TOPOLOGICAL ASPECTS OF SUITABLE THEORIES

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Roughly speaking a suitable theory is a theory T together with its formal provability predicate Prv (.). A pseudo-topological space is a boolean algebra B which carries a derivative operation d and its associated closure operation c. Thus we can pretend that B is a topological space. We show that the Lindenbaum algebra $B(T)$ of a suitable theory becomes, in a natural way, a pseudo-topological space, and hence we can translate properties of T into topological language, as properties of $B(T)$. We do this translation for several properties of T, including (1) satisfying Gödel's first theorem, (2) satisfying Löb's theorem and (3) asserting one's own inconsistency. These correspond to the topological properties (1) having an isolated point, (2) being scattered, (3) being discrete.

In Section 1 we define and discuss the relevant properties of suitable theories. In Section 2 we discuss pseudo-topological spaces, and in Section 3 we look at the various topological analogues of properties of suitable theories. Finally, in Section 4 we give various other remarks and open problems.

This paper is a partial continuation of (**1**), in particular we follow up the final paragraph of (**1**). We assume some familiarity, but not a detailed knowledge of (**1**).

1. Suitable theories

Let L be some fixed first-order language. We use σ, σ_1, σ_2 as variables over the set of L-sentences. We are concerned with certain pairs $\langle T, P\rangle$, where T is an L-theory and P is a function taking L-sentences σ to L-sentences $P(\sigma)$. (A theory is a consistent, deductively closed set of sentences.) We say two pairs $\langle T_1, P_1\rangle$, $\langle T_2, P_2\rangle$ are equivalent if $T_1 = T_2 = T$ (say) and $T \vdash P_1(\sigma) \leftrightarrow P_2(\sigma)$ holds for each sentence σ. It will not be necessary to distinguish between equivalent pairs since any fact we use (prove) about one pair will be true of all other pairs equivalent to that pair.

Consider the following three properties (which $\langle T, P\rangle$ may or may not have).

$$\text{(ADQ)} \quad (\forall\sigma)[T \vdash \sigma \Rightarrow T \vdash P(\sigma)]$$

$$\text{(SND)} \quad (\forall\sigma_1, \sigma_2)[T \cup \{P(\sigma_1 \rightarrow \sigma_2)\} \vdash P(\sigma_1) \rightarrow P(\sigma_2)]$$

$$\text{(PN)} \quad (\forall\sigma)[P(\sigma) \text{ is } \langle T, P\rangle\text{-nice}]$$

A sentence ν is $\langle T, P\rangle$-nice if $T \vdash \nu \rightarrow P(\nu)$. We make the following definition.

Definition. A suitable theory is a pair $\langle T, P\rangle$ satisfying (ADQ), (SND), (PN).

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The standard examples of suitable theories are those where T is a number theory and P is obtained from a formal provability predicate. A further discussion of the motivations for this definition occurs in Section 1 of (1), however, the following remarks should be noted.

In (1) it is assumed that P is obtained from a certain formula Prv, and names $\ulcorner \sigma \urcorner$ for each sentence σ, by putting $P(\sigma) = \text{Prv}(\ulcorner \sigma \urcorner)$. Nevertheless, only the function properties of $\text{Prv}(\ulcorner . \urcorner)$ are used (i.e. the results proved in (1) also hold for the present definition of suitable). The definition given here also covers the case where names $\ulcorner \sigma \urcorner$ for sentences do not exist but formulas $D_\sigma(v)$ defining them (i.e. $D_\sigma(v)$ means "$v = \ulcorner \sigma \urcorner$") do exist. In this case we can put

$$P(\sigma) = (\forall v)[D_\sigma(v) \rightarrow \text{Prv}(\,.\,)].$$

Finally note that the definition of (1) contains a clause (CNS) concerning the formal consistency statement CON(T). Most of the results of (1) and all those given here do not use (CNS), so we have dropped this clause.

We feel that the present definition is slightly better than the previous one, and since the results of (1) hold for the present definition (modulo the (CNS) clause) we may quote certain of these results.

There are certain properties of suitable theories which we will use all the time (often without saying so). These are Lemmas 1.1 and 1.2 of (1) and the following lemma. This lemma, which was pointed out to me by Angus Macintyre, we overlooked in (1), where it would have slightly simplified some of the arguments.

