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MULTIPLIERS FOR WEIGHTED HARDY SPACES ON LOCALLY COMPACT VILENKIN GROUPS

TOSHIYUKI KITADA

Let G be a locally compact Vilenkin group. We study multipliers which satisfy a generalised Hörmander condition from power-weighted Hardy space $H^p_\beta(G)$ to $H^q_{\beta'}(G)$ with $0 < p \leqslant q < \infty$, $0 < p \leqslant 1$, $-1 < \beta, \beta'$.

1. INTRODUCTION AND PRELIMINARY RESULTS

In [5] Kurtz gave weighted norm inequalities for kernel operators which map an $L^p(\mathbf{R}^n)$ space into an $L^q(\mathbf{R}^n)$ space with $1 < p < q < \infty$. Applying them to multiplier operators which satisfy a generalised Hörmander multiplier condition, he obtained a multiplier theorem between weighted $L^p(\mathbf{R}^n)$ spaces and weighted $L^q(\mathbf{R}^n)$ spaces. In [11] Vinogradova considered a multiplier condition which is stronger than that of Kurtz, and gave a multiplier theorem from weighted $L^p(\mathbf{R}^n)$ space to weighted $L^p(\mathbf{R}^n)$ space with different power-weights.

In this note we consider the case $0 < p \leqslant 1$, $p \leqslant q < \infty$ under the setting of the locally compact Vilenkin groups G, instead of $\mathbf{R}^n$. Let H^p_β $(0 < p < \infty,\ \beta > -1)$ be a power-weighted Hardy space on G. We give a sufficient condition for a function φ on Γ (the dual group of G) to be a multiplier from H^p_β to $H^q_{\beta'}$, $0 < p \leqslant 1$, $p \leqslant q < \infty$. Our main result is Theorem 2, which is showed by combining multiplier theorems on H^p_β $(0 < p < \infty,\ \beta > -1)$ of the present author [2, 4 and 3] with a weighted norm inequality for the fractional integral operator on G (Theorem 1).

Throughout this note G will denote a locally compact Vilenkin group, that is to say, G is a locally compact abelian topological group containing a strictly decreasing sequence of compact open subgroups $(G_n)_{-\infty}^{\infty}$ such that

(i) $\bigcup_{-\infty}^{\infty} G_n = G$ and $\bigcap_{-\infty}^{\infty} G_n = \{0\}$.

(ii) $\sup\{\text{ order } (G_n/G_{n+1}) : n \in \mathbf{Z}\} := B < \infty$.

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Examples of such groups are described in [**1**, Section 4.1.2]. Additional examples are the additive group of a local field (see [**10**]).

Let Γ be the dual group of G and let Γ_n be the annihilator of G_n for each $n \in \mathbf{Z}$. Then $(\Gamma_n)_{-\infty}^{\infty}$ is a strictly increasing sequence of compact open subgroups of Γ such that (i) $\bigcup_{-\infty}^{\infty} \Gamma_n = \Gamma$ and $\bigcap_{-\infty}^{\infty} \Gamma_n = \{1\}$, and (ii) order $(\Gamma_{n+1}/\Gamma_n) =$ order (G_n/G_{n+1}). We choose Haar measures dx on G and $d\gamma$ on Γ so that $|G_0| = |\Gamma_0| = 1$, where $|A|$ denotes the Haar measure of a measurable subset A of G, or Γ. Then $|G_n|^{-1} = |\Gamma_n| := m_n$ for each $n \in \mathbf{Z}$. For $x \in G$, we set $|x| = (m_n)^{-1}$ if $x \in G_n \setminus G_{n+1}$ and $|x| = 0$ if $x = 0$. Similarly, we set $|\gamma| = m_{n+1}$ if $\gamma \in \Gamma_{n+1} \setminus \Gamma_n$ and $|\gamma| = 0$ if $\gamma = 1$. Since $2m_n \leqslant m_{n+1}$ for each $n \in \mathbf{Z}$, it follows that $\sum_{n=k}^{\infty} (m_n)^{-\alpha} \leqslant C(m_k)^{-\alpha}$ and $\sum_{n=-\infty}^{k} (m_n)^{\alpha} \leqslant C(m_k)^{\alpha}$ for any $\alpha > 0$, $k \in \mathbf{Z}$.

