY 3.7. *Consider the following conditions for an R-module N.*

(1) *Every flat R-module is N-pure-subinjective.*
(2) *Every projective R-module is N-pure-subinjective.*
(3) *Every free R-module is N-pure-subinjective.*

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Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3). *If N is pure-projective, then* (3) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Obvious. (3) $\Rightarrow$ (2) Clear by Corollary 2.15(2).
Now assume that N is pure-projective and (3) holds. Let M be a flat module. Then M is a pure quotient of a free module F. Since F is N-pure-subinjective, by Theorem 3.6, M is N-pure-subinjective. Therefore, (1) holds. □

THEOREM 3.8. *Let N be a module and consider the following conditions:*

(1) *N is projective.*
(2) *Every quotient of an N-pure-subinjective module is N-pure-subinjective.*
(3) *Every quotient of a cotorsion module is N-pure-subinjective.*
(4) *Every quotient of a pure-injective module is N-pure-subinjective.*
(5) *Every quotient of an injective module is N-pure-subinjective.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (5) *and* (3) $\Rightarrow$ (4)*. If* $PE(N)$ *is projective, then all of them are equivalent. Moreover, consider*

(6) *every quotient of a flat cotorsion module is N-pure-subinjective;*
(7) *every cotorsion module is N-pure-subinjective.*

Then (3) $\Rightarrow$ (6) $\Rightarrow$ (7)*. Also, together* (1) *and* (7) *imply* (3).

Proof. (1) $\Rightarrow$ (2) and (1) and (7) $\Rightarrow$ (3) are similar to the proof of (1) $\Rightarrow$ (2) in Theorem 3.6.
(2) $\Rightarrow$ (4) $\Rightarrow$ (5), (3) $\Rightarrow$ (4) and (3) $\Rightarrow$ (6) are obvious.
(5) $\Rightarrow$ (1) It is proved as (4) $\Rightarrow$ (1) in Theorem 3.6.
(2) $\Rightarrow$ (3) $PE(N)$ being projective implies that $PE(N)/N$ is flat. So Theorem 3.1 completes the proof.
(6) $\Rightarrow$ (7) Let M be a cotorsion module. Then M has a flat cover $F \xrightarrow{f} M \to 0$. By Wakamatsu's Lemma (see [**16**, Lemma 2.1.1]), $\mathrm{Ker} f$ is cotorsion. It follows that F is also cotorsion. By (6), $M \cong F/\mathrm{Ker} f$ is N-pure-subinjective. □

COROLLARY 3.9. *Consider the following conditions for an R-module N:*

(1) *R is N-pure-subinjective as an R-module.*
(2) *Every finitely presented R-module is N-pure-subinjective.*
(3) *Every finitely generated R-module is N-pure-subinjective.*

Then (3) $\Rightarrow$ (2) $\Rightarrow$ (1)*. If N is projective, then* (1) $\Rightarrow$ (3).

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (1) Obvious.
Now assume that N is a projective module and (1) holds. Let M be a finitely generated module. Then M is a quotient of a finitely generated free module F. By Corollary 2.15(1), F is N-pure-subinjective, and so $N \in \underline{\mathcal{PI}}^{-1}(M)$ by Theorem 3.8. Therefore, (3) holds. □

COROLLARY 3.10. *The following are equivalent for an R-module M:*

(1) *M is R-pure-subinjective.*
(2) *Every finitely generated projective R-module belongs to* $\underline{\mathcal{PI}}^{-1}(M/K)$ *for each submodule K of M.*
(3) *Every finitely generated projective R-module belongs to* $\underline{\mathcal{PI}}^{-1}(M/K)$ *for each pure submodule K of M.*

Proof. (1) $\Rightarrow$ (2) Let $R \in \underline{\mathcal{PI}}^{-1}(M)$. By Theorem 2.16, every finitely generated free R-module belongs to $\underline{\mathcal{PI}}^{-1}(M)$, then so does every finitely generated projective R-module. For any submodule K of M and any finitely generated projective R-module P, due to Theorem 3.8, $P \in \underline{\mathcal{PI}}^{-1}(M/K)$.

(2) $\Rightarrow$ (3) Obvious.
(3) $\Rightarrow$ (1) Clear by the fact that 0 is a pure submodule of M. $\square$

In [**2**], a ring R is called *right pure hereditary* if every pure right ideal of R is projective. The next result is a characterization of right pure hereditary rings whose pure-injective hulls are projective in terms of pure-subinjectivity.

