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How Lipschitz Functions Characterize the Underlying Metric Spaces

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Abstract. Let X and Y be metric spaces and E, F be Banach spaces. Suppose that both X and Y are realcompact, or both E, F are realcompact. The zero set of a vector-valued function f is denoted by $z(f)$. A linear bijection T between local or generalized Lipschitz vector-valued function spaces is said to preserve zero-set containments or nonvanishing functions if

$$z(f) \subseteq z(g) \iff z(Tf) \subseteq z(Tg), \quad \text{or} \quad z(f) = \varnothing \iff z(Tf) = \varnothing,$$

respectively. Every zero-set containment preserver, and every nonvanishing function preserver when $\dim E = \dim F < +\infty$, is a weighted composition operator $(Tf)(y) = J_y(f(\tau(y)))$. We show that the map $\tau\colon Y \to X$ is a locally (little) Lipschitz homeomorphism.

1 Introduction

Let X and Y be metric spaces and let $C(X)$ and $C(Y)$ (resp. $C^b(X)$ and $C^b(Y)$) be the algebras of continuous (resp. bounded continuous) functions defined on X and Y, respectively. It is well known that every multiplicative linear bijection between $C(X)$ and $C(Y)$, or between $C^b(X)$ and $C^b(Y)$, gives rise to a homeomorphism between X and Y (see, *e.g.*, [11, 9.7 and 9.8]). Similar good conclusions hold for multiplicative linear bijections between various Lipschitz function spaces on X and Y. For example, if the spaces $\mathrm{Lip}(X)$ and $\mathrm{Lip}(Y)$ of Lipschitz functions are algebraic isomorphic, then the underlying metric spaces are Lipschitz homeomorphic [9].

In the vector-valued case, there is no multiplicative structure equipped with the vector spaces $C(X, E)$ and its various subspaces when E is a Banach space. Fortunately, we can still consider some structures related to zero sets. Denote the *zero set* and *cozero set*, respectively, of a (scalar or vector-valued) function f defined on X by

$$z(f) = \{x \in X : f(x) = 0\} \quad \text{and} \quad \mathrm{coz}(f) = \{x \in X : f(x) \neq 0\}$$

A linear map T between vector-valued function spaces defined on X and Y is said to be *separating* [3, 6, 12], or *disjointness preserving* [1, 2], if

$$\mathrm{coz}(f) \cap \mathrm{coz}(g) = \varnothing \implies \mathrm{coz}(Tf) \cap \mathrm{coz}(Tg) = \varnothing,$$

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and T is *biseparating* if the inverse implication also holds. In other words, T is biseparating exactly when

$$z(f) \cup z(g) = X \iff z(Tf) \cup z(Tg) = Y.$$

In recent years, disjointness structures have been intensely studied in many classes of function spaces (*cf.* [3–5, 7, 8, 13–15]). As a good substitute for multiplication preservers, (bijective) biseparating linear maps between (scalar or vector-valued) continuous functions always provide homeomorphisms when X and Y are compact (see, *e.g.*, [10]). However, it is a rather different case when X and Y are not compact. An example in [11, 4M] provides us with non-homeomorphic realcompact spaces X and Y such that $C^b(X)$ and $C^b(Y)$ are isometrically algebraic and lattice isomorphic.

In [16, 17], a bijective linear map T between (scalar or vector-valued) function spaces defined on completely regular spaces X and Y is called a (two directional) *zero-set containment preserver* if

$$z(f) \subseteq z(g) \iff z(Tf) \subseteq z(Tg),$$

and T is called a (two directional) *nonvanishing function preserver* if

$$z(f) = \varnothing \iff z(Tf) = \varnothing.$$

Li and Wong [16, 17] showed that every linear zero-set containment preserver, and every nonvanishing function preserver when $\dim E = \dim F < +\infty$, between (scalar or vector-valued) continuous functions defined on realcompact spaces X and Y provides a homeomorphism between X and Y. They also studied other classes of continuous functions including scalar Lipschitz functions.