The symbols $^\wedge$ and $^\vee$ will denote the Fourier transform and inverse Fourier transform, respectively. We have $(\xi_{G_n})^\wedge = |\Gamma_n|^{-1} \xi_{\Gamma_n} := F_n$ and, hence, $(\xi_{\Gamma_n})^\vee = |G_n|^{-1} \xi_{G_n} := \Delta_n$ for each $n \in \mathbf{Z}$, where ξ_A denote the indicator function of a set A.

The Lebesgue space on G with respect to the weight measure $|x|^\alpha dx$ will be denoted by $L^p_\alpha(G)$ or L^p_α, $0 < p < \infty$, $\alpha \in \mathbf{R}$, and we set $\|f\|_{p,\alpha} = \left(\int_G |f(x)|^p |x|^\alpha dx\right)^{1/p}$. When $\alpha = 0$, we write L^p and $\|f\|_p$ instead of L^p_0 and $\|f\|_{p,0}$, respectively. We set $|A|_\alpha = \int_A |x|^\alpha dx$ (hence, $|A|_0 = |A|$).

Following Taibleson's development of a distribution theory on local fields [**10**], we define $\mathcal{S}(G)$ or $\mathcal{S}$ to be the set of all functions φ on G such that φ has compact support and is constant on the cosets of some G_n, $n \in \mathbf{Z}$. A sequence $(\varphi_n)_1^\infty$ in $\mathcal{S}(G)$ converges to φ in $\mathcal{S}(G)$ if there are integers r, s so that each φ_n and φ are constant on the cosets of G_s and are supported on G_r and $(\varphi_n)_1^\infty$ tends to φ uniformly on G. The set of all continuous linear functionals on $\mathcal{S}(G)$ will be denoted by $\mathcal{S}'(G)$ or $\mathcal{S}'$. A sequence $(f_n)_1^\infty$ in $\mathcal{S}'(G)$ converges to f in $\mathcal{S}'(G)$ if for all $\varphi \in \mathcal{S}(G)$ we have $\lim_{n\to\infty}\langle f_n, \varphi\rangle = \langle f, \varphi\rangle$.

Similarly, $\mathcal{S}(\Gamma)$ and $\mathcal{S}'(\Gamma)$ are defined. For more details, see [**10**].

For $f \in \mathcal{S}'$ we define its maximal function f^* by $f^*(x) = \sup_n |f * \Delta_n(x)|$. The power-weighted Hardy spaces $H^p_\alpha := H^p_\alpha(G)$ are defined as the space of all $f \in \mathcal{S}'$ for which $\|f\|_{H^p_\alpha} := \|f^*\|_{p,\alpha} < \infty$, where $0 < p < \infty$, $\alpha \in \mathbf{R}$.

Let $0 < p < \infty$ and $\alpha > -1$. A function a on G is called a $(p,\infty)_\alpha$ atom if there exists an interval(coset) $I := x_0 + G_n$ such that (i) supp $a \subset I$, (ii) $\|a\|_\infty \leqslant |I|_\alpha^{-1/p}$, and (iii) $\int_G a(x)dx = 0$. The atomic characterisation of H^p_α spaces are given as follows, see [**6**, Theorem 3.5], [**3**, Theorem 3.2].

Lemma 1. *Let $0 < p \leqslant 1$ and $-1 < \alpha \leqslant 0$. Then $f \in H^p_\alpha$ if and only if*

$f = \sum_{i=1}^{\infty} \lambda_i a_i$ in S', where each $\lambda_i > 0$, a_i is a $(p,\infty)_\alpha$ atom and $\sum_{i=1}^{\infty} \lambda_i^p < \infty$. Furthermore, $\|f\|_{H_\alpha^p} \sim \inf\{(\sum \lambda_i^p)^{1/p}; f = \sum \lambda_i a_i\}$.

