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Abstract

A compact semisimple Lie algebra $\mathfrak{g}$ induces a Poisson structure $\pi_{\mathbb{S}}$ on the unit sphere $\mathbb{S}(\mathfrak{g}^*)$ in $\mathfrak{g}^*$. We compute the moduli space of Poisson structures on $\mathbb{S}(\mathfrak{g}^*)$ around $\pi_{\mathbb{S}}$. This is the first explicit computation of a Poisson moduli space in dimension greater or equal than three around a degenerate (i.e. not symplectic) Poisson structure.

Introduction

Recall that a *Poisson structure* on a manifold M is a Lie bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$, satisfying the Leibniz rule

$$\{f, gh\} = \{f, g\}h + \{f, h\}g.$$

Equivalently, a Poisson structure is given by bivector field $\pi \in \mathfrak{X}^2(M)$ satisfying $[\pi,\pi] = 0$ for the Schouten bracket; the bivector and the Lie bracket are related by

$$\{f, g\} = \langle \pi \mid df \wedge dg\rangle.$$

The *Hamiltonian* vector field of a function $f \in C^\infty(M)$ is $X_f := \{f,\cdot\} \in \mathfrak{X}(M)$. These vector fields span a singular involutive distribution, which integrates to a partition of M into regularly immersed submanifolds called *symplectic leaves*; each such leaf S carries canonically a symplectic structure: $\omega_S := \pi|_S^{-1} \in \Omega^2(S)$. A smooth function, constant along the symplectic leaves is called a *Casimir* function. We denote the space of Casimir functions by $\mathfrak{Casim}(M,\pi)$.

Lie theory provides interesting examples of Poisson manifolds. The dual vector space of a Lie algebra $(\mathfrak{g},[\cdot,\cdot])$ carries a canonical a Poisson structure $\pi_{\mathfrak{g}}$, given by

$$\langle(\pi_{\mathfrak{g}})_\xi \mid X \wedge Y\rangle := \xi([X,Y]), \quad \xi \in \mathfrak{g}^*, X, Y \in \mathfrak{g} = T^*_\xi\mathfrak{g}^*.$$

Assuming that $\mathfrak{g}$ is compact and semisimple, $\mathfrak{g}^*$ carries an $\mathrm{Aut}(\mathfrak{g})$-invariant inner product (e.g. induced by the Killing form). The corresponding unit sphere around the origin, denoted by $\mathbb{S}(\mathfrak{g}^*)$, inherits a Poisson structure $\pi_{\mathbb{S}} := \pi_{\mathfrak{g}}|_{\mathbb{S}(\mathfrak{g}^*)}$. We will call $(\mathbb{S}(\mathfrak{g}^*),\pi_{\mathbb{S}})$ the *Lie–Poisson sphere* corresponding to $\mathfrak{g}$. Lie algebra automorphisms of $\mathfrak{g}$ restrict to Poisson diffeomorphisms of $\pi_{\mathbb{S}}$, and the inner automorphisms act trivially on Casimir functions. Therefore $\mathrm{Out}(\mathfrak{g})$, the group of outer automorphisms of $\mathfrak{g}$, acts naturally on $\mathfrak{Casim}(\mathbb{S}(\mathfrak{g}^*),\pi_{\mathbb{S}})$.

Our main result describes all Poisson structures on $\mathbb{S}(\mathfrak{g}^*)$ near $\pi_{\mathbb{S}}$.

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Theorem 1. *For the Lie–Poisson sphere* $(\mathbb{S}(\mathfrak{g}^*), \pi_{\mathbb{S}})$*, corresponding to a compact semisimple Lie algebra* $\mathfrak{g}$*, the following hold.*

(a) *There exists a* C^p*-open set* $\mathcal{W} \subset \mathfrak{X}^2(\mathbb{S}(\mathfrak{g}^*))$ *around* $\pi_{\mathbb{S}}$*, such that every Poisson structure in* $\mathcal{W}$ *is isomorphic to one of the form* $f\pi_{\mathbb{S}}$*, where* f *is a positive Casimir function, by a diffeomorphism isotopic to the identity.*

(b) *For two positive Casimir functions* f *and* g*, the Poisson manifolds* $(\mathbb{S}(\mathfrak{g}^*), f\pi_{\mathbb{S}})$ *and* $(\mathbb{S}(\mathfrak{g}^*), g\pi_{\mathbb{S}})$ *are isomorphic precisely when* f *and* g *are related by an outer automorphism of* $\mathfrak{g}$*.*

The open set $\mathcal{W}$ will be constructed such that it contains all Poisson structures of the form $f\pi_{\mathbb{S}}$, with f a positive Casimir function. Therefore, the map $F \mapsto e^F\pi_{\mathbb{S}}$ induces a bijection between the space

$$\mathfrak{Casim}(\mathbb{S}(\mathfrak{g}^*), \pi_{\mathbb{S}})/\mathrm{Out}(\mathfrak{g})$$

and an open set around $\pi_{\mathbb{S}}$ in the Poisson moduli space of $\mathbb{S}(\mathfrak{g}^*)$. Using classical invariant theory, we show that this space is isomorphic to

$$C^\infty(\overline{B})/\mathrm{Out}(\mathfrak{g}),$$

where $B \subset \mathbb{R}^{l-1}$ is a bounded open set which is invariant under a linear action of $\mathrm{Out}(\mathfrak{g})$ on $\mathbb{R}^{l-1}$ and $l = \mathrm{rank}(\mathfrak{g})$.

The space of Casimir functions is the zeroth group of the *Poisson cohomology* of (M, π), computed by the complex of multivector fields on M with differential $d_\pi := [\pi, \cdot]$

$$(\mathfrak{X}^\bullet(M), d_\pi), \quad d_\pi(W) = [\pi, W].$$

The first cohomology group $H^1_\pi(M)$ represents the infinitesimal automorphisms of π modulo those coming from Hamiltonian vector fields and $H^2_\pi(M)$ has the heuristic interpretation of being the 'tangent space' to the Poisson moduli space at π. As our result suggests, for the Lie–Poisson sphere we have an isomorphism between $\mathfrak{Casim}(\mathbb{S}(\mathfrak{g}^*), \pi_{\mathbb{S}}) \cong H^2_{\pi_{\mathbb{S}}}(\mathbb{S}(\mathfrak{g}^*))$ given by multiplication with $[\pi_{\mathbb{S}}]$ (for a proof see [Măr13, § 7.2.1]).

There are only few descriptions, in the literature, of open sets in the Poisson moduli space of a compact manifold, and we recall below two such results.

