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A DIRECT ELLIPSOID METHOD FOR LINEAR PROGRAMMING

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This paper indicates how to apply the ellipsoid method directly to Linear Programming problems and proves that this kind of version of the ellipsoid method is almost as good as Karmarkar's type method in the theoretical sense.

0. INTRODUCTION

As several authors have noted, the ellipsoid method is in some sense equivalent to Karmarkar's method [**6, 7**]. Todd and Ye designed the ellipsoid according to the Karmarkar's potential function and proved that the ratio of the volume reduction of the ellipsoids is equal to the ratio of the potential function reduction. In this note, we apply the ellipsoid method directly to Karmarkar's canonical problem and get a nicer complexity bound than the traditional ellipsoid method. This shows, in another way, the relationship between the ellipsoid method and Karmarkar's method.

1. KARMARKAR'S PROBLEM AND ITS EQUIVALENCE

We shall consider the following problem

$(\overline{P})$ $$\min\{\overline{c}^T\overline{x} \mid \overline{x} \in \overline{R}\}$$

where $$\overline{R} = \{\overline{x} \mid \overline{A}\overline{x} = 0, \overline{e}^T\overline{x} = n, \overline{x} \geqslant 0\}$$

ASSUMPTIONS.

(1) $\overline{c}^T\overline{x}$ is not a constant on $\overline{R}$.
(2) $\overline{A} \in R^{m\times n}$ and $rank(\overline{A}) = m$.
(3) $\overline{A}\overline{e} = 0$ (where $\overline{e}$ denotes an n-dimension vector of 1's).
(4) $\overline{A}$ is an integer matrix.

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Now we suppose that z_1, z_2, ..., z_{n-m-1} are the basic solutions of

$$\overline{A}\overline{x} = 0$$
$$\overline{e}^T\overline{x} = 0$$

and A_1, A_2, ..., A_{n-m-1} are their normalised orthogonal vectors. Let $A = (A_1, A_2, \ldots, A_{n-m-1}) = (a_1, a_2, \ldots, a_n)^T$, $c = A^T\overline{c}$; then the linear mapping

$$\overline{x} := \overline{e} + Ax$$

transforms $(\overline{P})$ into

(P) $$\min\{c^T x \mid x \in R\}$$

where $R = \{x \mid Ax \geqslant -\overline{e}\} = \{x \mid a_1^T x \geqslant -1, a_2^T x \geqslant -1, \ldots, a_n^T x \geqslant -1\}$.

It is not difficult to see that to transform $(\overline{P})$ into (P) needs at most $(n+1) \times n \times m$ arithmetic operations. Generally speaking, to transform a standard linear programming into Karmarkar's canonical form needs to combine primal and dual problems (see [**2**]). So the reduced problem (P) has fewer variables than the problem $(\overline{P})$. The following method is actually designed for solving (P).

2. Direct Ellipsoid Method For Problem (P)

In the following, we use I and $\overline{c}_i$ to denote the $(n-m-1) \times (n-m-1)$ identity matrix and the ith component of $\overline{c}$.

Algorithm.

Step 1. Set $x_1 = 0$, $Q_1 = n(n-1)I$, $k = 1$;

Step 2. set $d = \begin{cases} -a_i, & \text{if there exists an } i \text{ such that } a_i^T x_k < -1, \\ c, & \text{otherwise;} \end{cases}$

Step 3. set $x_{k+1} = x_k - 1/(n-m)\left(Q_k d/\sqrt{d^T Q_k d}\right)$,

$$Q_{k+1} = \frac{(n-m-1)^2}{(n-m-1)^2 - 1}\left(Q_k - \frac{2}{n-m}\frac{(Q_k d)(Q_k d)^T}{d^T Q_k d}\right)$$

set $k = k+1$ and return to Step 2.