LEMMA 2. *Let $1 < p < \infty$ and $-1 < \alpha < p-1$. Then $H_\alpha^p \cong L_\alpha^p$.*

PROOF: One direction of $L_\alpha^p \subset H_\alpha^p$ follows from Hardy-Littlewood maximal inequality. The other direction is seen by a routine argument. We omit the details. ∎

LEMMA 3. *Let $0 < p \leqslant 1$ and $-1 < \alpha \leqslant 0$. Then $S_0 := \{f \in S, \int_G f(x)dx = 0\}$ is dense in H_α^p.*

PROOF: Let $f \in H_\alpha^p$ and $\varepsilon > 0$. Then, by Lemma 1, there is a function g, which is a finite linear combination of $(p,\infty)_\alpha$ atoms, such that $\|f-g\|_{H_\alpha^p}^p < \varepsilon$. Since supp g is compact and $\int g(x)dx = 0$, it is easily seen that $g * \Delta_n \in S_0$ for all $n \in \mathbf{Z}$. So if we show that $(g - g*\Delta_n)^* \to 0$ in L_α^p as $n \to \infty$, we have $\|f - g * \Delta_n\|_{H_\alpha^p}^p \leqslant \|f-g\|_{H_\alpha^p}^p + \|g - g*\Delta_n\|_{H_\alpha^p}^p < 2\varepsilon$ for large enough n, and this completes the proof of the lemma.

Since $g \in L^1, g * \Delta_n(x) \to g(x)$, for almost all x as $n \to \infty$. Therefore we have

$$\begin{aligned}(g - g*\Delta_n)^*(x) &= \sup_{m\in\mathbf{Z}} |(g - g*\Delta_n)*\Delta_m(x)| \\ &= \sup_{m>n} |g*\Delta_m(x) - g*\Delta_n(x)| \to 0 (n\to\infty),\end{aligned}$$

for almost all x. Since $(g - g*\Delta_n)^* \leqslant 2g^*$ and $g* \in L_\alpha^p$, the Lebesgue dominated convergence theorem implies that $(g - g*\Delta_n)^* \to 0$ in L_α^p. ∎

LEMMA 4. *Let $\alpha > 0$, $0 < p, q < \infty$ and $\beta, \beta' > -1$. Then there is a constant $C > 0$ such that*

$$|I|^\alpha \, |I|_{\beta'}^{1/q} \leqslant C\, |I|_\beta^{1/p} \text{ for any interval } I,$$

if and only if

$$\frac{\beta}{p} - \frac{\beta'}{q} = -\frac{1}{p} + \frac{1}{q} + \alpha \geqslant 0.$$

PROOF: For $\beta > -1$, it is easy to see that $|I|_\beta \sim (m_n)^{-\beta-1}$ if $I = G_n$, $n \in \mathbf{Z}$ and $|I|_\beta = (m_\ell)^{-\beta}(m_n)^{-1}$ if $I = x + G_n$, $x \in G_\ell \setminus G_{\ell+1}$, $\ell < n$. The proof of the lemma follows from this fact at once. ∎

2. FRACTIONAL INTEGRALS AND MULTIPLIERS

The fractional integral operator I_α on G is defined by $(I_\alpha f)^\wedge(\gamma) = |\gamma|^{-\alpha} \widehat{f}(\gamma)$, $f \in S_0$, $\alpha > 0$ (see [10, 7]). We set $k_\alpha(x) = |x|^{\alpha-1}$ for $\alpha \neq 1$, and $k_1(x) = \log|x|$. Then, unlike the case $\mathbf{R}^n$, $\widehat{k_\alpha}(\gamma)$ is not a constant times $|\gamma|^{-\alpha}$ in general.