For a compact symplectic manifold (M, ω), every Poisson structure C^0-close to ω^{-1} is symplectic as well. The Moser argument shows that two symplectic structures in the same cohomology class, and which are close enough to ω, are symplectomorphic by a diffeomorphism isotopic to the identity. This implies that the map $\pi \mapsto [\pi^{-1}] \in H^2(M)$ induces a bijection between an open set in the space of all Poisson structures modulo diffeomorphisms isotopic to the identity and an open set in $H^2(M)$. Also the heuristic prognosis holds, since $H^2(M) \cong H^2_{\omega^{-1}}(M)$. In general it is difficult to say more, that is, to determine whether two symplectic structures, different in cohomology, are symplectomorphic. In Corollary 2.5 we achieve this for the maximal coadjoint orbits of a compact semisimple Lie algebra.

Radko obtains in [Rad02] a description of the moduli space of topologically stable bivectors on a compact oriented surface Σ. These are bivectors $\pi \in \mathfrak{X}^2(\Sigma)$ that intersect the zero section of $\Lambda^2 T\Sigma$ transversally, and therefore form a dense C^1-open set in $\mathfrak{X}^2(\Sigma)$. The moduli space decomposes as a union of finite-dimensional manifolds (of different dimensions), and its tangent space at π is precisely $H^2_\pi(\Sigma)$. Since Σ is two-dimensional, every bivector $\pi \in \mathfrak{X}^2(\Sigma)$ is Poisson.

The main difficulty when studying deformations of Poisson structures on compact manifolds (in contrast, for example, to complex structures) is that the Poisson complex fails to be elliptic, unless the structure is symplectic. Therefore, in general, $H^2_\pi(M)$ and the Poisson moduli space are infinite-dimensional. This is also the case for the Lie–Poisson spheres, except for $\mathfrak{g} = \mathfrak{su}(2)$. The Lie algebra $\mathfrak{su}(2)$ is special also because it is the only one for which the Lie–Poisson sphere is symplectic (thus the result follows from Moser's theorem). Moreover, it is only the one for which the Lie–Poisson sphere is an integrable Poisson manifold (in the sense of [CF04]).

The outline of the paper. In the first section we prove part (a) of the theorem. This is done by realizing Poisson structures of the form $f\pi_{\mathbb{S}}$ as Poisson submanifolds of $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$, and then using the rigidity theorem around compact Poisson submanifolds from [Măr12]. In § 2 we discuss some standard results from Lie theory and a result from [Pap86], stating that all diffeomorphisms of a maximal coadjoint orbit are represented cohomologically by Lie group automorphisms. In § 3 we complete the proof of the theorem. Using that the regular part of a Poisson structure $f\pi_{\mathbb{S}}$, for f a positive Casimir function, is a trivial foliation with leaves diffeomorphic to a maximal orbit, we show that the symplectic structure on the leaves determines f up to an outer automorphism of $\mathfrak{g}$. Section 4 contains a description of the space of Casimir functions. In the last section we work out the case of $\mathfrak{g} = \mathfrak{su}(3)$.

1. Proof of part (a) of Theorem 1

Throughout this section we assume some familiarity with the theory of Lie algebroids and Lie groupoids. For definitions and basic properties we recommend [Mac05], and, for symplectic groupoids, see also [CF04].

Throughout the paper, we fix a compact semisimple Lie algebra $(\mathfrak{g}, [\cdot\,,\cdot])$. These assumptions on $\mathfrak{g}$ are equivalent to compactness of G, the 1-connected Lie group of $\mathfrak{g}$. It is well known that 1-connectedness of G also implies that $H^2(G) = 0$ (see e.g. [DK00]). We also fix an inner product on $\mathfrak{g}$ (hence also on $\mathfrak{g}^*$), which is not only G-invariant, but also $\mathrm{Aut}(\mathfrak{g})$-invariant; for example the negative of the Killing form. Notice that a G-invariant inner product is not automatically $\mathrm{Aut}(\mathfrak{g})$-invariant. For example, let $\mathfrak{g}$ be isomorphic to the direct product $\mathfrak{g} \cong \mathfrak{k} \times \mathfrak{k}$, where $\mathfrak{k}$ is simple Lie algebra of compact type. The G-invariant inner products on $\mathfrak{g}$ are of the form $s\mathrm{pr}_1^*(\kappa) + t\mathrm{pr}_2^*(\kappa)$, for $s, t > 0$, where κ is the negative of the Killing form on $\mathfrak{k}$, but the outer automorphism of $\mathfrak{g}$ that switches the two components stabilizes only inner products for which $s = t$.

A symplectic groupoid integrating the linear Poisson structure $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$ is

$$(T^*G, \omega_{\mathrm{can}}) \rightrightarrows \mathfrak{g}^*.$$

As a Lie groupoid, T^*G is isomorphic to the action groupoid $G \ltimes \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$, hence all its s-fibers are diffeomorphic to G. Since G is compact and $H^2(G) = 0$, the Poisson manifold $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$ satisfies the conditions of [Măr12, Theorem 2], and we state below conclusion (a) of this result.

Corollary 1.1 (of Theorem 2 in [Măr12]). *Let $S \subset \mathfrak{g}^*$ be a compact Poisson submanifold. There exists an integer $p \geqslant 0$ and there exist (arbitrarily small) open neighborhoods $U \subset \mathfrak{g}^*$ of S, such that for every open set O satisfying $S \subset O \subset \overline{O} \subset U$, there exist:*

- *an open neighborhood $\mathcal{V}_O \subset \mathfrak{X}^2(U)$ of $\pi|_U$ in the compact-open C^p-topology;*
- *a function $\widetilde{\pi} \mapsto \psi_{\widetilde{\pi}}$, which associates to a Poisson structure $\widetilde{\pi} \in \mathcal{V}_O$ an embedding*

$$\psi_{\widetilde{\pi}} : \overline{O} \longrightarrow \mathfrak{g}^*,$$

such that $\psi_{\widetilde{\pi}}$ is a Poisson diffeomorphism between

$$\psi_{\widetilde{\pi}} : (O, \widetilde{\pi}|_O) \longrightarrow (\psi_{\widetilde{\pi}}(O), \pi|_{\psi_{\widetilde{\pi}}(O)}),$$

and ψ is continuous at $\widetilde{\pi}=\pi$ (with $\psi_{\pi}=\mathrm{Id}_{\overline{O}}$), with respect to the C^p-topology on the space of Poisson structures and the C^1-topology on $C^{\infty}(\overline{O},\mathfrak{g}^*)$.