This paper works with vector-valued Lipschitz functions. In 2009, Araujo and Dubarbie [7] showed that if there is a linear biseparating map between spaces of vector-valued bounded Lipschitz functions on *complete* metric spaces X and Y, then X and Y are Lipschitz homeomorphic. In 2010, Leung [15] extended this to generalized Lipschitz function spaces, and provided a (topological) homeomorphism between X and Y. On the other hand, Jiménez-Vargas, Villegas-Vallecillos, and Wang [13, 14] worked on the same problem for spaces of vector-valued little Lipschitz functions defined on *locally compact* metric spaces X and Y and showed that X and Y are locally Lipschitz homeomorphic if the biseparating map is continuous. We note that the classical spaces Lip, lip^α, and Lip^b of Lipschitz, little Lipschitz and bounded Lipschitz functions, respectively, are special cases of generalized Lipschitz function spaces $\mathrm{Lip}_\Sigma(X, E)$. However, local Lipschitz function spaces $\mathrm{Lip}_{\mathrm{loc}}(X, E)$ are not generalized Lipschitz function spaces.

Suppose X and Y are (not necessarily complete or locally compact) metric spaces and E and F are Banach spaces such that both X and Y are realcompact, or both E and F are realcompact. Using results in [17] we see that every bijective linear zero-set containment preserver, and every nonvanishing function preserver when $\dim E = \dim F < +\infty$, between (local or generalized) Lipschitz function spaces is a weighted composition

$$(Tf)(y) = J_y\big(f(\tau(y))\big),$$

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where $J_y: E \to F$ is a linear bijection and $\tau: Y \to X$ is a homeomorphism between the underlying (not necessarily complete or locally compact) metric spaces. The preserver T is continuous if and only if all fibre linear maps J_y are bounded. In these cases, τ is a locally (little) Lipschitz homeomorphism (Theorems 3.5, 3.8, and 3.9) from Y onto X.

2 Various Preservers on Nicely Regular Subspaces

Definition 2.1 ([17, Definition 3.1]) Let X be a completely regular space and let E be a locally convex space. Let $\mathcal{A}(X,E)$ be a vector subspace of $C(X,E)$, and let

$$\mathcal{A}(X) := \{\psi \circ f : f \in \mathcal{A}(X,E), \psi \in E^*\}$$

be the subset of $C(X)$ consisting of coordinate functions of all f in $\mathcal{A}(X,E)$. We call $\mathcal{A}(X,E)$ *nicely regular* if the following conditions hold:

($\mathcal{A}1$) $\mathcal{A}(X)$ is self-adjoint if $\mathbb{K} = \mathbb{C}$ and its hermitian part $\mathrm{Re}\mathcal{A}(X)$ is a vector sublattice of $C(X)$ containing all constant functions.
($\mathcal{A}2$) For any f in $\mathcal{A}(X)$ and any e in E, the function $f \otimes e$ is in $\mathcal{A}(X,E)$. (Here, we denote by $f \otimes e$ the vector-valued function $x \mapsto f(x)e$ for the scalar-valued function f and the vector $e \in E$.)
($\mathcal{A}3$) $Z(X) = Z(\mathcal{A}(X))$.
($\mathcal{A}4$) If $h_n \geq 0$ is a bounded function in $\mathcal{A}(X)$ for $n = 1, 2, \dots$, then there is a strictly positive sequence $\{\alpha_n\}$ such that the sum $\sum_n \alpha_n h_n$ converges pointwise to a function in $\mathcal{A}(X)$.

Recall that the *support* $\mathrm{supp}(f)$ of a function f is the closure of its cozero set, $\mathrm{coz}(f)$. A map T between function spaces is called a *support containment preserver* if

$$\mathrm{supp}(f) \subseteq \mathrm{supp}(g) \quad \Longrightarrow \quad \mathrm{supp}(Tf) \subseteq \mathrm{supp}(Tg).$$

Proposition 2.2 *Suppose that $\mathcal{A}(X,E)$ and $\mathcal{A}(Y,F)$ are nicely regular subspaces of $C(X,E)$ and $C(Y,F)$, respectively. Assume that $T: \mathcal{A}(X,E) \to \mathcal{A}(Y,F)$ is a linear bijection. Consider the following conditions:*

(i) *T preserves zero-set containments;*
(ii) *T preserves nonvanishing functions;*
(iii) *T is biseparating;*
(iv) *T preserves support containment and separating;*
(v) *T and T^{-1} preserve support containments.*

Then we have that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v).