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Lemma 5. *Let $\alpha > 0$. Then, in the sense of distributions, $\widehat{k_\alpha}$ is a radial function on Γ and $\widehat{k_\alpha}(\gamma) \sim |\gamma|^{-\alpha}$, that is, there exist constants $C_1, C_2 > 0$ such that*

$$C_2 |\gamma|^{-\alpha} \leqslant \left|\widehat{k_\alpha}(\gamma)\right| \leqslant C_1 |\gamma|^{-\alpha} \quad \text{for } \gamma \in \Gamma.$$

Proof: Consider first $\alpha \neq 1$. Since $|x|^{\alpha-1}$ is locally integrable, we have, for each $\psi \in \mathcal{S}(\Gamma)$,

$$\begin{aligned}\langle \widehat{k_\alpha}, \psi \rangle &= \langle |x|^{\alpha-1}, \psi^\vee \rangle \\ &= \sum_{n=s}^{\infty} (m_n)^{1-\alpha} \int_G \xi_{G_n \setminus G_{n+1}}(x) \psi^\vee(x) dx \\ &= \sum_{n=s}^{\infty} (m_n)^{1-\alpha} \int_\Gamma (F_n - F_{n+1})(\gamma) \psi(\gamma) d\gamma,\end{aligned}$$

where $s \in \mathbf{Z}$ is an integer such that ψ is constant on each cosets of Γ_s, but not on a coset of Γ_{s+1} in Γ. We set $F = \sum_{n=-\infty}^{\infty} (m_n)^{1-\alpha}(F_n - F_{n+1})$ and define

$$\langle F, \psi \rangle := \sum_{n=s}^{\infty} (m_n)^{1-\alpha} \int_\Gamma (F_n - F_{n+1})(\gamma) \psi(\gamma) d\gamma.$$

Then $\widehat{k_\alpha} = F$ in $\mathcal{S}'$. And if $\gamma \in \Gamma_{\ell+1} \setminus \Gamma_\ell$, $\ell \in \mathbf{Z}$, then

$$\begin{aligned} F(\gamma) &= \sum_{n=\ell+1}^{\infty} (m_n)^{1-\alpha}(F_n - F_{n+1})(\gamma) - (m_\ell)^{1-\alpha} F_{\ell+1}(\gamma) \\ &= \sum_{n=\ell}^{\infty} \left((m_{n+1})^{1-\alpha} - (m_n)^{1-\alpha} \right) F_{n+1}(\gamma) \\ &= \sum_{n=\ell}^{\infty} \frac{(m_{n+1})^{1-\alpha} - (m_n)^{1-\alpha}}{m_{n+1}} \\ &= |\gamma|^{-\alpha} \sum_{n=\ell}^{\infty} (m_{\ell+1})^{\alpha} \frac{(m_{n+1})^{1-\alpha} - (m_n)^{1-\alpha}}{m_{n+1}} \\ &= |\gamma|^{-\alpha} C_{\ell,\alpha}, \text{ say },\end{aligned}$$

where the second equality follows from the fact that $(m_n)^{-\alpha} \to 0$ $(n \to \infty)$. It is easy to see that

$$\frac{1 - 2^{\alpha-1}}{1 - B^{-\alpha}} \leqslant C_{\ell,\alpha} \leqslant \frac{1 - B^{\alpha-1}}{1 - 2^{-\alpha}}, \quad \text{if } \alpha < 1$$

and

$$\frac{1 - B^{\alpha-1}}{1 - 2^{-\alpha}} \leqslant C_{\ell,\alpha} \leqslant \frac{1 - 2^{\alpha-1}}{1 - B^{-\alpha}}, \quad \text{if } \alpha > 1.$$

When $\alpha = 1$, a similar argument for $k_1(x) = \log |x|$ holds and we have the conclusion of lemma. ∎

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REMARK 1. When $m_n = p^n$, $n \in \mathbf{Z}$ ($p \geqslant 2$ is a prime integer), we have $\widehat{k_\alpha}(\gamma) = \left((1-p^{\alpha-1})/(1-p^{-\alpha})\right)|\gamma|^{-\alpha}$, $\alpha \neq 1$ and $\widehat{k_1}(\gamma) = \left((\log p)/(1-p^{-1})\right)|\gamma|^{-1}$.