We will apply this result to spheres in $\mathfrak{g}^*$. For $f\in C^{\infty}(\mathbb{S}(\mathfrak{g}^*))$, with $f>0$, consider the following embedded sphere S_f in $\mathfrak{g}^*\backslash\{0\}$,

$$S_f:=\left\{\frac{1}{f(\xi)}\xi \mid \xi\in\mathbb{S}(\mathfrak{g}^*)\right\},$$

and denote by $\varphi_f:\mathbb{S}(\mathfrak{g}^*)\to S_f$, $\varphi_f(\xi):=\xi/f(\xi)$ the map parameterizing S_f. The spheres of type S_f form a C^1-open set in the space of all (unparameterized) spheres, namely every sphere $S\subset\mathfrak{g}^*$, for which $0\notin S$ and the map

$$\mathrm{pr}:S\longrightarrow\mathbb{S}(\mathfrak{g}^*),\quad \xi\mapsto\frac{1}{|\xi|}\xi$$

is a diffeomorphism, is of the form S_f for some positive function f on $\mathbb{S}(\mathfrak{g}^*)$.

Lemma 1.2. *The sphere S_f is a Poisson submanifold if and only f is a Casimir function. In this case, the following map is a Poisson diffeomorphism*

$$\widetilde{\varphi}_f:(\mathbb{S}(\mathfrak{g}^*)\times\mathbb{R}_+,tf\pi_{\mathbb{S}})\longrightarrow(\mathfrak{g}^*\backslash\{0\},\pi_{\mathfrak{g}}),\quad (\xi,t)\mapsto\frac{1}{tf(\xi)}\xi.$$

Proof. Compact Poisson submanifolds of $\mathfrak{g}^*$ are the same as G-invariant submanifolds, and Casimir functions of $\pi_{\mathbb{S}}$ are the same as G-invariant functions on $\mathbb{S}(\mathfrak{g}^*)$. This implies the first part. The second part follows if we show that $\widetilde{\varphi}_f^*$ preserves the Lie bracket of $X,Y\in\mathfrak{g}\subset C^{\infty}(\mathfrak{g}^*)$. Using that Casimir functions go inside the bracket, and that $\mathbb{S}(\mathfrak{g}^*)$ is a Poisson submanifold, this is straightforward:

$$\widetilde{\varphi}_f^*(\{X,Y\})=\frac{1}{tf}\{X,Y\}|_{\mathbb{S}(\mathfrak{g}^*)}=tf\left\{\frac{1}{tf}X|_{\mathbb{S}(\mathfrak{g}^*)},\frac{1}{tf}Y|_{\mathbb{S}(\mathfrak{g}^*)}\right\}\Big|_{\mathbb{S}(\mathfrak{g}^*)}=tf\{\widetilde{\varphi}_f^*(X),\widetilde{\varphi}_f^*(Y)\}|_{\mathbb{S}(\mathfrak{g}^*)}.\qquad\square$$

We are now ready to prove part (a) of Theorem 1.

Proof of part (a) of Theorem 1. To every Casimir function $f>0$ we associate a C^p-open set $\mathcal{W}_f\subset\mathfrak{X}^2(\mathbb{S}(\mathfrak{g}^*))$ containing $f\pi_{\mathbb{S}}$, such that every Poisson structure in $\mathcal{W}_f$ is isomorphic to one of the form $g\pi_{\mathbb{S}}$, for $g>0$ a Casimir function, by a diffeomorphism isotopic to the identity. Then $\mathcal{W}:=\bigcup_f\mathcal{W}_f$ satisfies the conclusion.

We will apply Corollary 1.1 to the sphere S_f. Let $S_f\subset O\subset U$ be open sets as in the corollary, with $0\notin U$. Denote by $\mathcal{U}_f$ the set of functions $\chi\in C^{\infty}(\overline{O},\mathfrak{g}^*)$ satisfying $0\notin\chi(S_f)$, and for which the map

$$\mathrm{pr}\circ\chi\circ\varphi_f:\mathbb{S}(\mathfrak{g}^*)\longrightarrow\mathbb{S}(\mathfrak{g}^*)$$

is a diffeomorphism isotopic to the identity. The first condition is C^0-open and the second is C^1-open. For the inclusion $\mathrm{Id}_{\overline{O}}$ of $\overline{O}$ in $\mathfrak{g}^*$, we have that $\mathrm{pr}\circ\mathrm{Id}_{\overline{O}}\circ\varphi_f=\mathrm{Id}$, and thus $\mathcal{U}_f$ is a C^1-neighborhood of $\mathrm{Id}_{\overline{O}}$. By continuity of ψ, there exists a C^p-neighborhood $\mathcal{V}_f\subset\mathfrak{X}^2(U)$ of $\pi_{\mathfrak{g}}|_U$, such that $\psi_{\widetilde{\pi}}\in\mathcal{U}_f$, for every Poisson structure $\widetilde{\pi}$ in $\mathcal{V}_f$. We define the C^p-open set $\mathcal{W}_f$ as

$$\mathcal{W}_f:=\{W\in\mathfrak{X}^2(\mathbb{S}(\mathfrak{g}^*))\mid\widetilde{\varphi}_{f,*}(tW)|_U\in\mathcal{V}_f\}.$$

By Lemma 1.2, we have that $\widetilde{\varphi}_{f,*}(tf\pi_{\mathbb{S}})|_U=\pi_{\mathfrak{g}}|_U$, and thus $f\pi_{\mathbb{S}}\in\mathcal{W}_f$. Let $\overline{\pi}$ be a Poisson structure in $\mathcal{W}_f$. Then for $\widetilde{\pi}:=\widetilde{\varphi}_{f,*}(t\overline{\pi})|_U\in\mathcal{V}_f$, we have that $\psi_{\widetilde{\pi}}\in\mathcal{U}_f$ is a Poisson map between

$$\psi_{\widetilde{\pi}}:(O,\widetilde{\pi}|_O)\longrightarrow(U,\pi_{\mathfrak{g}}|_U).$$