Proof For the equivalences, we need only to verify two claims.

Claim 1 If T is biseparating, then T preserves support containments.

Let $f, g \in \mathcal{A}(X,E)$ with $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$. If there exists y_0 in $\mathrm{supp}(Tf)$ such that $y_0 \notin \mathrm{supp}(Tg)$, then there is an open neighborhood U_0 of y_0 such that

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$U_0 \cap \operatorname{supp}(Tg) = \varnothing$. Therefore, we can choose y' in $U_0 \cap \operatorname{coz}(Tf)$ and a function h in $\mathcal{A}(X, E)$ with $z(Th) = Y \setminus U_0$, and thus $(Th)(y') \neq 0$. Note that $\operatorname{coz}(Th) \cap \operatorname{coz}(Tg) = \varnothing$. Then we can derive that $\operatorname{coz}(h) \cap \operatorname{coz}(g) = \varnothing$, since T^{-1} is separating, and hence $\operatorname{coz}(h) \cap \operatorname{coz}(f) = \varnothing$. Since T is separating, $\operatorname{coz}(Th) \cap \operatorname{coz}(Tf) = \varnothing$. This is a contradiction, because $y' \in \operatorname{coz}(Th) \cap \operatorname{supp}(Tf)$. This asserts that

$$\operatorname{supp}(f) \subseteq \operatorname{supp}(g) \Longrightarrow \operatorname{supp}(Tf) \subseteq \operatorname{supp}(Tg).$$

Claim 2 If T preserves support containments, then T^{-1} is separating.

Let $\operatorname{coz}(Tf) \cap \operatorname{coz}(Tg) = \varnothing$ and suppose $x_0 \in \operatorname{coz}(f) \cap \operatorname{coz}(g)$. Then there is an open neighborhood V_0 of x_0 such that $V_0 \subseteq \operatorname{coz}(f) \cap \operatorname{coz}(g)$. So we can find k in $\mathcal{A}(X, E)$ such that $k(x_0) \neq 0$ and $z(k) = X \setminus V_0$. That is, $\operatorname{coz}(k) \subseteq \operatorname{coz}(f) \cap \operatorname{coz}(g)$. Since T preserves support containments,

$$\operatorname{supp}(Tk) \subseteq \operatorname{supp}(Tf) \cap \operatorname{supp}(Tg).$$

Since $\operatorname{coz}(Tf) \cap \operatorname{coz}(Tg) = \varnothing$, we see that

$$\operatorname{coz}(Tk) \subseteq \operatorname{supp}(Tf) \subseteq Y \setminus \operatorname{coz}(Tg),$$

and hence $\operatorname{coz}(Tk) \cap \operatorname{coz}(Tg) = \varnothing$. This implies that $Tk = 0$ and hence $k = 0$, which derives a contradiction. This tells us that T^{-1} is separating.

The other implications follow from [17, Lemmas 3.3 and 3.6]. ∎

3 Establishing Lipschitz Homeomorphisms Between Underlying Metric Spaces

Recall that a mapping $\tau\colon X \to Y$ between metric spaces X and Y is said to be *locally Lipschitz* if each point of X has a neighborhood on which τ is Lipschitz. If τ is bijective, and both τ and τ^{-1} are locally Lipschitz (respectively, Lipschitz), then τ is said to be a *locally Lipschitz homeomorphism* (respectively, *Lipschitz homeomorphism*). In [19, Theorem 2.1], Scanlon showed that $\tau\colon X \to Y$ is locally Lipschitz if and only if τ is Lipschitz on each compact subset of X.

