BULL. AUSTRAL. MATH. SOC. 47H30, 47D03
VOL. 74 (2006) [411–418]

ON STRONG CONVEX COMPACTNESS PROPERTY OF SPACES OF NONLINEAR OPERATORS

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The strong convex compactness property is important for property persistence of operator semigroups under perturbations. It has been investigated in the linear setting. In this paper, we are concerned with the property in the nonlinear setting. We prove that the following spaces of (nonlinear) operators enjoy the strong convex compactness property: the space of compact operators, the space of completely continuous operators, the space of weakly compact operators, the space of conditionally weakly compact operators, the space of weakly completely continuous operators, the space of demicontinuous operators, the space of weakly continuous operators and the space of strongly continuous operators. Moreover, we prove the property persistence of operator semigroups under nonlinear perturbation.

1. Introduction

It is well-known that the space $K(X, Y)$ of linear compact operators from Banach space X to Banach space Y is not complete in the strong operator topology. However, in most cases we are really interested in compactness persistence in a certain sense of convexity. For example, in practical application of operator semigroups, we often have to know if an operator semigroup preserves its compactness under a certain perturbation (see below). Motivated by this, in the frame of bounded linear operators, Voigt [**15**] introduced a relevant notion, named the strong convex compactness property, and proved that the following several spaces enjoy the property: the space of completely continuous operators, the space of Dieudonné operators and the space of unconditionally summing operators. As early as in 1988, Weis [**16**] effectively proved the strong convex compact property for the space of strictly singular operators from $L^p(\Omega)$ (or $C(\Omega)$) into itself, although he did not specifically give the definition. Recently, Mokhter-Kharroubi [**10**] presented direct proofs of the convex compactness property for the space of compact operators as well as the space of weakly compact operators, and applied the property to the spectral theory of perturbed semigroups. Almost at the same time as [**15**], Schlüchtermann [**14**] proved that both the space of weakly compact operators and the space of conditionally weakly compact

Received 5th June, 2006
This work was supported by the Natural Science Foundation of China under the contact no. 10531030.

operators enjoy the property. Moreover, he remarked that the mentioned conclusions can be extended to the nonlinear case if the uniform boundedness in Voigt's definition of the strong convex compactness property is replaced with the uniform integrability property (see, [**14**, Remark 2.4] or (2.3) below). This implies that some spaces of nonlinear operators may also enjoy the strong convex compactness property. The significance of this remark is, that, inter alia, an operator semigroup may still preserve its compactness under some nonlinear perturbation. Indeed, since the perturbed semigroup $\{S_t\}_{t\geqslant 0}$ and the unperturbed linear semigroup $\{T_t\}_{t\geqslant 0}$ bear the following relation:

$$S_t x = T_t x + \int_0^t T_{t-r} B S_r x \, dr, \ \forall x \in X, t \geqslant 0 \tag{1.1}$$

where A is the infinitesimal generator of $\{T_t\}_{t\geqslant 0}$ and B the nonlinear perturbation operator, the compactness of S_t is completely determined by that of the integral term of (1.1). Noticing that, for each $t > 0$, the operator defined by the integral term is a convex combination of the operators $\{T_{t-r}BS_r\}_{0<r<t}$ in the strong operator topology, we derive the compactness of the perturbed semigroup $\{S_t\}_{t\geqslant 0}$ if each integral element $T_{t-r}BS_r$ belongs to a compact operator space that enjoys the strong convex compactness property. Therefore, it is significant to probe what spaces of nonlinear operators enjoys the strong convex compactness property.

The purpose of this paper is to investigate the strong convex compactness property for several spaces of nonlinear operators, and then to derive property persistence of operator semigroup under nonlinear perturbation. Nonlinear operators are not assumed to be continuous unless specified.

2. Preliminaries

Throughout this paper we assume that X and Y are two Banach spaces on the same scalar field $\mathbb{K}(=\mathbb{R}$ or $\mathbb{C})$, and denote by $B(X,Y)$ the space of bounded linear operators from X to Y. Below we introduce the definitions of the strong convex compactness property for spaces of operators from X to Y.

Definition 1: ([**15**]) A closed linear subspace E of $B(X,Y)$ is said to have the strong convex compactness property if, for any finite measure space $(\Omega, \mathcal{A}, \mu)$ and bounded strongly measurable function $U : \Omega \to E$, the strong integral $\int U \, d\mu$ defined by

$$\int_\Omega U d\mu(\omega)x := \int_\Omega U(\omega)x d\mu(\omega), \ x \in X.$$

belongs to E.