We consider the (generalised fractional integral) operator T_α as follows:

DEFINITION 1: Let $\alpha > 0$ and $\tau(\gamma)$ be a radial function on Γ such that $\tau(\gamma) \sim |\gamma|^{-\alpha}$. We define the operator T_α by $(T_\alpha f)^\wedge(\gamma) = \tau(\gamma)\widehat{f}(\gamma)$, $f \in \mathcal{S}_0$. We set $\tau_n := \tau(\gamma)$, $\gamma \in \Gamma_{n+1} \setminus \Gamma_n$, for each $n \in \mathbf{Z}$.

Note that if $\alpha > 0$ and $f \in \mathcal{S}_0$ then $T_\alpha f \in \mathcal{S}_0$. If $0 < \alpha < 1$ and $f \in \mathcal{S}$ then $T_\alpha f$ is well defined and locally integrable. For either case, we have

$$T_\alpha f = \sum_{n=-\infty}^{\infty} \tau_n(\Delta_{n+1} - \Delta_n) * f.$$

In what follows, we assume that $0 < p \leqslant 1$ and $-1 < \beta \leqslant 0$.

THEOREM 1. *Let $\alpha > 0$, $0 < p \leqslant q < \infty$ and $\beta' > -1$. Then the following conditions are equivalent:*

$$\|T_\alpha f\|_{H^q_{\beta'}} \leqslant C\,\|f\|_{H^p_\beta} \quad \text{for all } f \in \mathcal{S}_0, \tag{1}$$

$$\frac{\beta+1}{p} = \frac{\beta'+1}{q} + \alpha \quad \text{and } 0 \leqslant \frac{1}{p} - \frac{1}{q} \leqslant \alpha. \tag{2}$$

This theorem is similar to Theorem(1.5) in [8] for the fractional integral operator on $\mathbf{R}^n$. Since our weights are power-weights, a necessary and sufficient condition for the inequality (1) is given precisely as (2). By Lemma 3, the inequality (1) has a continuous extension to all of H^p_β.

PROOF: For simplicity of notation, we write T for T_α.
(1) $\Rightarrow$ (2): For any interval $I := x_0 + G_{n_0}$, $x_0 \in G$, $n_0 \in \mathbf{Z}$, we define $a \in \mathcal{S}_0$ by

$$a(x) = (B+1)^{-1}\,|I|\,|I|_\beta^{-1/p}\,(\Delta_{n_0+1} - \Delta_{n_0})(x - x_0).$$

Then a is a $(p,\infty)_\beta$ atom and $\|a\|_{H^p_\beta} \leqslant 1$. And for $x \in I$,

$$\begin{aligned} |Ta(x)| &= \left|\sum_{n=-\infty}^{\infty} \tau_n(\Delta_{n+1} - \Delta_n) * a(x)\right| \\ &= |\tau_{n_0} a(x)| \geqslant C(m_{n_0})^{-\alpha}\,|I|_\beta^{-1/p} \\ &= C\,|I|^\alpha\,|I|_\beta^{-1/p}. \end{aligned}$$

Since $(Ta)^* \geqslant |Ta|$ on I, we have

$$\begin{aligned} 1 \geqslant \|a\|_{H^p_\beta} &\geqslant C\,\|Ta\|_{H^q_{\beta'}} = C\,\|(Ta)^*\|_{q,\beta'} \\ &\geqslant C\,\|Ta\|_{q,\beta'} \geqslant C\,|I|^\alpha\,|I|_\beta^{-1/p}\,|I|_{\beta'}^{1/q}. \end{aligned}$$

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Hence, Lemma 4 implies (2).
(2) $\Rightarrow$ (1): We first show that for $(p,\infty)_\alpha$ atom a, Ta is a $(q,\infty)_{\beta'}$ atom up to a constant which is independent of a. Let a be a $(p,\infty)_\beta$ atom such that supp $a \subset I := x_0 + G_{n_0}$, $x_0 \in G$, $n_0 \in \mathbf{Z}$. If $x \notin I$, then $x - x_0 \in G_\ell \setminus G_{\ell+1}$ for some $\ell \in \mathbf{Z}$, $\ell < n_0$. Then $(x+G_n) \cap I = \emptyset$ for $n > \ell$, and $x + G_n \supset I$ for $n \leqslant \ell$. So $\Delta_n * a(x) = 0$ for all $n \in \mathbf{Z}$. This shows $Ta(x) = 0$. Hence, supp $Ta \subset I$.