Let (X, d_X) be a metric space and E be a Banach space. The vector space of all locally Lipschitz functions from X into E is denoted by $\operatorname{Lip}_{\text{loc}}(X, E)$. For each nonempty compact subset K of X, a seminorm $\rho_K\colon \operatorname{Lip}_{\text{loc}}(X, E) \to [0, +\infty)$ is given by

$$\rho_K(f) = L_K(f) + \|f\|_K,$$

where

$$L_K(f) = \sup\Big\{ \frac{\|f(x) - f(y)\|}{d_X(x, y)} : x, y \in K, x \neq y \Big\} \quad \text{and} \quad \|f\|_K = \sup_{x \in K} \|f(x)\|.$$

The set of all seminorms ρ_K, where K runs through all nonempty compact subsets of X, generates a Hausdorff complete locally convex vector topology on $\operatorname{Lip}_{\text{loc}}(X, E)$.

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For any closed subset C of a metric space X, it is obvious that the function

$$f_C(x) = \min\{1, d_X(x, C)\}$$

is a bounded Lipschitz function and $z(f_C) = C$. Therefore,

$$Z\big(\operatorname{Lip}(X,E)\big) = Z\big(\operatorname{Lip}_{\mathrm{loc}}(X,E)\big) = Z\big(\operatorname{Lip}^b(X,E)\big) = Z\big(\operatorname{Lip}(X)\big) = Z(X).$$

Here, $Z(A)$ denotes the collection of all the zero sets of functions in A, and we write $Z(X) = Z(C(X))$ for simplicity.

In [15], Leung defined a new class of spaces, the generalized Lipschitz function space $\operatorname{Lip}_\Sigma(X, E)$. We say that $\sigma\colon [0, +\infty) \to [0, +\infty]$ is a *modulus function* if σ is nondecreasing, $\sigma(0) = 0$, and σ is continuous at 0. A nonempty set Σ of modulus functions is called a *modulus set* if the following two conditions are satisfied:

(MS1) For any $\sigma_1, \sigma_2 \in \Sigma$, there exist $\sigma \in \Sigma$ and $K < +\infty$ such that $\sigma_1 + \sigma_2 \le K\sigma$.
(MS2) For every sequence $\{\sigma_n\}$ in Σ and every non-negative summable real sequence $\{a_n\}$, there are $\sigma \in \Sigma$ and $K < +\infty$ such that $\sum a_n(\sigma_n \wedge 1) \le K\sigma$.

Let Σ be a modulus set. Let X be a metric space and E be a Banach space. The *generalized Lipschitz function space* $\operatorname{Lip}_\Sigma(X, E)$ is the set of all vector-valued functions $f\colon X \to E$ such that $\omega_f \le K\sigma$ for some $\sigma \in \Sigma$ and $K < +\infty$. Here, $\omega_f\colon [0, +\infty) \to [0, +\infty]$ is defined by

$$\omega_f(\varepsilon) = \sup\{\|f(x_1) - f(x_2)\| : d_X(x_1, x_2) \le \varepsilon\}.$$

When E is the scalar field, $\operatorname{Lip}_\Sigma(X, E)$ is abbreviated to $\operatorname{Lip}_\Sigma(X)$. The spaces of Lipschitz, little Lipschitz, bounded Lipschitz, and uniformly continuous functions are special cases of the generalized Lipschitz function space.

A generalized Lipschitz function space $\operatorname{Lip}_\Sigma(X)$ is said to be *Lipschitz normal* if for every pair of subsets U, V of X with $d(U, V) > 0$, there exists f in $\operatorname{Lip}_\Sigma(X)$ such that $0 \le f \le 1$, $f = 0$ on U, and $f = 1$ on V. We will say that $\operatorname{Lip}_\Sigma(X, E)$ is Lipschitz normal if $\operatorname{Lip}_\Sigma(X)$ is. In the sequel, all generalized Lipschitz function spaces $\operatorname{Lip}_\Sigma(X, E)$ are assumed to be Lipschitz normal. In particular, $Z(\operatorname{Lip}_\Sigma(X)) = Z(X)$ ([15, Lemma 3]).

Both $\operatorname{Lip}_{\mathrm{loc}}(X, E)$ and $\operatorname{Lip}_\Sigma(X, E)$ are nicely regular. In other words, they satisfy the conditions $(\mathcal{A}1)$–$(\mathcal{A}4)$ of Definition 2.1.