In Definition 1, the boundedness of U means that $\sup_{\omega\in\Omega}\|U(\omega)\| < \infty$, where $\|U(\omega)\|$ is the operator norm of the bounded linear operator $U(\omega)$. Since the operator norm is excessive for a nonlinear operator, we have to modify Definition 1 so that the notion of convex compactness property is also valid for nonlinear operator spaces.

DEFINITION 2: Let E be a linear space of operators from X to Y enjoying some specified property. Then, E is said to have the strong convex compactness property if, for any finite measure space $(\Omega, \mathcal{A}, \mu)$ and strongly measurable function $U : \Omega \to E$ that satisfies for any bounded subset B of X,

$$\sup_{\omega\in\Omega}\sup_{x\in B}\|U(\omega)x\| < \infty, \tag{2.1}$$

the strong integral $\int U\, d\mu$ defined by

$$\int_\Omega U d\mu(\omega)x := \int_\Omega U(\omega)x d\mu(\omega),\ x \in X \tag{2.2}$$

belongs to E.

By standard technique it can be shown that if E is a closed subspace of $B(X,Y)$ then Definition 2 is reduced to Definition 1. It is worth mentioning that, in order to make the definition valid for nonlinear weakly compact operators as well as nonlinear conditionally weakly compact operators, Schlüchtermann suggests in [**14**] that the assumption $\sup_{\omega\in\Omega}\|U(\omega)\| < \infty$ in Definition 1 should be replaced with the uniform integrability

$$\sup_{x\in B}\int_\Omega \|U(\omega)x\| d\mu(\omega) < \infty,\ \lim_{\mu(A)\to 0}\sup_{x\in B}\int_A \|u(\omega)x\| d\mu(\omega) = 0, \tag{2.3}$$

for any bounded subset B of X. It is clear to see that the condition (2.1) implies the uniform integrability (2.3). Hence, the definition suggested by Schlüchtermann seems to be more general than Definition 2. Initially, we defined the convex compactness property using the condition (2.3) instead of (2.1). However we now adopt the present definition is because: firstly, as the nonlinear extension of uniform boundedness in Definition 1, the condition (2.1) is more direct than (2.3); and secondly, in most cases, to verify the condition (2.3) is more difficult than (2.1). Moreover, it has been suggested that the conditions (2.1) and (2.3) might be equivalent in the cases treated by us. However, we are not able to give the equivalent proof.

Below we introduce several types of nonlinear operators, which can be found in, for example, [**4, 14, 15, 17**].

An operator T from X into Y is *compact* (*weakly compact*) if for every bounded subset M of X, its range $T(M)$ is relatively compact (relatively weakly compact) in Y. A continuous compact operator is called *completely continuous*. We denote by $K_{nl}(X,Y)$, $WK_{nl}(X,Y)$ and $CC_{nl}(X,Y)$ the space of all compact operators, the space of weakly compact operators and the space of completely continuous operators, respectively. Here the subscript "nl" is the abbreviation for nonlinear, which is only used to distinguish them from their respective subspaces consisting of corresponding linear operators.

An operator T from X into Y is said to be *conditionally weakly compact* (CWC) if every sequence in the range $T(M)$ admits a weak Cauchy subsequence for any bounded subset M of X. The space of all CWC operators is denoted by $CWK_{nl}(X,Y)$. It is easy to show that a linear CWC operator from X into Y must be bounded and hence belongs to $B(X,Y)$.

We call a mapping T from X into Y *a weakly completely continuous* (WCC) operator if it maps weak Cauchy sequences of X to weakly convergent sequences. In [**15**], a weakly completely continuous operator is also called *Dieudonné* operator. It is easy to show that a linear WCC operator must be bounded and hence continuous. We denote by $WCC_{nl}(X,Y)$ the space of all weakly completely continuous operators from X into Y.

An operator T from X into Y is called *demicontinuous* if Tx_n is weakly convergent to Tx for all $x \in X$ and sequences $\{x_n\}_{n=1}^{\infty} \subset X$ converging to x. We denote by $DC_{nl}(X,Y)$ the space of all demicontinuous operators from X into Y.

An operator T from X into Y is said to be *weakly continuous* if, for all $x \in X$ and sequences $\{x_n\}_{n=1}^{\infty}$ weakly converging to x, Tx_n is weakly convergent to Tx. It is obvious that a linear weakly continuous operator must be continuous. The space of all weakly continuous operators from X into Y is denoted by $WC_{nl}(X,Y)$.