Let $x \in I$. If $n < n_0$, then $x + G_{n+1} \supset I$ and $\Delta_{n+1} * a(x) = 0$. Hence, by Lemma 4,

$$\begin{aligned} |Ta(x)| &\leqslant \sum_{n=n_0}^{\infty} |\tau_n(\Delta_{n+1} - \Delta_n) * a(x)| \\ &\leqslant C \sum_{n=n_0}^{\infty} (m_n)^{-\alpha} \|a\|_\infty \leqslant C (m_{n_0})^{-\alpha} \|a\|_\infty \\ &\leqslant C\,|I|^\alpha\,|I|_\beta^{-1/p} \leqslant C\,|I|_{\beta'}^{-1/q}. \end{aligned}$$

The cancellation property of Ta follows from that of a. Therefore Ta is a $(q,\infty)_{\beta'}$ atom up to a constant such that

$$\|Ta\|_{H^q_{\beta'}} \leqslant C, \tag{3}$$

where C is independent of a (we note that under the condition (2), $\beta' \leqslant q(1/p - \alpha) - 1 \leqslant 0$, so by Lemma 2, $\|Ta\|_{H^q_{\beta'}} \sim \|Ta\|_{q,\beta'}$, if $q > 1$).

The inequality (3) also holds for the modified operator T^N ($N \in \mathbf{Z}$) defined by $(T^N f)^\wedge = \tau \xi_{\Gamma \setminus \Gamma_N} \widehat{f}$. This is checked easily and we emphasise that the constant C in (3) for T^N is the same as the one in (3) for T.

Let us go on to prove (1). We consider the case $q \leqslant 1$ and $q > 1$ separately. In either case, for $f \in \mathcal{S}_0$, let $f(x) = \sum_{i=1}^{\infty} \lambda_i a_i$ be a possible atomic decomposition of f (as an element of H^p_β). Since $f \in \mathcal{S}_0$, there is an $N \in \mathbf{Z}$ such that $\widehat{f} = 0$ on Γ_N. Then we have $Tf = T^N f$.

If $q \leqslant 1$, then it follows from (3) that

$$\left\| \sum_{i=1}^{\infty} \lambda_i T^N a_i \right\|_{H^q_{\beta'}} \leqslant C \left(\sum_{i=1}^{\infty} \lambda_i^q \right)^{1/q} \leqslant C \left(\sum_{i=1}^{\infty} \lambda_i^p \right)^{1/p},$$

since $p \leqslant q$. This means that $\sum_{i=1}^{\infty} \lambda_i T^N a_i$ converges in $H^q_{\beta'}$. Hence, for any $\psi \in \mathcal{S}(\Gamma)$, we have

$$\left\langle \left(\sum_{i=1}^{\infty} \lambda_i T^N a_i \right)^\wedge, \psi \right\rangle = \left\langle \sum_{i=1}^{\infty} \lambda_i \widehat{a}_i \tau \xi_{\Gamma \setminus \Gamma_N}, \psi \right\rangle = \left\langle \sum_{i=1}^{\infty} \lambda_i \widehat{a}_i, \tau \xi_{\Gamma \setminus \Gamma_N} \psi \right\rangle$$

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because $\tau\xi_{\Gamma\backslash\Gamma_N}$ is locally constant on Γ,

$$= \left\langle \left(\sum_{i=1}^{\infty} \lambda_i a_i \right)^{\wedge}, \tau\xi_{\Gamma\backslash\Gamma_N}\psi \right\rangle = \langle \tau\xi_{\Gamma\backslash\Gamma_N}\widehat{f}, \psi \rangle$$
$$= \langle (T^N f)^{\wedge}, \psi \rangle = \langle \widehat{Tf}, \psi \rangle.$$

Therefore we have $Tf = \sum_{i=1}^{\infty} \lambda_i T^N a_i$ and

$$\|Tf\|_{H^q_{\beta'}} \leqslant C \left(\sum_{i=1}^{\infty} \lambda_i^p \right)^{1/p}.$$

By taking the infimum on the right hand side above, we have the inequality (1).