Lemma 1. *For each suitable theory $\langle T, P \rangle$ and sentences σ_1, σ_2,*

$$T \vdash (P(\sigma_1) \wedge P(\sigma_2)) \leftrightarrow P(\sigma_1 \wedge \sigma_2).$$

Proof. Use the tautologies $\sigma_1 \rightarrow (\sigma_2 \rightarrow \sigma_1 \wedge \sigma_2)$, $\sigma_1 \wedge \sigma_2 \rightarrow \sigma_1$, $\sigma_1 \wedge \sigma_2 \rightarrow \sigma_2$.

Let $\perp$ be some fixed refutable sentence, i.e. $T \vdash \neg\perp$ for each theory T.

Although the following lemma is not strictly relevant here, it is worth noting and its corollary will motivate some of our results.

Lemma 2. *Let $\langle T, P \rangle$ be a suitable theory. Then $\langle T, P \rangle$ has* (CNS) (of (1)) *if and only if $T \vdash \text{CON(T)} \rightarrow \neg P(\perp)$, and $\langle T, P \rangle$ has* (SNC) (of (1)) *if and only if $T \vdash \neg P(\perp) \rightarrow \text{CON(T)}$.*

Corollary. *If $\langle T, P \rangle$ has* (CNS, SNC) *then $T \vdash \neg P(\perp) \leftrightarrow \text{CON(T)}$.*

The sentence $P(\perp)$ will occur a lot, it is worth remembering this corollary whenever it does.

As in (1), given a suitable theory $\langle T, P \rangle$ we put, for each sentence σ,

$$\sigma^+ = P(\sigma) \rightarrow \sigma,$$

$$\sigma^- = P(\neg\sigma) \rightarrow \sigma.$$

The following theorem (which is essentially Theorem 2.1 of (**1**)) shows the significance of σ^+. The significance of σ^- will become clear in Section 3.

Theorem 1. *Let $\langle T, P\rangle$ be a suitable theory and σ a fixed sentence. There is a sentence δ such that*

$$T \vdash \delta \leftrightarrow (P(\delta) \rightarrow \sigma)$$

if and only if

$$T \vdash P(\sigma^+) \rightarrow P(\sigma).$$

Also if such a δ does exist then $T \vdash \delta \leftrightarrow \sigma^+$.

This theorem is a local version of Gödel's diagonalisation lemma. As an example of its use let us consider Gödel's first theorem.

Let $\langle T, P\rangle$ be a suitable theory. If $\langle T, P\rangle$ satisfies certain other conditions then we can use the diagonalisation lemma to obtain a sentence γ such that

$$T \vdash \gamma \leftrightarrow \neg P(\gamma).$$

We can then show that not $T \vdash \gamma$, or, more precisely,

$$T \vdash \gamma \Rightarrow T \vdash \bot.$$

Now $\neg P(\gamma)$ and $P(\gamma) \rightarrow \bot$ are logically equivalent so the theorem shows that (for a general suitable theory) such a sentence γ exists if and only if

$$T \vdash P(\bot^+) \rightarrow P(\bot)$$

and when γ does exist

$$T \vdash \gamma \leftrightarrow \bot^+.$$

(Notice that $\bot^+$ is $P(\bot) \rightarrow \bot$, i.e. $\neg P(\bot)$.) Thus we have

$$\text{(G)} \quad T \vdash \bot^+ \Rightarrow T \vdash \bot$$

as a version of Gödel's first theorem. The formalisation of this, namely

$$\text{(FG)} \quad T \vdash P(\bot^+) \rightarrow P(\bot)$$

is a necessary and sufficient condition for the existence of a sentence γ.

This gives us two properties, (G), (FG), which a suitable theory may or may not have. (These are not the same as the (G), (FG) of (**1**), although they are related. The (G), (FG) of (**1**) are concerned with Gödel's second theorem.)

Two other properties are Löb's property

$$\text{(L)} \quad (\forall \sigma)[T \vdash \sigma^+ \Rightarrow T \vdash \sigma]$$

and its formalisation

$$\text{(FL)} \quad (\forall \sigma)[T \vdash P(\sigma^+) \rightarrow P(\sigma)].$$

To show that certain suitable theories have (L) Löb first used the diagonalisation lemma to obtain a sentence δ (as in the theorem) and then argued as follows. Clearly $T \cup \{P(\delta)\} \vdash \delta \rightarrow \sigma$ so that (by Lemma 1.2 (iii) of (**1**)) $T \vdash P(\delta) \rightarrow P(\sigma)$. Hence if $T \vdash \sigma^+$ then $T \vdash P(\delta) \rightarrow \sigma$, i.e. $T \vdash \delta$. Thus $T \vdash P(\delta)$, and so $T \vdash \sigma$. However given the theorem, we can restate this argument as (FL)$\Rightarrow$(L). (This implication was proved in (**1**).)