An operator T from X into Y is said to be *strongly continuous* if, for all $x \in X$ and sequences $\{x_n\}_{n=1}^{\infty}$ weakly converging to x, Tx_n is convergent to Tx. Denote by $SC_{nl}(X,Y)$ the space of all strongly continuous operators from X into Y.

3. Strong Convex Compactness Property

In this section we prove the strong convex compact property for the operator spaces defined in the previous section.

Proposition 1. *$K_{nl}(X,Y)$ has the strong convex compactness property provided Y is separable.*

Proof: It can be shown that an operator T from X into Y is compact if and only if its restriction to any separable subspace of X is also compact. Indeed, for any bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ of X, $\{Tx_n\}_{n\in\mathbb{N}}$ has an accumulation point in X if and only if this is also true for $\{T|_V x_n\}_{n\in\mathbb{N}}$ in the separable subspace $V = \overline{\operatorname{span}}\{x_n : n \in \mathbb{N}\}$ spanned by $\{x_n\}_{n\in\mathbb{N}}$, where $T|_V$ is the restriction of T to the subspace V. Hence, without loss of generality we assume that X is separable.

Since Y is separable, by [**1**, Chapter XI, Theorem 9] we know that Y can be embedded isomorphically into the Banach space $C[0,1]$ of continuous real functions on the interval $[0,1]$. Because it does not affect the compactness of operators to enlarge the range space, we assume, without loss of generality, that $Y = C[0,1]$. Since $C[0,1]$ possesses a Schauder basis, there exists a sequence $\{P_n\}_{n\in\mathbb{N}}$ of finite dimensional projections such that $\{P_n\}_{n\in\mathbb{N}}$ strongly converges to the identity operator I of Y. Thus, it is easy to show that a nonlinear operator K is compact if and only if, for any bounded set B of X, $\lim_{n\to+\infty}\sup_{x\in B}\|Kx-P_nKx\|=0$. In fact, if K is compact, for any bounded set B, the range $K(B)$ is relative compact (that is, the closure $\overline{K(B)}$ is compact). Hence, it follows that

$$\lim_{n\to+\infty}\sup_{x\in B}\|Kx-P_nKx\|=\lim_{n\to+\infty}\sup_{y\in K(B)}\|y-P_ny\|=0$$

if and only if $\lim_{n\to+\infty}\|y-P_ny\|=0$ for each $y\in\overline{K(B)}$, which is guaranteed by the strong convergence of $\{P_n\}_{n\in\mathbb{N}}$.

Conversely, if for any bounded set B holds $\lim_{n\to+\infty}\sup_{x\in B}\|Kx-P_nKx\|=0$, for any $\varepsilon>0$ there exists a sufficiently large $n_\varepsilon\in\mathbb{N}$ such that $\sup_{x\in B}\|Kx-P_{n_\varepsilon}Kx\|\leqslant\varepsilon/2$. Because $P_{n_\varepsilon}K(B)$ is relatively compact, there exists a finite set Z_ε such that $P_{n_\varepsilon}K(B)$ is contained in the $\varepsilon/2$-neighbourhood of Z_ε. Hence, $K(B)$ is contained in the ε-neighbourhood of Z_ε. This implies that $K(B)$ is relatively compact, that is, K is compact.

Now, we turn to the operator $\int U\,d\mu$ defined by (2.2). By the definition it follows that, for any bounded set B,

$$\sup_{x\in B}\left\|\int Ud\mu x-P_n\int Ud\mu x\right\|\leqslant\int_\Omega\sup_{x\in B}\left\|U(\omega)x-P_nU(\omega)x\right\|\mathrm{d}\mu(\omega).$$

From the strong convergence of $\{P_n\}_{n\in\mathbb{N}}$ and the uniform boundedness theorem, we derive that there exists a constant $M>0$ such that $\|P_n\|\leqslant M$ for all $n\in\mathbb{N}$. Since $(\Omega,\mathcal{A},\mu)$ is finite and $U(\omega)$ satisfies the condition (2.1), we have $\left\|P_nU(\omega)x\right\|\leqslant\|P_n\|\left\|U(\omega)x\right\|\leqslant M\left\|U(\omega)x\right\|$ and $\left\|U(\cdot)x\right\|:\Omega\to\mathbb{R}$ is integrable for $x\in B$. Combined with the compactness of $U(\omega)$, the dominated convergence theorem gives that

$$\lim_{n\to+\infty}\sup_{x\in B}\left\|\int Ud\mu x-P_n\int Ud\mu x\right\|\leqslant\int_\Omega\lim_{n\to+\infty}\sup_{x\in B}\left\|U(\omega)x-P_nU(\omega)x\right\|\mathrm{d}\mu(\omega)=0.$$

Hence, $\int Ud\mu$ is compact, that is, $\int Ud\mu\in K_{nl}(X,Y)$. ∎

REMARK 1. In the above proof, we adopt the proof idea of [**15**, Theorem 1.3].