3. Main Theorem and Lemmas

Main Theorem. *Let $\{x_k\}$ be generated by the algorithm, l^* be the optimal* *value of (P); then*

$$\min\{c^T x_t - l^* \mid x_t \in R, t \leqslant k\} \leqslant n(n-1)\left(\max_i \bar{c}_i - \min_i \bar{c}_i\right) \exp\left(-\frac{k}{2(n-m)^2}\right).$$

So as $$k \geqslant 2(n-m)^2 L(\overline{A}) + 2[ln(n) + ln\left(\max_i \bar{c}_i - \min_i \bar{c}_i\right)](n-m)^2$$

we have $$\min\{c^T x_t - l^* \mid x_t \in R, t \leqslant k\} \leqslant 2^{-L(\overline{A})}.$$

In order to prove this theorem, we need the following notions and lemmas. Given an integer $q \neq 0$, we need one cell for the sign and $\lceil \log_2(|q|+1) \rceil$ cells for the $\{0,1\}$-string in order to represent its absolute value in binary. For "0"or "1" we need only one cell. So to encode an integer q we need

$$L(q) := 1 + \lceil \log_2(|q|+1) \rceil$$

cells. We call $L(q)$ the encoding length of q. Hence the encoding length of a rational number $r = s/t$ is $L(r) = L(s) + L(t)$.

LEMMA 1.

(a) *For every rational number* r, $|r| \leqslant 2^{L(r)} - 1$.
(b) *For every rational vector* $x \in R^n$, $\|x\| \leqslant \|x\|_1 \leqslant 2^{(L(x)-n)} - 1$.
(c) *For every rational matrix* $D \in R^{n\times n}$, $|\det(D)| \leqslant 2^{(L(D)-n^2)} - 1$.

PROOF: See [3].

LEMMA 2. *Without loss of generality we assume that* $\bar{c}_1 = \min_i \bar{c}_i$, $\bar{c}_n = \max_i \bar{c}_i$.

(1) $n\bar{c}_1 \leqslant \bar{c}^T\bar{x} \leqslant n\bar{c}_n$ *for* $\bar{x} \in \overline{R}$.
(2) $l_1 := (n-1)\bar{c}_1 - \sum_{i=2}^{n} \bar{c}_i \leqslant c^T x \leqslant l_2 := (n-1)\bar{c}_n - \sum_{i=1}^{n-1} \bar{c}_i$ *for* $x \in R$.
(3) $R_1 = \{x \mid \|x\|^2 \leqslant n/(n-1)\} \subset R \subset R_2 = \{x \mid \|x\|^2 \leqslant n(n-1)\}$.

PROOF: The proof is very direct, so we omit it here.

LEMMA 3. *Let* $E_k = \{x \mid (x-x_k)^T Q_k^{-1}(x-x_k) \leqslant 1\}$; *then*

$$\frac{vol(E_{k+1})}{vol(E_k)} \leqslant q_{n-m} = \exp\left(-\frac{1}{2(n-m)}\right).$$

PROOF: See [3].

LEMMA 4. (Brünn-Minkowski)
Suppose C_1, $C_2 \subset R^n$ *are two compact convex subsets; then*

$$[vol(\theta C_1 + (1-\theta)C_2)]^{1/n} \geqslant \theta[vol(C_1)]^{1/n} + (1-\theta)[vol(C_2)]^{1/n}.$$

PROOF: See [4, Theorem 46].

Now let us prove the main theorem. For $k \geqslant 1$, denote

$$\varepsilon_k = \min\{c^T x_j - l^* \mid j \leqslant k \text{ and } x_j \in R\}.$$

Since $x_1 \in R$, ε_k is well defined.

For $x_j \in R$, denote

$$H_j = \{x \mid x \in R, c^T x \leqslant c^T x_j\}$$
$$\varphi(t) = \{x \mid x \in R, c^T x \leqslant t\}.$$

It follows that, $l^* + \varepsilon_k \leqslant c^T x_j$. So we have

(1) $$\varphi(l^* + \varepsilon_k) \subset R \cap H_j \text{ for any } j \leqslant k \text{ with } x_j \in R.$$

Denote

(2) $$U_{k+1} = R \cap_{1 \leqslant j \leqslant k} H_j.$$

By the construction of the ellipsoids we know that, for any $k \geqslant 1$, the following conclusions hold:

(3) $$E_k \cap \{x \mid c^T x \leqslant c^T x_k\} \subset E_{k+1}, \text{ if } x_k \in R;$$