Informally we say a theory is strange if it asserts its own inconsistency. Remembering the Corollary of Lemma 2 this gives us a fifth property.

$$\text{(S)} \quad T \vdash P(\bot).$$

We note the following.

Theorem 2. *The following implications hold.*

$$\text{(S)} \Rightarrow \text{(FL)} \Leftrightarrow \text{(L)} \Rightarrow \text{(FG)} \Rightarrow \text{(G)}.$$

Proof. For each sentence σ the sentence $\bot \rightarrow \sigma$ is a tautology and so (for each suitable theory $\langle T, P \rangle$) $T \vdash P(\bot) \rightarrow P(\sigma)$. Thus $(S) \Rightarrow$(FL). We noted above that (FL)$\Rightarrow$(L), and similarly (FG)$\Rightarrow$(G). Clearly we have (FL)$\Rightarrow$(FG) so it is sufficient to show (L)$\Rightarrow$(FL). This implication follows from the next lemma.

Lemma 3. *For each suitable theory $\langle T, P \rangle$ and sentence σ, $T \vdash (P(\sigma^+) \rightarrow P(\sigma))^+$.*

Proof. Let $\tau = P(\sigma^+) \rightarrow P(\sigma)$ so that

$$T \cup \{P(\tau)\} \vdash P^2(\sigma^+) \rightarrow P^2(\sigma)$$

and

$$T \cup \{P(\sigma^+)\} \vdash P^2(\sigma) \rightarrow P(\sigma).$$

But $T \vdash P(\sigma^+) \rightarrow P^2(\sigma^+)$ so that

$$T \cup \{P(\tau), P(\sigma^+)\} \vdash P(\sigma)$$

which gives $T \vdash \tau^+$, as required.

2. Pseudo topological spaces

A pseudo topological space is a boolean algebra $B = \langle B, \wedge, \vee, -, \leqq, 0, 1 \rangle$ (we assume that $0 \neq 1$) which carries an operation d satisfying the following.

$$(0) \quad d(0) = 0.$$

$$(1) \quad (\forall x, y \in B)[d(x \vee y) = d(x) \vee d(y)].$$

$$(2) \quad (\forall x \in B)[d^2(x) \leqq d(x)].$$

Such an operation is called a derivative operation. A closure operation is an operation c such that (0, 1, 2) hold (with d replaced by c) as well as the following.

$$(3) \quad (\forall x \in B)[x \leqq c(x)].$$

The following lemma is easily proved.

Lemma 4. *Each pseudo topological space $\langle B, d \rangle$ carries an associated closure operation c given by $c(x) = x \wedge d(x)$ (for each $x \in B$).*

Let U be a set carrying a topology T, let $B = P(U)$ with the obvious boolean algebra on B, and let d be the derivative operation of T. Then d satisfies (0, 1) but not necessarily (2). In fact, d satisfied (2) if an only if T is T_D (see **(2)** for details). Every T_1 space is T_D and every T_D space is T_0. These inclusions are strict. Thus T_D spaces are the standard examples of pseudo-topological spaces; for these the associated closure operation given by Lemma 4 is the closure operation of the given topology.

We will be concerned with pseudo-topological spaces which are definitely not genuine topological spaces.

Many definitions, theorems, etc., for topological spaces make sense for pseudo-topological spaces. For instance we have the "set" of limit points $L = d(1)$, and the "set" of isolated points $I = 1-L$. Also, for each $x \in B$, the element $is(x) = x-d(x)$ is the analogue of the set of isolated points of x.

By analogy with genuine topological spaces we make the following definitions (given a pseudo-topological space $\langle B, d\rangle$).

(*a*) $\langle B, d\rangle$ has an isolated point if $I \neq 0$.

(*b*) $\langle B, d\rangle$ has a dense set of isolated points if $c(I) = 1$.

(*c*) $\langle B, d\rangle$ is scattered if $(\forall x \neq 0)[is(x) \neq 0]$.

(*d*) $\langle B, d\rangle$ is discrete if $I = 1$.

3. Topological analogues

Let $B(T)$ be the Lindenbaum algebra of the theory T. The elements of $B(T)$ are equivalence classes of sentences, where two sentences σ_1, σ_2 are equivalent if $T \vdash \sigma_1 \leftrightarrow \sigma_2$. The connectives $\wedge$, $\vee$, $\neg$ then induce in a natural way operations $\wedge$, $\vee$, $-$ on $B(T)$, making $B(T)$ into a boolean algebra. We will confuse the elements of $B(T)$ with the sentences they contain, thus at any one time σ may be a sentence or the corresponding equivalence class of sentences. We note that the ordering of $B(T)$ is given by

$$\sigma_1 \leqq \sigma_2 \Leftrightarrow T \vdash \sigma_1 \rightarrow \sigma_2$$

and 0,1 are given by $0 = \perp$, $1 = \neg\perp$.