THEOREM 3.11. *Let R be a ring and $\mathcal{S}$ denote the set $\{I \leq R_R : I$ is pure in $R\}$. Consider the following conditions:*

(1) *R is right pure hereditary.*
(2) *For every $I \in \mathcal{S}$, every quotient of an I-pure-subinjective module is I-pure-subinjective.*
(3) *For every $I \in \mathcal{S}$, every quotient of a cotorsion module is I-pure-subinjective.*
(4) *For every $I \in \mathcal{S}$, every quotient of a pure-injective module is I-pure-subinjective.*
(5) *For every $I \in \mathcal{S}$, every quotient of an injective module is I-pure-subinjective.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (5) *and* (3) $\Rightarrow$ (4). *If $PE(R_R)$ is projective, then all of them are equivalent.*

Proof. Let $I \in \mathcal{S}$. By Remark 1.3, $PE(I)$ is a direct summand of $PE(R_R)$. If $PE(R_R)$ is projective, then so is $PE(I)$. The rest is clear by Theorem 3.8. $\square$

4. Further results. This section is devoted to further results on the concept of pure-subinjectivity. We show that the pure-subinjectivity domain of any non-singular module is closed under the essential pure-extensions. It is also obtained that endomorphism rings of pure-injective right modules are right pure-injective. In the next, we completely determine the pure submodules which belong to pure-subinjectivity domain of a module.

THEOREM 4.1. *Every pure submodule N of a module M with $N \in \underline{\mathcal{PI}}^{-1}(M)$ is pure-essential in a direct summand of M, also this direct summand is $PE(N)$.*

Proof. Let M be a module, N a pure submodule of M and $N \in \underline{\mathcal{PI}}^{-1}(M)$. Then there exists $f : PE(N) \to M$ such that $f|_N = i$ where $i : N \to M$ is the inclusion. We claim that f is monic. To see this, let $p \in PE(N)$ with $f(p) = 0$. If $p \in N$, then $p = 0$. Assume that $p \notin N$. Since $f|_N = i$, clearly, $pR \cap N = 0$. Let $\sum_{i=1}^{n} x_i r_{ij} = n_j + pR$ be a finite system of equations over $(pR + N)/pR$ where $r_{ij} \in R$, $n_j \in N$ for $j = 1, \ldots, m$ and $\{a_i + pR : i = 1, \ldots, n\}$ a solution in $PE(N)/pR$. Then $\sum_{i=1}^{n}(a_i + pR)r_{ij} = n_j + pR$, and so $\sum_{i=1}^{n} a_i r_{ij} = n_j + pr_j$ for some $r_j \in R$ and $j = 1, \ldots, m$. Since $f|_N = i$ and $f(p) = 0$, we obtain $\sum_{i=1}^{n} f(a_i) r_{ij} = n_j$ for $j = 1, \ldots, m$. Due to purity of N in M, there exists $\{b_i \in N : i = 1, \ldots, n\}$ such that $\sum_{i=1}^{n} b_i r_{ij} = n_j$ for $j = 1, \ldots, m$. It follows that $\sum_{i=1}^{n}(b_i + pR)r_{ij} = n_j + pR$ for $j = 1, \ldots, m$. Hence, $(pR + N)/pR$ is pure in $PE(N)/pR$. Since N is pure-essential in $PE(N)$, we have $pR = 0$, and so $p = 0$. Thus, $PE(N) \leq M$. On the other hand, since N is pure in M, $PE(N)$ is a direct summand of $PE(M)$. By the modularity condition, the pure-essential extension $PE(N)$ of N is a direct summand of M. $\square$

The next result is a consequence of Theorems 4.1 2.14.

COROLLARY 4.2. *Let M be a module and N a submodule of M. Then the following are equivalent:*

(1) *N is pure in M and $N \in \underline{\mathcal{PI}}^{-1}(M)$.*
(2) *$M = PE(N) \oplus K$ for some submodule K of M and $N \in \underline{\mathcal{PI}}^{-1}(K)$.*

For a module M, $Z(M) = \{m \in M \mid mI = 0$ for some essential right ideal I in $R\}$ is called *singular submodule* of M. Equivalently, $m \in Z(M)$ if and only if the right annihilator $r_R(m)$ of m in R is an essential right ideal of R. Recall that a module M with $Z(M) = 0$ is called *non-singular*.

THEOREM 4.3. *Let M be a non-singular module and $N \in \underline{\mathcal{PI}}^{-1}(M)$. Then every essential pure extension of N belongs to $\underline{\mathcal{PI}}^{-1}(M)$.*

Proof. Let K be an essential pure extension of N, L a pure extension of K, and $f: K \to M$ a homomorphism. Then N is pure in L. Since $N \in \underline{\mathcal{PI}}^{-1}(M)$, there exists $g: L \to M$ with $g|_N = f|_N$. We claim that $r_R(f(k) - g(k))$ is essential in R for any $k \in K$. Let $k \in K$ and $0 \neq a \in R$. If $ka = 0$, then $a \in r_R(f(k) - g(k))$. If $ka \neq 0$, then there exists $b \in R$ such that $0 \neq kab \in N$ by the essentiality of N in K. Hence, $f(kab) = g(kab)$, and so $0 \neq ab \in r_R(f(k) - g(k))$. Thus, we proved the assertion. This implies $f(k) - g(k) \in Z(M) = 0$. It follows that g extends f. Therefore, $K \in \underline{\mathcal{PI}}^{-1}(M)$. □

In the following, we deal with an R-module structure of an R/I-module where I is an ideal of a ring R.