Recall that a metrizable space is realcompact if and only if its cardinality is a non-measurable cardinal. In particular, all separable metric spaces and separable Banach spaces are realcompact. See, *e.g.*, [11, Theorem 15.24].

For the rest of this paper, X and Y are metric spaces and E, F are Banach spaces such that both X and Y are realcompact, or both E and F are realcompact.

Theorem 3.1 *Suppose that $\mathcal{A}(X, E)$ and $\mathcal{A}(Y, F)$ are nicely regular subspaces of $C(X, E)$ and $C(Y, F)$, respectively. Let T be a linear bijection from $\mathcal{A}(X, E)$ onto $\mathcal{A}(Y, F)$.*

(i) *If T preserves zero-set containments, then there exists a homeomorphism $\tau\colon Y \to X$ such that*

$$T(f)(y) = J_y\big(f(\tau(y))\big). \tag{3.1}$$

Here, all the fiber maps $J_y(e) = T(1 \otimes e)(y)$ are bijective and linear from E onto F.

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(ii) *The same conclusions hold provided T preserves nonvanishing functions instead and one (and thus both) of E and F is of finite dimension.*

Proof These are applications of [17, Theorems 3.5, 4.4, and 5.1]. ■

Remark 3.2 For the generalized Lipscthiz spaces, a form of Lipschitz continuity of the map τ in (3.1) has been obtained by Leung [15, Proposition 26].

Furthermore, we want to obtain some metric properties of the homeomorphic map $\tau\colon Y \to X$. Recall that, for an arbitrary but fixed point x_0 in X, a complete norm can be defined on $\mathrm{Lip}(X)$ by

$$\|f\|_{\mathrm{Lip}} = \max\{|f(x_0)|, L_f\}.$$

Here, L_f is the Lipschitz constant of f. When the base point x_0 is changed to another point x in X, we might get a different, but equivalent, norm on $\mathrm{Lip}(X)$. In particular, the point evaluation $f \mapsto f(x)$ is a bounded linear functional of $\mathrm{Lip}(X)$.

Lemma 3.3 *Let X and Y be metric spaces. Any composition map $T\colon \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ defined by $Tf = f \circ \tau$ is automatically continuous for the respective Lipschitz norm.*

Proof Let $\{f_n\}$ be a sequence in $\mathrm{Lip}(X)$ with $\|f_n\|_{\mathrm{Lip}} \to 0$ and $\|Tf_n - g\|_{\mathrm{Lip}} \to 0$ for some g in $\mathrm{Lip}(Y)$. For any y in Y, the point evaluations at y and $x = \tau(y)$ are continuous on $\mathrm{Lip}(Y)$ and $\mathrm{Lip}(X)$, respectively. Since

$$|f_n(x) - g(y)| = |f_n(\tau(y)) - g(y)| = |(Tf_n - g)(y)| \to 0 \quad \text{and} \quad f_n(x) \to 0,$$

we derive that $g(y) = 0$, for all $y \in Y$. Having a closed graph, T is bounded. ■

By arguments similar to those in [9, Theorem 3.9 and Lemma 3.15], we obtain the following lemma. We will give a short sketch of the proof in the interest of self-containment.

Lemma 3.4 *Let X and Y be metric spaces, and let $\tau\colon Y \to X$ be a homeomorphism.*

(i) *If $f \circ \tau \in \mathrm{Lip}(Y)$ for all f in $\mathrm{Lip}(X)$, then τ is Lipschitz.*
(ii) *If $f \circ \tau \in \mathrm{Lip}^b_{\mathrm{loc}}(Y)$ for each f in $\mathrm{Lip}^b_{\mathrm{loc}}(X)$, then τ is locally Lipschitz.*

Proof (i) Fix $x_0 \in X$ and $y_0 = \tau^{-1}(x_0) \in Y$ to define the Lipschitz norm on $\mathrm{Lip}(X)$ and $\mathrm{Lip}(Y)$, respectively. The map $T\colon \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ given by $Tf = f \circ \tau$ is a unital vector lattice isomorphism. Then T is continuous by [9, Theorem 3.8]. For any $x_1, x_2 \in X$, we have that

$$d_X(x_1, x_2) = \sup\Big\{ \frac{|f(x_1) - f(x_2)|}{L_f} : f \in \mathrm{Lip}(X), L_f \neq 0, f(x_0) = 0 \Big\}.$$

Indeed, when we choose $f = d_X(\,\cdot\,, x_1) - d_X(x_0, x_1)$, we can prove the above equality.