By the discussion before Lemma 1.2, the condition that $\mathrm{pr} \circ \psi_{\widetilde{\pi}} \circ \varphi_f$ is a diffeomorphism, implies that $\psi_{\widetilde{\pi}}(S_f) = S_g$, for some $g > 0$. Since $(\mathbb{S}(\mathfrak{g}^*) \times \{1\}, \overline{\pi})$ is a Poisson submanifold of $(\mathbb{S}(\mathfrak{g}^*) \times \mathbb{R}_+, t\overline{\pi})$, it follows that $S_f = \widetilde{\varphi}_f(\mathbb{S}(\mathfrak{g}^*) \times \{1\})$ is a Poisson submanifold of $(O, \widetilde{\pi}|_O)$, and, since $\psi_{\widetilde{\pi}}$ is a Poisson map, we also have that $S_g = \psi_{\widetilde{\pi}}(S_f)$ is a Poisson submanifold of $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$. So, by Lemma 1.2, g is a Casimir function and

$$\varphi_g : (\mathbb{S}(\mathfrak{g}^*), g\pi_{\mathbb{S}}) \longrightarrow (S_g, \pi_{\mathfrak{g}}|_{S_g})$$

is a Poisson diffeomorphism. Therefore also the map

$$\varphi_g^{-1} \circ \psi_{\widetilde{\pi}} \circ \varphi_f : (\mathbb{S}(\mathfrak{g}^*), \overline{\pi}) \longrightarrow (\mathbb{S}(\mathfrak{g}^*), g\pi_{\mathbb{S}}),$$

is a Poisson diffeomorphism. This map is isotopic to the identity, because $\varphi_g^{-1} = \mathrm{pr}|_{S_g}$, and by construction $\mathrm{pr} \circ \psi_{\widetilde{\pi}} \circ \varphi_f$ is isotopic to the identity. □

2. Some standard Lie theoretical results

In this section we recall some results on semisimple Lie algebras, which will be used in the proof. Most of these can be found in standard textbooks like [Dix96, DK00, Kna02].

2.1 Automorphisms

The group $\mathrm{Aut}(\mathfrak{g})$, of Lie algebra automorphisms of $\mathfrak{g}$, contains the normal subgroup $\mathrm{Ad}(G)$, of inner automorphisms. Below we recall two descriptions of the group of *outer automorphisms* $\mathrm{Out}(\mathfrak{g}) := \mathrm{Aut}(\mathfrak{g})/\mathrm{Ad}(G)$.

We fix $T \subset G$ a maximal torus with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. Let $\mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ be the subgroup of $\mathrm{Aut}(\mathfrak{g})$ consisting of elements which send $\mathfrak{t}$ to itself. Since every two maximal tori are conjugated, $\mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ intersects every component of $\mathrm{Aut}(\mathfrak{g})$; hence $\mathrm{Out}(\mathfrak{g}) \cong \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})/\mathrm{Ad}(N_G(T))$, where $N_G(T)$ is the normalizer of T in G.

Denote by $\Phi \subset i\mathfrak{t}^*$ the corresponding root system, and its symmetry group by

$$\mathrm{Aut}(\Phi) := \{f \in Gl(i\mathfrak{t}^*) : f(\Phi) = \Phi\}.$$

For $\sigma \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$, we have that $(\sigma|_{\mathfrak{t}})^* \in \mathrm{Aut}(\Phi)$. This gives a group homomorphism

$$\tau : \mathrm{Aut}(\mathfrak{g}, \mathfrak{t}) \longrightarrow \mathrm{Aut}(\Phi), \quad \sigma \mapsto (\sigma^{-1}|_{\mathfrak{t}})^*.$$

Let $W \subset \mathrm{Aut}(\Phi)$ be the *Weyl group* of Φ. Knapp [Kna02, Theorem 7.8] gives the following.

Lemma 2.1. *The map $\tau : \mathrm{Aut}(\mathfrak{g}, \mathfrak{t}) \to \mathrm{Aut}(\Phi)$ is surjective, and*

$$\tau^{-1}(W) = \mathrm{Ad}(N_G(T)) \subset \mathrm{Aut}(\mathfrak{g}, \mathfrak{t}).$$

Therefore, τ induces an isomorphism between $\mathrm{Out}(\mathfrak{g}) \cong \mathrm{Aut}(\Phi)/W$.

Moreover, if $\mathfrak{c}$ is an open Weyl chamber, then $\mathrm{Aut}(\Phi) = W \rtimes \mathrm{Aut}(\Phi, \mathfrak{c})$, where

$$\mathrm{Aut}(\Phi, \mathfrak{c}) := \{f \in \mathrm{Aut}(\Phi) \mid f(\mathfrak{c}) = \mathfrak{c}\},$$

and $\mathrm{Aut}(\Phi, \mathfrak{c})$ is isomorphic to the symmetry group of the Dynkin diagram of Φ.

The last part of the lemma allows us to compute $\mathrm{Out}(\mathfrak{g})$ for all semisimple compact Lie algebras. First, it is enough to consider simple Lie algebras, since if $\mathfrak{g}$ decomposes into simple components as $n_1\mathfrak{s}_1 \oplus \cdots \oplus n_k\mathfrak{s}_k$, then

$$\mathrm{Out}(\mathfrak{g}) \cong S_{n_1} \ltimes \mathrm{Out}(\mathfrak{s}_1)^{n_1} \times \cdots \times S_{n_k} \ltimes \mathrm{Out}(\mathfrak{s}_k)^{n_k}.$$

Further, for the simple Lie algebras, a glimpse at their Dynkin diagram reveals that the only ones with nontrivial outer automorphism group are $A_{n\geqslant 2}$, $D_{n\geqslant 5}$, E_6 with $\mathrm{Out}\cong\mathbb{Z}_2$, and D_4 with $\mathrm{Out}\cong S_3$.

2.2 The coadjoint action and its symplectic orbits

We denote the adjoint action of G on $\mathfrak{g}$ by $\mathrm{Ad}_g(X)$, and the coadjoint action by $\mathrm{Ad}^{\dagger}_g(\xi):=\xi\circ\mathrm{Ad}_{g^{-1}}$.

The symplectic leaves of the Poisson manifold $(\mathfrak{g}^*,\pi_{\mathfrak{g}})$ are the coadjoint orbits. For $\xi\in\mathfrak{g}^*$, denote by (O_ξ,Ω_ξ) the symplectic leaf through ξ, by $G_\xi\subset G$ the stabilizer of ξ and by $\mathfrak{g}_\xi\subset\mathfrak{g}$ the Lie algebra of G_ξ. The pullback of Ω_ξ to G, via the map $g\mapsto\mathrm{Ad}^{\dagger}_g(\xi)$, is $d\widetilde{\xi}$, where $\widetilde{\xi}\in\Omega^1(G)$ is the left invariant extension of ξ.

The adjoint representation is isomorphic to the coadjoint representation; an isomorphism between them is induced by the Killing form. We restate here some standard results about the adjoint action in terms of the coadjoint (as a reference see [DK00, § 3.2]). We are interested especially in the set $\mathfrak{g}^*_{\mathrm{reg}}$ of *regular* elements. An element $\xi\in\mathfrak{g}^*$ is regular if and only if it satisfies any of the following equivalent conditions:

- $\mathfrak{g}_\xi$ is a maximal abelian subalgebra;
- the leaf O_ξ has maximal dimension among all leaves;
- G_ξ is a maximal torus in G.