For the standard suitable theories $\langle T, P\rangle$ the algebra $B(T)$ itself is not very interesting. To see this we use the following theorem of Tarski.

Theorem 3. *For each theory T the following are equivalent.*

(i) *$B(T)$ is atomless.*

(ii) *For each sentence σ which is consistent with T, the deductive closure of $T \cup \{\sigma\}$ is not complete.*

The standard suitable theories $\langle T, P\rangle$ all satisfy (ii) and so $B(T)$ is atomless. Also (assuming the countability of the language L) $B(T)$ is countable and so is uniquely determined.

The function P can be considered as an operation on $B(T)$, since

$$T \vdash \sigma_1 \leftrightarrow \sigma_2 \Rightarrow T \vdash P(\sigma_1) \leftrightarrow P(\sigma_2).$$

Thus we also have an operation d on $B(T)$ given by $d(\sigma) = -P(\neg\sigma)$. This gives us the following.

Theorem 4. *Each suitable theory $\langle T, P\rangle$ gives rise to a pseudo-topological space $\langle B(T), d\rangle$.*

Proof. (ADQ) gives $T \vdash P(\neg\bot)$ so that $d(0) = -P(\neg\bot) = -1 = 0$. For each pair of sentences σ_1, σ_2 Lemma 1 gives

$$\begin{aligned} d(\sigma_1 \vee \sigma_2) &= -P(\neg(\sigma_1 \vee \sigma_2)) \\ &= -P(\neg\sigma_1 \wedge \neg\sigma_2) \\ &= -(P(\neg\sigma_1) \wedge P(\neg\sigma_2)) \\ &= d(\sigma_1) \vee d(\sigma_2) \end{aligned}$$

as required.

Finally (PN) gives, for each sentence σ,

$$T \vdash \neg P(\neg\neg P(\neg\sigma)) \rightarrow \neg P(\neg\sigma)$$

so that $d^2(\sigma) \leqq d(\sigma)$, which completes the proof.

The derivative operation d gives us a certain closure operation c (as in Lemma 4). We have already met this for

$$\begin{aligned} c(\sigma) &= \sigma \vee d(\sigma) \\ &= \sigma \vee -P(\neg\sigma) \\ &= P(\neg\sigma) \rightarrow \sigma \\ &= \sigma^{-}. \end{aligned}$$

This observation gives us a more instructive proof of Theorem 5.1 of (**1**).

We can now interpret properties of the suitable theory $\langle T, P\rangle$ as properties of the pseudo-topological space $\langle B(T), d\rangle$. This gives us some interesting analogies. For instance, we have the " set " of limit points

$$L = d(1) = \neg P(\neg\neg\bot) = \neg P(\bot) = \bot^{+}.$$

and the " set " of isolated points $I = -L = P(\bot)$. Notice also that, for each sentence σ,

$$is(\sigma) = \sigma - d(\sigma) = \sigma \wedge P(\sigma) = \neg(\neg\sigma)^{+}.$$

Remembering the definition (a, b, c, d) given at the end of Section 2, we have the following theorem.

Theorem 5. *For each suitable theory $\langle T, P\rangle$ the following equivalences hold.*

(*a*) *$\langle T, P\rangle$ has* (G)$\Leftrightarrow$*$\langle B(T), d\rangle$ has an isolated point.*

(*b*) *$\langle T, P\rangle$ has* (FG)$\Leftrightarrow$*$\langle B(T), d\rangle$ has a dense set of isolated points.*

(*c*) *$\langle T, P\rangle$ has* (L)$\Leftrightarrow$*$\langle B(T), d\rangle$ is scattered.*

(*d*) *$\langle T, P\rangle$ has* (S)$\Leftrightarrow$*$\langle B(T), d\rangle$ is discrete.*

Proof. (*a*) We have

$$\langle T, P\rangle \text{ has (G)} \Leftrightarrow \text{not } [T \vdash \bot^+]$$
$$\Leftrightarrow \quad L \neq 1$$
$$\Leftrightarrow \quad I \neq 0$$

as required.

(*b*) We have

$$\langle T, P\rangle \text{ has (FG)} \Leftrightarrow T \vdash P(\bot)^-$$
$$\Leftrightarrow c(I) = 1$$

as required.