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Thus for each $f \in \mathrm{Lip}(X)$ with $f(x_0) = 0$, we have $\|f\|_{\mathrm{Lip}} = L_f$. By the continuity of T, we can derive that

$$L_{Tf} \leq \|Tf\|_{\mathrm{Lip}} \leq \|T\| \cdot \|f\|_{\mathrm{Lip}} = \|T\| \cdot L_f.$$

So for any $y_1, y_2 \in Y$ we obtain

$$\begin{aligned} d_X\big(\tau(y_1), \tau(y_2)\big) &= \sup\Big\{ \frac{|f(\tau(y_1)) - f(\tau(y_2))|}{L_f} : f \in \mathrm{Lip}(X), L_f \neq 0, f(x_0) \neq 0 \Big\} \\ &\leq \sup\Big\{ \|T\| \frac{|(Tf)(y_1) - (Tf)\tau(y_2)|}{L_{Tf}} : f \in \mathrm{Lip}(X), L_{Tf} \neq 0, (Tf)(y_0) \neq 0 \Big\} \\ &\leq \|T\| \cdot d_Y(y_1, y_2). \end{aligned}$$

(ii) For any compact subset K of Y, it suffices to show that τ is Lipschitz on K. Indeed, for each $f \in \mathrm{Lip}(\tau^{-1}(K))$, we can extend it to be a Lipschitz funtion on Y, which is also denoted by f. By the assumption we can derive that $f \circ \tau \in \mathrm{Lip}^b_{\mathrm{loc}}(Y)$, and hence $f \circ \tau$ is Lipschitz on K. So by (i) we have that τ is Lipschitz on K. ■

Theorem 3.5 *Assume that* $T\colon \mathrm{Lip}_{\mathrm{loc}}(X, E) \to \mathrm{Lip}_{\mathrm{loc}}(Y, F)$ *is a bijective weighted composition operator*

$$T(f)(y) = J_y\big(f(\tau(y))\big).$$

Then $\tau\colon Y \to X$ *is a locally Lipschitz homeomorphism. Moreover,* T *is continuous if and only if all fiber linear bijections* J_y *are bounded.*

Proof Recall that the continuity of T means that for any compact subset K in Y and real number $C > 0$, there exist a compact subset W in X and a real scalar $M > 0$ such that $\rho_K(Tf) < C$ whenever $\rho_W(f) < M$. It is then obvious that the continuity of T ensures the boundedness of all J_y.

Conversely, for any compact subset $K \subset Y$ and $C > 0$, if $\tau\colon Y \to X$ is a homeomorphism, then $W = \tau^{-1}(Y)$ is a compact subset of X. So we can see that

$$\mathrm{Lip}(W, E) = \{f|_W : f \in \mathrm{Lip}_{\mathrm{loc}}(X, E)\} \text{ and } \mathrm{Lip}(K, F) = \{g|_K : g \in \mathrm{Lip}_{\mathrm{loc}}(Y, F)\}$$

are Banach spaces. When we define $U\colon \mathrm{Lip}(W, E) \to \mathrm{Lip}(K, F)$ by

$$(Uf)(y) = J_y(f(\tau(y))), \quad \forall\, y \in K, f \in \mathrm{Lip}(W, E),$$

it follows from the closed graph theorem that U is a bounded linear bijection and $U(f|_W) = (Tf)|_K$ for all $f \in \mathrm{Lip}_{\mathrm{loc}}(X, E)$. Therefore, for any $f \in \mathrm{Lip}_{\mathrm{loc}}(X, E)$ with $\rho_W(f) < \frac{C}{\|U\|}$, we can derive that $\|f|_W\| = \rho_W(f) < \frac{C}{\|U\|}$, and then

$$\rho_K(Tf) = \|(Tf)|_K\| = \|U(f|_W)\| \leq \|U\| \cdot \|f|_W\| < C.$$

This implies that $T\colon \mathrm{Lip}_{\mathrm{loc}}(X, E) \to \mathrm{Lip}_{\mathrm{loc}}(Y, F)$ is continuous.