We regard $\mathfrak{t}^*$ as a subspace of $\mathfrak{g}^*$, by identifying it with $\mathfrak{t}^*=\{\xi\in\mathfrak{g}^*\mid\mathfrak{t}\subset\mathfrak{g}_\xi\}\subset\mathfrak{g}^*$. Consider $\mathfrak{t}^*_{\mathrm{reg}}:=\mathfrak{t}^*\cap\mathfrak{g}^*_{\mathrm{reg}}$, the regular part of $\mathfrak{t}^*$. Then $\mathfrak{t}^*_{\mathrm{reg}}$ is the union of the open Weyl chambers. If $\mathfrak{c}$ is such a chamber, the global structure of $\mathfrak{g}^*_{\mathrm{reg}}$ is described by the equivariant diffeomorphism [DK00, Proposition 3.8.1]

$$\Psi: G/T\times\mathfrak{c}\xrightarrow{\sim}\mathfrak{g}^*_{\mathrm{reg}},\quad \Psi([g],\xi)=\mathrm{Ad}^{\dagger}_g(\xi). \tag{1}$$

2.3 The maximal coadjoint orbits

The manifold G/T is called a *generalized flag manifold*, and it is diffeomorphic to all maximal leaves of the linear Poisson structure on $\mathfrak{g}^*$. Their cohomology is well understood [Bor53]; we recall here the following lemma.

LEMMA 2.2. *For $\xi\in\mathfrak{t}^*$, $d\widetilde{\xi}$ is the pullback of a 2-form ω_ξ on G/T. Moreover, the assignment $\mathfrak{t}^*\ni\xi\mapsto[\omega_\xi]\in H^2(G/T)$ is a linear isomorphism.*

Proof. Since $d\widetilde{\xi}$ is the pullback of the symplectic structure on the symplectic leaf O_ξ via the map $G\to G/G_\xi\cong O_\xi$, and since $T\subset G_\xi$, the first part follows.

Viewing G as a principal T-bundle over G/T, the long exact sequence for the homotopy groups gives $\pi_1(T)\cong\pi_2(G/T)$ and $\pi_1(G/T)=0$; thus, using also the Hurewicz theorem, we obtain that the second Betti number of G/T equals $\dim(\mathfrak{t})$. So it suffices to show injectivity of the map $\xi\mapsto[\omega_\xi]$. Let $\xi\in\mathfrak{t}^*$, with $\xi\neq 0$. Then we can find an element $X\in\mathfrak{t}$ such that $\xi(X)\neq 0$ and $\exp(X)=1$. Since G is simply connected, the loop $\gamma_X(t):=\exp(tX)$ is the boundary of some disc D_X; and D_X projects to some sphere S_X in G/T. Using Stokes theorem, we obtain that

$$\int_{S_X}\omega_\xi=\int_{D_X}d\widetilde{\xi}=\int_{\gamma_X}\widetilde{\xi}=\int_0^1\xi\left(\frac{d}{ds}(\exp((s-t)X))|_{s=t}\right)dt=\xi(X)\neq 0.$$

This shows that ω_ξ is nontrivial in cohomology, and finishes the proof. □

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An element $\sigma \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ integrates to a Lie group isomorphism of G, denoted by the same symbol, which satisfies $\sigma(T) = T$. Therefore it induces a diffeomorphism $\overline{\sigma}$ of G/T. This diffeomorphism has the following property.

LEMMA 2.3. *We have that* $\overline{\sigma}^*(\omega_\xi) = \omega_{\sigma^*(\xi)}$.

Proof. Using that $l_{\sigma(g)^{-1}} \circ \sigma = \sigma \circ l_{g^{-1}}$, the following computation implies the result:

$$\sigma^*(\widetilde{\xi})(X) = \xi(dl_{\sigma(g)^{-1}} \circ d\sigma(X)) = \xi(d\sigma \circ dl_{g^{-1}}(X)) = \widetilde{\sigma^*(\xi)}(X), \quad \forall X \in T_gG. \qquad \square$$

Every diffeomorphism of G/T induces an algebra automorphism of $H^\bullet(G/T)$, and the possible outcomes are covered by the maps $\overline{\sigma}$. This follows from [Pap86, Theorem 1.2], the conclusion we state below (for a self contained exposition see [Măr13, Ch. 7]).

PROPOSITION 2.4. *For every diffeomorphism* $\varphi : G/T \to G/T$*, there exists* $\sigma \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ *such that* $\overline{\sigma} : G/T \to G/T$ *induces the same map on* $H^\bullet(G/T)$ *as* φ*, i.e.*

$$\varphi^* = \overline{\sigma}^* : H^\bullet(G/T) \longrightarrow H^\bullet(G/T).$$

Proof. By Lemma 2.1, we can choose $k := |\mathrm{Aut}(\Phi)|$ elements $\sigma_1, \ldots, \sigma_k \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$, such that $\mathrm{Aut}(\Phi) = \{(\sigma_1)|^*_{\mathfrak{t}^*}, \ldots, (\sigma_k)|^*_{\mathfrak{t}^*}\}$. By Lemma 2.3, we have that $\overline{\sigma}_i^*(\omega_\xi) = \omega_{\sigma_i^*(\xi)}$, and, since the map $\xi \mapsto [\omega_\xi]$ is an isomorphism (Lemma 2.2), it follows that $\overline{\sigma}_1^*, \ldots, \overline{\sigma}_k^*$ have different actions on $H^\bullet(G/T)$. Now, by Theorem 1.2 in [Pap86] the group of graded automorphisms of $H^\bullet(G/T, \mathbb{Z})$ is isomorphic to the group $\mathrm{Aut}(\Phi)$, so it has k elements, and this implies that the σ_i cover all the possible such automorphisms. This finishes the proof. $\square$

The following consequence will not be used in the proof of Theorem 1.

COROLLARY 2.5. *The map* $\xi \mapsto \omega_\xi$ *induces a bijection between* $\mathfrak{t}^*_{\mathrm{reg}}/\mathrm{Aut}(\Phi)$ *and an open set in the moduli space of all symplectic structures on* G/T.