If $q > 1$, then by using Minkowsky's inequality,

$$\left(\int_G \left| \sum_{i=1}^{\infty} \lambda_i T^N a_i(x) \right|^q |x|^{\beta'} \, dx \right)^{1/q} \leqslant \sum_{i=1}^{\infty} \lambda_i \left(\int_G |T^N a_i(x)|^q \, |x|^{\beta'} \, dx \right)^{1/q}$$
$$\leqslant C \sum_{i=1}^{\infty} \lambda_i \leqslant C \left(\sum_{i=1}^{\infty} \lambda_i^p \right)^{1/p},$$

because $p \leqslant 1$. Hence, $\sum_{i=1}^{\infty} \lambda_i T^N a_i$ converges in $L^q_{\beta'}$. The remainder of the proof is the same as the case $q \leqslant 1$. This completes the proof of theorem. $\square$

REMARK 2. Compared with the proof of Theorem(1.5) in [**9**], our proof of Theorem 1 is simple as above. It is due to the fact that $\mathcal{S}_0(G) = \{f \in \mathcal{S}(G),\ \mathrm{supp}\, \widehat{f} \not\ni 1\}$. In [**9**] Strömberg and Wheeden also deal with the case $p > 1$, and obtain [**9**, Theorem (1.1)]. For the groups G, by using other methods as in [**8**], we can get the following result:

Let $1 < p \leqslant q < \infty$, $0 < \alpha < 1$ and $-1 < \beta < p - 1$, $-1 < \beta'$. Then

$$\|T_\alpha f\|_{q,\beta'} \leqslant C \, \|f\|_{p,\beta} \qquad \text{for all } f \in \mathcal{S}$$

if and only if

$$\frac{\beta + 1}{p} = \frac{\beta' + 1}{q} + \alpha \quad \text{ and } 0 \leqslant \frac{1}{p} - \frac{1}{q} \leqslant \alpha.$$

Before stating Theorem 2, we need to introduce a generalised Hörmander class of multipliers space, $M(s, \lambda, \alpha)$ (see [**5**]).

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DEFINITION 2: Let $\lambda > 0$, $1 \leqslant s \leqslant \infty$ and $\alpha \in \mathbf{R}$. For a function φ on Γ, we set $\varphi_j := \varphi\xi_{\Gamma_{j+1}\setminus\Gamma_j}$, $j \in \mathbf{Z}$. A function φ on Γ belongs to $M(s,\lambda,\alpha)$ if there is a constant C such that

$$|\varphi(\gamma)| \leqslant C\,|\gamma|^{-\alpha} \text{ and } \sup_{j\in\mathbf{Z}}\left\{(m_j)^{\lambda-1/s+\alpha}\left\|D^\lambda\varphi_j\right\|_s\right\} < \infty,$$

where $D^\lambda\varphi_j := \left(|x|^\lambda(\varphi_j)^\vee\right)^\wedge$.

$M(s,\lambda,0)$ is $M(s,\lambda)$ introduced in [**2**, and **3**]. Notice that if we let $\varphi(\gamma) = \psi(\gamma)\,|\gamma|^{-\alpha}$, then $\varphi \in M(s,\lambda,\alpha)$ if and only if $\psi \in M(s,\lambda)$. Also, $|\gamma|^{-\alpha} \in M(s,\lambda,\alpha)$ for all $\lambda > 0$ and $1 \leqslant s \leqslant \infty$.