THEOREM 4.4. *Let R be a ring, I an ideal of R, M an R/I-module and N an R-module. If M is an $N/(NI)$-pure-subinjective R/I-module, then it is an N-pure-subinjective R-module.*

Proof. Let $N/(NI) \in \underline{\mathcal{PI}}^{-1}(M_{R/I})$, an R-module K be a pure extension of N and $f: N \to M$ an R-homomorphism. By Remark 1.4, N/NI can be embedded in K/KI as a pure submodule via $g: N/NI \to K/KI$ defined by $g(n + NI) = n + KI$ for any $n \in N$. Since $NI \leq \operatorname{Ker} f$, by Factor Theorem, there exists $\bar{f}: N/(NI) \to M$ such that $\bar{f}\pi_N = f$ where $\pi_N: N \to N/NI$ is natural projection. By assumption, there exists an R/I-homomorphism $h: K/KI \to M$ such that $hg = \bar{f}$. Since h is also an R-homomorphism and $h\pi_K i_N = f$ where $\pi_K: K \to K/KI$ is the natural projection and $i_N: N \to K$ is the inclusion, $h\pi_K$ extends f. Thus, $N \in \underline{\mathcal{PI}}^{-1}(M_R)$. □

COROLLARY 4.5. *Let R be a ring and I an ideal of R. Then the following hold:*
(1) *Let M and N be R/I-modules. Then M is an N-pure-subinjective R-module if and only if it is an N-pure-subinjective R/I-module.*
(2) [**6**, Theorem 3.7] *Let M be an R/I-module. Then M is a pure-injective R-module if and only if it is a pure-injective R/I-module.*
(3) *Let M be an R-module. Then $PE(M/MI)$ has an R/I-module structure.*

Proof. (1) The necessity is clear. The sufficiency holds by Theorem 4.4.
(2) This known result is a consequence of (1) by using Theorem 2.3.
(3) Due to (2), $PE(M/MI)_{R/I}$ is pure-injective as an R-module. By the definition of the pure-injective hull of a module, $PE(M/MI)_R$ is contained in $PE(M/MI)_{R/I}$. This implies that $PE(M/MI)_R$ has an R/I-module structure. □

We close this paper by observing some results about the functor $\operatorname{Hom}_R(-, -)$, in particular, the endomorphism rings of modules.

THEOREM 4.6. *Let M be a right R-module and N a left S- right R-bimodule. If M is N-pure-subinjective, then $Hom_R(N, M)$ is an S-pure-subinjective right S-module.*

Proof. Let K be a pure extension of S as a right S-module. We need to show that $\mathrm{Hom}_S(K, \mathrm{Hom}_R(N, M)) \to \mathrm{Hom}_S(S, \mathrm{Hom}_R(N, M))$ is epic. For any left R-module A, since S is pure in K as a right S-module, we have an exact sequence

$$0 \to S \otimes_S (N \otimes_R A) \to K \otimes_S (N \otimes_R A)$$

and so $0 \to (S \otimes_S N) \otimes_R A \to (K \otimes_S N) \otimes_R A$. Hence, $S \otimes_S N$ is pure in the right R-module $K \otimes_S N$. Since $S \otimes_S N \cong N$ as a right R-module and $N \in \underline{\mathcal{PI}}^{-1}(M)$, we obtain the exact sequence $\mathrm{Hom}_R(K \otimes_S N, M) \to \mathrm{Hom}_R(S \otimes_S N, M) \to 0$. By the Adjoint Isomorphism, we have the exact sequence

$$\mathrm{Hom}_S(K, \mathrm{Hom}_R(N, M)) \to \mathrm{Hom}_S(S, \mathrm{Hom}_R(N, M)) \to 0$$

as desired. Therefore, $S \in \underline{\mathcal{PI}}^{-1}(\mathrm{Hom}_R(N, M))$. □

Corollary 4.7(3) is known from [**13**, Lemma 33.4], also it is obtained as a consequence of Theorems 2.3 and 4.6.

COROLLARY 4.7. *The following hold:*

(1) *Let M and N be right R-modules. If $N \in \underline{\mathcal{PI}}^{-1}(M)$, then $\mathrm{End}_R(N) \in \underline{\mathcal{PI}}^{-1}(\mathrm{Hom}_R(N, M))$ as a right $\mathrm{End}_R(N)$-module.*

(2) *Let M be a module and N a submodule of M with $N \in \underline{\mathcal{PI}}^{-1}(M)$. If $f(N) \subseteq N$ for every homomorphism $f : N \to M$, then $\mathrm{End}_R(N)$ is right pure-injective.*

(3) *Endomorphism ring of any pure-injective module is right pure-injective.*

Proof. (1) Clear from Theorem 4.6. (3) The proof follows from (2).
(2) By (1), the right $\mathrm{End}_R(N)$-module $\mathrm{End}_R(N)$ belongs to $\underline{\mathcal{PI}}^{-1}(\mathrm{Hom}_R(N, M))$. On the other hand, $\mathrm{Hom}_R(N, M) = \mathrm{End}_R(N)$ by hypothesis. Hence, $\mathrm{End}_R(N)$ is right pure-injective due to Theorem 2.3. □

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