Proof. First, Proposition 2.4, Lemma 2.3 and Lemma 2.1 imply that, for $\xi_1, \xi_2 \in \mathfrak{t}^*_{\mathrm{reg}}$, we have that ω_{ξ_1} and ω_{ξ_2} are symplectomorphic, if and only if $\xi_1 = f(\xi_2)$ for some $f \in \mathrm{Aut}(\Phi)$. This shows that the map $\xi \mapsto \omega_\xi$ induces a bijection

$$\Theta : \mathfrak{t}^*_{\mathrm{reg}}/\mathrm{Aut}(\Phi) \longrightarrow \mathcal{S}/\mathrm{Diff}(G/T),$$

where $\mathcal{S}$ denotes the space of all symplectic form on G/T which are symplectomorphic to one of the type ω_ξ, for some $\xi \in \mathfrak{t}^*_{\mathrm{reg}}$. The Moser argument implies that $\mathcal{S}$ is C^0-open in the space of all symplectic forms. $\square$

3. Proof of part (b) of Theorem 1

In Poisson geometric terms, $\mathfrak{g}^*_{\mathrm{reg}}$ is described as the regular part of $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$, i.e. the open subset consisting of leaves of maximal dimension. The regular part of $\mathbb{S}(\mathfrak{g}^*)$ is $\mathbb{S}(\mathfrak{g}^*)_{\mathrm{reg}} = \mathfrak{g}^*_{\mathrm{reg}} \cap \mathbb{S}(\mathfrak{g}^*)$. Let $\mathfrak{c} \subset \mathfrak{t}^*$ be an open Weyl chamber and denote by $\mathbb{S}(\mathfrak{c}) := \mathfrak{c} \cap \mathbb{S}(\mathfrak{g}^*)$. From (1) it follows that $\mathbb{S}(\mathfrak{g}^*)_{\mathrm{reg}}$ is described by the diffeomorphism

$$\Psi : G/T \times \mathbb{S}(\mathfrak{c}) \xrightarrow{\sim} \mathbb{S}(\mathfrak{g}^*)_{\mathrm{reg}}, \quad \Psi([g], \xi) := \mathrm{Ad}^\dagger_g(\xi),$$

and the symplectic leaves correspond to the slices $(G/T \times \{\xi\}, \omega_\xi)$, $\xi \in \mathfrak{c}$.

Proof of part (b) of Theorem 1. Let $\phi : (\mathbb{S}(\mathfrak{g}^*), f\pi_{\mathbb{S}}) \longrightarrow (\mathbb{S}(\mathfrak{g}^*), g\pi_{\mathbb{S}})$ be a Poisson diffeomorphism, where f, g are positive Casimir functions. Now, the symplectic leaves of $f\pi_{\mathbb{S}}$ and $g\pi_{\mathbb{S}}$ are also

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the coadjoint orbits O_ξ, for $\xi \in \mathbb{S}(\mathfrak{g}^*)$, but with symplectic structures $1/f(\xi)\omega_\xi$, respectively $1/g(\xi)\omega_\xi$. In particular, they have the same regular part $\mathbb{S}(\mathfrak{g}^*)_{\mathrm{reg}}$. So, after conjugating with Ψ, the Poisson diffeomorphism on the regular parts takes the form

$$\Psi^{-1}\circ\phi\circ\Psi : (G/T\times\mathbb{S}(\mathfrak{c}),\Psi^*(f\pi_{\mathbb{S}})) \xrightarrow{\sim} (G/T\times\mathbb{S}(\mathfrak{c}),\Psi^*(g\pi_{\mathbb{S}})),$$
$$(x,\xi)\mapsto(\phi_\xi(x),\theta(\xi)),$$

for a diffeomorphism $\theta : \mathbb{S}(\mathfrak{c}) \xrightarrow{\sim} \mathbb{S}(\mathfrak{c})$ and a symplectomorphism

$$\phi_\xi : (G/T, \omega_{\xi/f(\xi)}) \xrightarrow{\sim} (G/T, \omega_{\theta(\xi)/g(\theta(\xi))}).$$

Since $\mathbb{S}(\mathfrak{c})$ is connected it follows that the maps ϕ_ξ for $\xi\in\mathbb{S}(\mathfrak{c})$ are isotopic to each other, so they induce the same map on $H^2(G/T)$ and, by Proposition 2.4, this map is also induced by a diffeomorphism $\overline{\sigma}$ corresponding to some $\sigma\in\mathrm{Aut}(\mathfrak{g},\mathfrak{t})$. Lemma 2.3 implies the following equality in $H^2(G/T)$ for all $\xi\in\mathbb{S}(\mathfrak{c})$:

$$[\omega_{\xi/f(\xi)}] = [\phi_\xi^*(\omega_{\theta(\xi)/g(\theta(\xi))})] = [\overline{\sigma}^*(\omega_{\theta(\xi)/g(\theta(\xi))})] = [\omega_{\sigma^*(\theta(\xi)/g(\theta(\xi)))}].$$

Using Lemma 2.2 we obtain that $\xi/f(\xi) = \sigma^*(\theta(\xi))/g(\theta(\xi))$. Since σ^* preserves the norm, we get that $f(\xi) = g(\theta(\xi))$. This shows that $\xi = \sigma^*(\theta(\xi))$, so σ^* preserves $\mathbb{S}(\mathfrak{c})$ and, on this space, $\theta = (\sigma^{-1})^*$. So $f\circ\sigma^*(\xi) = g(\xi)$ for all $\xi\in\mathbb{S}(\mathfrak{c})$. Since the regular leaves are dense and all hit $\mathbb{S}(\mathfrak{c})$, and, since both $f\circ\sigma^*$ and g are Casimir functions, it follows that $f\circ\sigma^* = g$. □

4. The space of Casimir functions

By the main theorem, the map which associates to $F\in\mathfrak{Casim}(\mathbb{S}(\mathfrak{g}^*),\pi_{\mathbb{S}})$ the Poisson structure $e^F\pi_{\mathbb{S}}$ on $\mathbb{S}(\mathfrak{g}^*)$ induces a parameterization of an open set in the Poisson moduli space of $\mathbb{S}(\mathfrak{g}^*)$ around $\pi_{\mathbb{S}}$ by the space

$$\mathfrak{Casim}(\mathbb{S}(\mathfrak{g}^*),\pi_{\mathbb{S}})/\mathrm{Out}(\mathfrak{g}).$$

In this section we describe this space using classical invariant theory.