THEOREM 2. *Let $\alpha > 0$ and $0 \leqslant 1/p - 1/q \leqslant \alpha$. Suppose that $\varphi \in M(s,\lambda,\alpha)$ for $1 \leqslant s \leqslant \infty$, $\lambda > \max(1,1/q) - 1/\max(2,s')$. Then*

$$\left\|\left(\varphi\hat{f}\right)^\vee\right\|_{H^q_{\beta'}} \leqslant C\,\|f\|_{H^p_\beta} \text{ for all } f \in \mathcal{S}_0, \tag{4}$$

if $-1 < \beta \leqslant 0$, $\max(-1,-q\lambda) < \beta'$ and

$$\frac{\beta+1}{p} = \frac{\beta'+1}{q} + \alpha.$$

PROOF: Let $\psi(\gamma) := \varphi(\gamma)\,|\gamma|^\alpha$ and $f \in \mathcal{S}_0$. Then $\psi \in M(s,\lambda)$.

If $q \leqslant 1$, then, by Theorem 4.5 in [3] and Theorem 1, we have

$$\begin{aligned}\left\|\left(\varphi\hat{f}\right)^\vee\right\|_{H^q_{\beta'}} &= \left\|\left(\psi(I_\alpha f)^\wedge\right)^\vee\right\|_{H^q_{\beta'}}\\ &\leqslant C\,\|I_\alpha f\|_{H^q_{\beta'}}\\ &\leqslant C\,\|f\|_{H^p_\beta}.\end{aligned}$$

If $q > 1$, then, by Theorem 1 in [2] or Theorem(3.6) in [4] and Theorem 1, we have,

$$\left\|\left(\varphi\hat{f}\right)^\vee\right\|_{H^q_{\beta'}} \leqslant C\,\|I_\alpha f\|_{H^q_{\beta'}} \leqslant C\,\|f\|_{H^p_\beta}.$$

This completes the proof of theorem. □

By Lemma 3, the inequality (4) in Theorem 2 has a continuous extension to all of H^p_β. When $0 < \alpha < 1$, we can prove Theorem 2 directly by the method as in the proof of [**3**, Theorem 4.4 and Theorem 4.5].

For the case $p > 1$, we can also get a similar result to Theorem 2 by the same idea as in the proof above (see Remark 2). This will appear elsewhere.

REFERENCES

[1] R.E. Edwards and G.I. Gaudry, *Littlewood-Paley and multiplier theory* (Springer Verlag, Berlin, Heidelberg, New York, 1977).

[2] T. Kitada, 'Weighted L^p multipliers on locally compact Vilenkin groups', *Sci. Rep. Hirosaki Univ.* **36** (1989), 135–143.

[3] T. Kitada, 'Weighted H^p multipliers on locally compact Vilenkin groups', *Monatsh. Math.* **110** (1990), 283–295.

[4] T. Kitada and C.W. Onneweer, 'Hörmander-type multipliers on locally compact Vilenkin groups', in *Theory and applications of Gibbs derivatives* (Institute of Mathematica, Belgrade, 1989).

[5] D.S. Kurtz, 'Sharp function estimates for fractional integrals and related operators', *J. Austral. Math. Soc. Ser. A* **49** (1990), 129–137.

[6] C.W. Onneweer and T.S. Quek, 'Multipliers on weighted Hardy spaces over locally compact Vilenkin groups, I', *J. Austral. Math. Soc. Ser. A* **48** (1990), 472–496.

[7] L. Saloff-Coste, 'Opérateurs pseudo diifférentiels sur certains groupes totalement discontinus', *Studia Math.* **83** (1986), 205–228.

[8] E.T. Sawyer and R.L. Wheeden, 'Weighted inequalities for fractional integrals on Euclidian spaces and homogeneous spaces'. (Preprint 1989).

[9] J-O. Strömberg and R.L. Wheeden, 'Fractional integrals on weighted H^p and L^p spaces', *Trans. Amer. Math. Soc.* **287** (1985), 293–321.

[10] M.H. Taibleson, *Fourier analysis on local fields*, Mathematical Notes 15 (Princeton University Press, 1975).

[11] G.A. Vinogradova, 'On multipliers in weighted spaces', *Anal. Math.* **16** (1990), 215–226.

Department of Mathematics
Faculty of General Education
Hirosaki University
Hirosaki 036
Japan