(*c*) We have

$$\langle T, P\rangle \text{ has (L)} \Leftrightarrow (\forall \sigma)[T \vdash (\neg\sigma)^+ \Rightarrow T \vdash \neg\sigma]$$
$$\Leftrightarrow (\forall \sigma)[\neg\sigma \neq 1 \Rightarrow (\neg\sigma)^+ \neq 1]$$
$$\Leftrightarrow (\forall \sigma \neq 0)[is(\sigma) \neq 0]$$

as required.

(*d*) Finally we have

$$\langle T, P\rangle \text{ has (S)} \Leftrightarrow T \vdash P(\bot)$$
$$\Leftrightarrow I = 1$$

as required.

Let us now look at the separation properties of $\langle B(T), d\rangle$. First we note the following.

Lemma 5. *For each suitable theory $\langle T, P\rangle$, a sentence σ is $\langle T, P\rangle$-nice if and only if σ is open in $\langle B(T), d\rangle$.*

Proof. Clearly σ is open if and only if $\neg\sigma$ is closed, i.e.

$$\neg\sigma = c(\neg\sigma) = \neg\sigma \vee \neg P(\sigma)$$

and this occurs exactly when $\neg P(\sigma) \leqq \neg\sigma$, i.e. $T \vdash \sigma \rightarrow P(\sigma)$. Hence the result.

As we remarked in Section 2, there is some justification for considering $\langle B(T), d\rangle$ to be a T_D (and hence T_0) pseudo-space. However, in most cases it will not be T_1.

Clearly, a space is T_1 if and only if each non-empty set has a non-empty closed subset, equivalently if each non-universal set has a non-universal open superset. Thus we have the following.

Corollary. *For each suitable theory $\langle T, P\rangle$, the pseudo-space $\langle B(T), d\rangle$ is T_1 if and only if for each non-provable (in T) sentence σ there is a $\langle T, P\rangle$-nice, non-provable sentence ν such that $T \vdash \sigma \rightarrow \nu$.*

In the next theorem we use N to suggest complete number theory.

Theorem 6. *Let $\langle T, P\rangle$ be a suitable theory such that T is a proper subtheory of some theory N, where*

$$N\vdash P(\sigma)\Rightarrow T\vdash\sigma$$

holds for each sentence σ. Then $\langle B(T), d\rangle$ is not T_1.

Proof. Suppose, on the contrary, that $\langle B(T), d\rangle$ is T_1, and consider any $\sigma\in N-T$. The above Corollary gives us a sentence ν such that

$$T\vdash\sigma\rightarrow\nu,\ T\vdash\nu\rightarrow P(\nu),$$

but not $T\vdash\nu$. The first and second of these give $N\vdash P(\nu)$, and so $T\vdash\nu$, which is a contradiction.

4. Further remarks

Given any suitable theory we have two operations $(\cdot)^+, (\cdot)^-$. We can form various composites of these, but this gives us just one new operation $(\cdot)^\vee$, where (as in (1)) $\sigma^\vee=\sigma^+\vee\sigma^-$. To see this we remember that $(\cdot)^+, (\cdot)^-$ are idempotent and note the following theorem.

Theorem 7. *For each suitable theory $\langle T, P\rangle$ and sentence σ, the sentences $\sigma^\vee, \sigma^{+-}, \sigma^{-+}$ are each provably equivalent to $P(\bot)\rightarrow\sigma$.*

This theorem gives us the following

Corollary. *The properties* (*P*) (*of* (**1**)) *and* (FG) *are the same.*

The results given here can be viewed as results about certain modal algebras. Consider the propositional modal system based on the symbols $\neg, \rightarrow, \wedge, \vee, \Box$ whose axioms are all formulas of the form

(i) tautology,

(ii) $\Box(A\rightarrow B)\rightarrow(\Box A\rightarrow\Box B)$

(iii) $\Box A\rightarrow\Box\Box A$

and whose rules are modus ponens and $A/\Box A$. The modal algebras for this system are exactly the pseudo-topological spaces, so each class of suitable theories gives a modal system extending the above system. This suggests several questions. What is the modal system corresponding to the class of all suitable theories? (The above?) What is (are) the modal system(s) corresponding to Peano number theory? Which modal systems can be characterised by classes of suitable theories?

How far does the pseudo-space of a suitable theory characterise that theory? For instance, suppose $\langle T_1, P_1\rangle$, $\langle T_1, P_2\rangle$ have isomorphic pseudo-spaces; how different can the two theories be?

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