Let $P[\mathfrak{g}^*]$ and $P[\mathfrak{t}^*]$ denote the algebras of polynomials on $\mathfrak{g}^*$ and $\mathfrak{t}^*$ respectively. A classical result (see e.g. [Dix96, Theorem 7.3.5]) states that the restriction map $P[\mathfrak{g}^*]\to P[\mathfrak{t}^*]$ induces an isomorphism between the algebras of invariants

$$P[\mathfrak{g}^*]^G \cong P[\mathfrak{t}^*]^W. \tag{2}$$

A theorem of Schwarz [Sch75] extends this result to the smooth setting

$$C^\infty(\mathfrak{g}^*)^G \cong C^\infty(\mathfrak{t}^*)^W. \tag{3}$$

To explain this, first recall that the algebra $P[\mathfrak{g}^*]^G$ is generated by $l := \dim(\mathfrak{t})$ algebraically independent homogeneous polynomials $p_1,\ldots,p_l$ (Theorem 7.3.8 [Dix96]). Hence, by (2), $P[\mathfrak{t}^*]^W$ is generated by $q_1 := p_1|_{\mathfrak{t}^*},\ldots,q_l := p_l|_{\mathfrak{t}^*}$. Consider the maps

$$p = (p_1,\ldots,p_l) : \mathfrak{g}^* \longrightarrow \mathbb{R}^l \quad\text{and}\quad q = (q_1,\ldots,q_l) : \mathfrak{t}^* \longrightarrow \mathbb{R}^l,$$

and let $\Delta := p(\mathfrak{g}^*)$. Since the inclusion $\mathfrak{t}^*\subset\mathfrak{g}^*$ induces a bijection between the W-orbits and the G-orbits, it follows that $q(\mathfrak{t}^*) = \Delta$. The theorem of Schwarz [Sch75] applied to the action of G on $\mathfrak{g}^*$ and to the action of W on $\mathfrak{t}^*$ shows that the pullbacks by p and q give isomorphisms between

$$C^\infty(\mathfrak{g}^*)^G \cong C^\infty(\Delta) \quad\text{and}\quad C^\infty(\mathfrak{t}^*)^W \cong C^\infty(\Delta) \tag{4}$$

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(hence we obtain (3)). Schwarz's result asserts that p, respectively q, induce homeomorphisms between the orbit spaces and Δ,

$$\mathfrak{g}^*/G \cong \Delta \cong \mathfrak{t}^*/W. \tag{5}$$

We can describe the orbit space also using an open Weyl chamber $\mathfrak{c} \subset \mathfrak{t}^*$.

LEMMA 4.1. *The map $q : \overline{\mathfrak{c}} \to \Delta$ is a homeomorphism and restricts to a diffeomorphism between the interiors $q : \mathfrak{c} \to \mathrm{int}(\Delta)$.*

Proof. It is well known that $\overline{\mathfrak{c}}$ intersects each orbit of W exactly once (see e.g. [DK00]) and so, by (5), the map is a bijection. Since $q : t^* \to \mathbb{R}^l$ is proper, it follows that also $q|_{\overline{c}}$ is proper, and this implies the first part. We are left to check that $q|_{\mathfrak{c}}$ is an immersion. Let $V \in T_\xi\mathfrak{c}$ be a nonzero vector. Consider χ a smooth, compactly supported function on $\mathfrak{c}$ satisfying $d\chi_\xi(V) \neq 0$. Since the action gives a homeomorphism $W \times \mathfrak{c} \cong \mathfrak{t}^*_{\mathrm{reg}}$, χ has a unique W-invariant extension to $\mathfrak{t}^*$, which is defined on $w\mathfrak{c}$ by $\widetilde{\chi} = w^*(\chi)$, and extended by zero on $\mathfrak{t}^*\backslash\mathfrak{t}^*_{\mathrm{reg}}$. Then, by (4), $\widetilde{\chi}$ is of the form $\widetilde{\chi} = h \circ q$, for some $h \in C^\infty(\Delta)$. Differentiating in the direction of V, we obtain that $d_\xi q(V) \neq 0$, and this finishes the proof. □

The polynomials $p_1, \ldots, p_l$ are not unique; a necessary and sufficient condition for a set of homogeneous polynomials to be such a generating system is that their image in I/I^2 forms a basis, where $I \subset P[\mathfrak{g}^*]^G$ denotes the ideal of polynomials vanishing at 0. Since I^2 is $\mathrm{Out}(\mathfrak{g})$ invariant, it is easy to see that we can choose $p_1, \ldots, p_l$ such that $p_1(\xi) = |\xi|^2$ and the linear span of $p_2, \ldots, p_l$ is $\mathrm{Out}(\mathfrak{g})$ invariant. This choice endows $\mathbb{R}^l$ with a linear action of $\mathrm{Out}(\mathfrak{g})$, for which p is $\mathrm{Aut}(\mathfrak{g})$ equivariant. Moreover, the action is trivial on the first component and $\{0\} \times \mathbb{R}^{l-1}$ is invariant. The isomorphism $\mathrm{Aut}(\Phi)/W \cong \mathrm{Out}(\mathfrak{g})$ from Lemma 2.1 shows that also q is equivariant with respect to the actions of $\mathrm{Aut}(\Phi)$ and $\mathrm{Out}(\mathfrak{g})$. Thus we have isomorphisms between the spaces

$$C^\infty(\mathfrak{g}^*)^G/\mathrm{Aut}(\mathfrak{g}) \cong C^\infty(\mathfrak{t}^*)^W/\mathrm{Aut}(\Phi) \cong C^\infty(\Delta)/\mathrm{Out}(\mathfrak{g}).$$

Notice that every Casimir function f on $\mathbb{S}(\mathfrak{g}^*)$ can be extended to a G-invariant smooth function on $\mathfrak{g}^*$, and therefore

$$\mathfrak{Casim}(\mathbb{S}(\mathfrak{g}^*), \pi_{\mathbb{S}}) \cong C^\infty(\mathfrak{g}^*)^G|_{\mathbb{S}(\mathfrak{g}^*)}.$$

Since $p_1(\xi) = |\xi|^2$, it follows that $p(\mathbb{S}(\mathfrak{g}^*)) = (\{1\} \times \mathbb{R}^{l-1}) \cap \Delta$. Writing $p' := (p_2, \ldots, p_l) : \mathfrak{g}^* \to \mathbb{R}^{l-1}$ and $\Delta' := p'(\mathbb{S}(\mathfrak{g}^*))$, we have that $C^\infty(\Delta') = C^\infty(\Delta)|_{\{1\}\times\Delta'}$. Lemma 4.1 implies that $q' := p'|_{\mathbb{S}(\mathfrak{t}^*)}$ is a homeomorphism between $\mathbb{S}(\overline{\mathfrak{c}}) \cong \Delta'$, which restricts to a diffeomorphism between $\mathbb{S}(\mathfrak{c}) \cong \mathrm{int}(\Delta')$. This shows that $\Delta' = \overline{B}$, where B is a bounded open set, diffeomorphic to an open ball.

With these, we have the following description of the Casimir functions.