(4) $$E_k \cap \{x \mid a_i^T x \geqslant a_i^T x_k\} \subset E_{k+1}, \text{ if } x_k \notin R.$$

We assert that

(5) $$U_{k+1} \subset E_{k+1} \text{ for } k \geqslant 1.$$

In fact, for $k = 1$, since $x_1 \in R$ implies $E_1 \cap \{x \mid c^T x \leqslant c^T x_1\} \subset E_2$ and $R \subset E_1$, it follows that

$$U_2 = R \cap H_1 \subset E_1 \cap H_1 \subset E_2.$$

Now we assume that for $k = h - 1$ the conclusion is true, that is,

(6) $$U_h \subset E_h.$$

If $x_h \in R$, by (3) and (6) we have

$$U_{h+1} = U_h \cap H_h \subset E_h \cap H_h \subset E_{h+1}.$$

If $x_h \notin R$, then it is clear that $R \subset \{x \mid a_i^T x \geqslant a_i^T x_h\}$. By (4) and (6) we get

$$U_{h+1} = U_h \subset E_h \cap \{x \mid a_i^T x \geqslant a_i^T x_h\} \subset E_{h+1},$$

and the induction is finished.

By (1), (2) and (5) we get

$$\varphi(l^* + \varepsilon_k) \subset U_{k+1} \subset E_{k+1}. \tag{7}$$

From Lemma 3

$$\begin{aligned}[\mathrm{vol}(\varphi(l^* + \varepsilon_k))]^{1/(n-m-1)} &\leqslant [\mathrm{vol}(E_{k+1})]^{1/(n-m-1)} \\ &\leqslant (q_{n-m})^{k/(n-m-1)}[\mathrm{vol}(E_1)]^{1/(n-m-1)}.\end{aligned} \tag{8}$$

On the other hand, for any $x \in R$, by Lemma 2 we have $l^* \leqslant l^* + \varepsilon_k \leqslant l_2$. Hence there exists a $\lambda_k \in [0,1]$ such that

$$l^* + \varepsilon_k = \lambda_k l_2 + (1 - \lambda_k) l^*.$$

Furthermore,

$$(1 - \lambda_k)\varphi(l^*) + \lambda_k \varphi(l_2) \sqsubseteq \varphi(l^* + \varepsilon_k),$$

and the assumption (1) implies $\mathrm{vol}(\varphi(l^*)) = 0$ and $R = \varphi(l_2)$. From Lemma 4 it follows that

$$\begin{aligned}[\mathrm{vol}(\varphi(l^* + \varepsilon_k))]^{1/(n-m-1)} &\geqslant (1 - \lambda_k)[\mathrm{vol}(\varphi(l^*))]^{1/(n-m-1)} + \lambda_k[\mathrm{vol}(\varphi(l_2))]^{1/(n-m-1)} \\ &= \lambda_k[\mathrm{vol}(\varphi(l_2))]^{1/(n-m-1)} \\ &= \lambda_k[\mathrm{vol}(R)]^{1/(n-m-1)}.\end{aligned}$$

Therefore

$$\begin{aligned}\varepsilon_k &= \lambda_k(l_2 - l^*) \\ &\leqslant \left[\frac{\mathrm{vol}(\varphi(l^* + \varepsilon_k))}{\mathrm{vol}(R)}\right]^{1/(n-m-1)} (l_2 - l^*) \\ &\leqslant \left[\frac{\mathrm{vol}(E_{k+1})}{\mathrm{vol}(R)}\right]^{1/(n-m-1)} (l_2 - l^*) \\ &\leqslant (q_{n-m})^{k/(n-m-1)} \left[\frac{\mathrm{vol}(R_2)}{\mathrm{vol}(R_1)}\right]^{1/(n-m-1)} (l_2 - l_1) \\ &\leqslant exp\left(-\frac{k}{2(n-m)^2}\right)(n-1)(l_2 - l_1) \\ &= exp\left(-\frac{k}{2(n-m)^2}\right) n(n-1)(\bar{c}_n - \bar{c}_1),\end{aligned}$$

and the proof is completed. $\square$

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