PROPOSITION 2. *$CC_{nl}(X,Y)$ enjoys the strong convex compactness property.*

PROOF: According to the proof of Proposition 1 we can assume that X is separable and without loss of generality, let $X=\overline{\{x_n:n\in\mathbb{N}\}}$. Let $U:\Omega\to CC_{nl}(X,Y)$ be strongly measurable. Then, for every x_n, there exists a μ-null set F_n such that $\{U(\omega)x_n:\omega\in\Omega\backslash F_n\}$ is contained in a separable subspace of Y. Since $U(\omega)$ is continuous for all $\omega\in\Omega$, it is easy to show that the μ-null set $F=\bigcup_{n=1}^{\infty}F_n$ satisfies

$$\{U(\omega)x:\ \omega\in\Omega\backslash F,x\in X\}\subseteq\overline{\cup\{U(\omega)x_n:\ \omega\in\Omega\backslash F_n,n=1,2,\dots\}}.$$

The set on the right side is obviously contained in a separable subspace of Y. Hence, we can assume that Y is separable, without loss of generality. The remainder proof is similar to that of Proposition 1. $\square$

REMARK 2. From the above proofs it can be seen that Proposition 1 and Proposition 2 still hold even if the condition (2.1) in Definition 2 is replaced with (2.3).

PROPOSITION 3. ([14]) *$WK_{nl}(X,Y)$ has the strong convex compactness property.*

PROPOSITION 4. ([14]) *$CWK_{nl}(X,Y)$ has strong the convex compactness property.*

PROPOSITION 5. *$WCC_{nl}(X,Y)$ enjoys the strong convex compactness property.*

PROOF: Let $U : \Omega \to WCC_{nl}(X,Y)$ be a strongly measurable function on the finite measure space $(\Omega, \mathcal{A}, \mu)$ such that the condition (2.1) is satisfied. By the definition of the WCC operator we know that the sequence $\{U(\omega)x_n\}_{n\in\mathbf{N}}$ is weakly convergent for all $\omega \in \Omega$ and any given weak Cauchy sequence $\{x_n\}_{n\in\mathbf{N}}$. If we denote by $y(\omega)$ the corresponding weak limit, namely,

$$\lim_{n\to\infty} \langle U(\omega)x_n, y^*\rangle = \langle y(\omega), y^*\rangle, \text{ for all } \omega \in \Omega \text{ and } y^* \in Y^*,$$

then, by [**3**, Theorem 2, p. 42], the function $\omega \mapsto y(\omega)$ is strongly measurable. Moreover, by the condition (2.1) we have that, for all $y^* \in Y^*$,

$$\sup_{\omega\in\Omega}\left|\langle y(\omega), y^*\rangle\right| \leqslant \sup_{\omega\in\Omega}\sup_{n\in\mathbf{N}}\left|\langle U(\omega)x_n, y^*\rangle\right| < \infty.$$

That is, $\sup\limits_{\omega\in\Omega}\|y(\omega)\| < \infty$. Hence, the function $\omega \mapsto y(\omega)$ is Bochner integrable.

Since there holds that, for all $y^* \in Y^*$,

$$\begin{aligned}\left|\left\langle \int_\Omega U(\omega)x_n\, d\mu(\omega), y^*\right\rangle - \left\langle \int_\Omega y(\omega)\, d\mu(\omega), y^*\right\rangle\right| &= \left|\int_\Omega \langle U(\omega)x_n, y^*\rangle - \langle y(\omega), y^*\rangle\, d\mu(\omega)\right| \\ &\leqslant \int_\Omega \left|\langle U(\omega)x_n, y^*\rangle - \langle y(\omega), y^*\rangle\right| d\mu(\omega),\end{aligned}$$

applying the dominated convergence theorem in [**3**], we conclude that $\int_\Omega U(\omega)x_n d\mu(\omega)$ weakly converges to $\int_\Omega y(\omega)d\mu(\omega)$ as $n \to \infty$. Therefore, the proof is completed. $\square$