COROLLARY 4.2. *The polynomial map $p' : \mathfrak{g}^* \to \mathbb{R}^{l-1}$ is equivariant with respect to the actions of $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Out}(\mathfrak{g})$, and $q' := p'|_{\mathfrak{t}^*}$ is equivariant with respect to the actions of $\mathrm{Aut}(\Phi)$ and $\mathrm{Out}(\mathfrak{g})$. These maps induce isomorphisms between*

$$\mathfrak{Casim}(\mathbb{S}(\mathfrak{g}^*), \pi_{\mathbb{S}})/\mathrm{Out}(\mathfrak{g}) \cong C^\infty(\mathbb{S}(\mathfrak{t}^*))^W/\mathrm{Out}(\mathfrak{g}) \cong C^\infty(\Delta')/\mathrm{Out}(\mathfrak{g}),$$

and $\mathrm{Out}(\mathfrak{g})$-equivariant homeomorphisms between the spaces

$$\mathbb{S}(\mathfrak{g}^*)/G \cong \mathbb{S}(\mathfrak{t}^*)/W \cong \Delta'.$$

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5. The case of $\mathfrak{su}(3)$

In this section, we describe our result for the Lie algebra $\mathfrak{g} = \mathfrak{su}(3)$, whose 1-connected Lie group is $G = \mathbf{SU}(3)$. Recall that

$$\mathfrak{su}(3) = \{A \in M_3(\mathbb{C}) \mid A + A^* = 0, \operatorname{tr}(A) = 0\},$$
$$\mathbf{SU}(3) = \{U \in M_3(\mathbb{C}) \mid UU^* = I, \det(U) = 1\}.$$

We use the invariant inner product given by the negative of the trace form $(A, B) := -\operatorname{tr}(AB)$. Let $\mathfrak{t}$ be the space of diagonal matrices in $\mathfrak{su}(3)$

$$\mathfrak{t} := \left\{ D(ix_1, ix_2, ix_3) := \begin{pmatrix} ix_1 & 0 & 0 \\ 0 & ix_2 & 0 \\ 0 & 0 & ix_3 \end{pmatrix} \middle| \, x_j \in \mathbb{R}, \sum_j x_j = 0 \right\}.$$

The corresponding maximal torus is

$$T := \left\{ D(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \mid \theta_j \in \mathbb{R}, \prod_j e^{i\theta_j} = 1 \right\}.$$

The Weyl group is $W = S_3$. It acts on $\mathfrak{t}$ as follows:

$$\sigma D(ix_1, ix_2, ix_3) = D(ix_{\sigma(1)}, ix_{\sigma(2)}, ix_{\sigma(3)}), \quad \sigma \in S_3.$$

The Dynkin diagram of $\mathfrak{su}(3)$ is A_2 (a graph with one edge), so its symmetry group is $\mathbb{Z}_2$. A generator of $\operatorname{Out}(\mathfrak{su}(3))$ is complex conjugation

$$\gamma \in \operatorname{Aut}(\mathfrak{su}(3), \mathfrak{t}), \quad \gamma(A) = \overline{A}.$$

On $\mathfrak{t}$, γ acts by multiplication with -1.

Under the identification of $\mathfrak{t} \cong \mathfrak{t}^*$ given by the inner product, the invariant polynomials $P[\mathfrak{t}]^{S_3}$ are generated by the symmetric polynomials

$$q_1(D(ix_1, ix_2, ix_3)) = x_1^2 + x_2^2 + x_3^2, \quad q_2(D(ix_1, ix_2, ix_3)) = \sqrt{6}(x_1^3 + x_2^3 + x_3^3).$$

Identifying also $\mathfrak{su}(3) \cong \mathfrak{su}^*(3)$, q_1 and q_2 are the restriction to $\mathfrak{t}$ of the invariant polynomials p_1, $p_2 \in P[\mathfrak{su}^*(3)]^{\mathbf{SU}(3)}$ (which generate $P[\mathfrak{su}^*(3)]^{\mathbf{SU}(3)}$),

$$p_1(A) = -\operatorname{tr}(A^2), \quad p_2(A) = i\sqrt{6}\operatorname{tr}(A^3).$$

Clearly $p_2 \circ \gamma = -p_2$. The inner product on $\mathfrak{t}$ is

$$(D(ix_1, ix_2, ix_3), D(ix_1', ix_2', ix_3')) = x_1x_1' + x_2x_2' + x_3x_3',$$

and we have that $\mathbb{S}(\mathfrak{t}^*) \cong \mathbb{S}(\mathfrak{t})$ is a circle, isometrically parameterized by

$$A(\theta) := \frac{\cos(\theta)}{\sqrt{2}} D(i, -i, 0) + \frac{\sin(\theta)}{\sqrt{6}} D(i, i, -2i), \quad \theta \in [0, 2\pi].$$

In polar coordinates on $\mathfrak{t}$, the polynomials q_1 and q_2 become

$$q_1(rA(\theta)) = r^2, \quad q_2(rA(\theta)) = r^3 \sin(3\theta).$$

This implies that the space Δ is given by

$$\Delta = \{(r^2, r^3 \sin(3\theta)) \mid r \geqslant 0, \theta \in [0, 2\pi]\} = \{(x, y) \in \mathbb{R}^2 \mid x^3 \geqslant y^2\}.$$

The map $q := (q_1, q_2) : \mathfrak{t} \to \mathbb{R}^2$, restricted to the open Weyl chamber

$$\mathfrak{c} := \{rA(\theta) \mid r > 0, \theta \in (-\pi/6, \pi/6)\},$$

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is a diffeomorphism onto $\mathrm{int}(\Delta)$. The linear action of $\mathbb{Z}_2 = \mathrm{Out}(\mathfrak{su}(3))$ on $\mathbb{R}^2$, for which q is equivariant, is multiplication by -1 on the second component. Therefore $q' := q_2$ is a $\mathbb{Z}_2$-equivariant homeomorphism between $\mathbb{S}(\bar{\mathfrak{c}})$ and Δ':

$$q' : \mathbb{S}(\bar{\mathfrak{c}}) = \{A(\theta) \mid \theta \in [-\pi/6, \pi/6]\} \xrightarrow{\sim} \Delta' := [-1, 1],$$

which restricts to a diffeomorphism between the interiors.

We conclude that the Poisson moduli space of the seven-dimensional sphere $\mathbb{S}(\mathfrak{su}(3)^*)$ is parameterized around $\pi_{\mathbb{S}}$ by the space

$$C^\infty([-1, 1])/\mathbb{Z}_2,$$

where $\mathbb{Z}_2$ acts on $C^\infty([-1, 1])$ by the involution

$$\gamma(f)(x) = f(-x), \quad f \in C^\infty([-1, 1]).$$

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