PROPOSITION 6. *$DC_{nl}(X,Y)$ possesses the strong convex compactness property.*

PROOF: Let $U : \Omega \to DC_{nl}(X,Y)$ be a strongly measurable function on the finite measure space $(\Omega, \mathcal{A}, \mu)$ such that the condition (2.1) is satisfied. For any given $x \in X$

and sequences $\{x_n\}_{n=1}^{\infty}$ converging to x, we have that $\langle U(\omega)x_n, y^*\rangle \to \langle U(\omega)x, y^*\rangle$ for all $\omega \in \Omega$ and $y^* \in Y^*$. Moreover, we have that, for all $y \in Y^*$,

$$\begin{aligned}\left|\left\langle \int_\Omega U(\omega)x_n \, d\mu(\omega), y^*\right\rangle - \left\langle \int_\Omega U(\omega)x \, d\mu(\omega), y^*\right\rangle\right| \\ = \left|\int_\Omega \langle U(\omega)x_n, y^*\rangle - \langle U(\omega)x, y^*\rangle \, d\mu(\omega)\right| \\ \leqslant \int_\Omega \left|\langle U(\omega)x_n, y^*\rangle - \langle U(\omega)x, y^*\rangle\right| \, d\mu(\omega).\end{aligned}$$

Hence, by the dominated convergence theorem in [3] we conclude that $\int_\Omega U(\omega)x_n d\mu(\omega)$ weakly converges to $\int_\Omega y(\omega) d\mu(\omega)$ as $n \to \infty$. ∎

PROPOSITION 7. *$WC_{nl}(X, Y)$ possesses the strong convex compactness property.*

PROPOSITION 8. *$SC_{nl}(X, Y)$ possesses the strong convex compactness property.*

We omit the proofs of Propositions 7 and 8 because they are similar to that of Proposition 6.

In the remainder of the paper, we apply the above propositions to the property persistence of operator semigroups. For the properties of operator semigroups, see [**6, 9, 12**]. By [**12, 13**], we know that the perturbed operator $A + K$ generates a strongly continuous semigroup $\{S_t\}_{t\geqslant 0}$ of Lipschitz continuous operators if A is a generator of a C_0-semigroup $\{T_t\}_{t\geqslant 0}$ and K is Lipschitz continuous, where the two semigroups bear the following relation

$$S_t x = T_t x + \int_0^t T_{t-r} K S_r x \, dr, \ \forall x \in X, t \geqslant 0. \tag{3.1}$$

An operator K from X into itself is said to be Lipschitz continuous if there exists a nonnegative constant M such that $\|Kx - Ky\| \leqslant M\|x - y\|$ for all $x, y \in X$.

PROPOSITION 9. *Let $\{T_t\}_{t\geqslant 0}$ be a C_0-semigroup on X with its generator A, and let K be a Lipschitz continuous operator from X into itself. Then, so is the Lipschitz operator semigroup $\{S_t\}_{t\geqslant 0}$ generated by $A+K$ whenever $\{T_t\}_{t\geqslant 0}$ is compact, completely continuous, weakly compact, or conditionally weakly compact.*

PROOF: By the equality (3.1) and Propositions 1-4, it suffices to verify that $T_{t-r}KS_r$ inherits the specified property of T_{t-r} for all $t, r > 0$. Because it is Lipschitz continuous, KS_r maps any bounded set into bounded set. Hence, for all $t, r \geqslant 0$ with $t > r$, so is $T_{t-r}KS_r$ if T_{t-r} is compact, completely continuous, weakly compact or conditionally weakly compact. ∎

https://doi.org/10.1017/S0004972700040466 Published online by Cambridge University Press

REMARK 3. If the perturbation operator K is linear, then it is easy to show that so is the perturbed semigroup $\{S_t\}_{t\geqslant 0}$ whenever the unperturbed semigroup $\{T_t\}_{t\geqslant 0}$ is strongly continuous or weakly completely continuous. See [**2, 5, 6, 7, 8, 11**] for details about property persistence of operator semigroups under linear perturbation. However, in the nonlinear case, we do not know if the perturbed $\{S_t\}_{t\geqslant 0}$ inherits the weak continuity, strong continuity or weakly complete continuity of the unperturbed semigroup $\{T_t\}_{t\geqslant 0}$.

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