We next show that τ is a locally Lipschitz homeomorphism without assuming the continuity of T. By Lemma 3.4 and [19, Theorem 2.1], τ is local Lipschitz if for each $f \in \mathrm{Lip}^b_{\mathrm{loc}}(X)$, $f \circ \tau$ is Lipschitz on every compact subset K of Y containing at least two points.

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Claim 1 For any e in E with $\|e\| = 1$, we have $\inf_{y\in K} \|T(1\otimes e)(y)\| > 0$.

This is obvious, as the continuous function $y \mapsto \|T(1\otimes e)(y)\|$ never vanishes on the compact subset $K \subseteq Y$.

It follows from Claim 1 that $y \mapsto 1/\|T(1\otimes e)(y)\|$ defines a function in $\mathrm{Lip}(K)$, and then, by [18], it can be extended to a scalar-valued bounded Lipschitz function on Y.

Claim 2 For each f in $\mathrm{Lip}^b_{\mathrm{loc}}(X)$, the function $f\circ\tau$ is Lipschitz on $K \subseteq Y$.

Observe that for all $f \geq 0$ on X, we have $f\circ\tau \geq 0$ on Y and

$$\begin{aligned}
&\big|\, f\circ\tau(y)\|T(1\otimes e)(y)\| - f\circ\tau(y')\|T(1\otimes e)(y')\|\,\big| \\
&\quad = \big|\, \|J_y(f(\tau(y))e)\| - \|J_{y'}(f(\tau(y'))e)\|\,\big| \\
&\quad = \big|\, \|T(f\otimes e)(y)\| - \|T(f\otimes e)(y')\|\,\big| \\
&\quad \leq \|T(f\otimes e)(y) - T(f\otimes e)(y')\| \\
&\quad \leq L_K(T(f\otimes e))d_Y(y,y'), \quad \forall\, y, y' \in K.
\end{aligned}$$

That is, $f\circ\tau(\cdot)\|T(1\otimes e)(\cdot)\|$ is Lipschitz on K. Hence $f\circ\tau$ is Lipschitz on K. For any f in $\mathrm{Lip}^b_{\mathrm{loc}}(X)$, by writing f as a linear combination of at most four positive functions, we derive that $f\circ\tau$ is Lipschitz on K.

By a similar argument, τ^{-1} is also locally Lipschitz on X. We thus see that τ is a locally Lipschitz homeomorphism, which completes the proof. ■

We say that a modulus function σ is *of bounded type $O(t)$* if there is a finite positive constant L_σ such that $\sigma(t) \leq L_\sigma t$ for all $t \geq 0$.

Theorem 3.6 *Assume that Σ and Σ' consist of modulus functions σ of bounded type $O(t)$. Assume also that $\mathrm{Lip}^b(X) \subseteq \mathrm{Lip}_\Sigma(X)$ and $\mathrm{Lip}(Y)^b \subseteq \mathrm{Lip}_{\Sigma'}(Y)$. If $T\colon \mathrm{Lip}_\Sigma(X,E) \to \mathrm{Lip}_{\Sigma'}(Y,F)$ is a bijective weighted composition operator*

$$T(f)(y) = J_y\big(f(\tau(y))\big),$$

then $\tau\colon Y \to X$ is a locally Lipschitz homeomorphism.

Proof For any compact subset K of Y containing at least two points and each f in $\mathrm{Lip}^b_{\mathrm{loc}}(X)$, by [18], the restricted function $f|_{\tau(K)}$ in $\mathrm{Lip}(\tau(K))$ can be extended to a g in $\mathrm{Lip}^b(X) \subseteq \mathrm{Lip}_\Sigma(X)$. Since $\mathrm{Lip}_\Sigma(X,E)$ is Lipschitz normal, there exists h in $\mathrm{Lip}_\Sigma(X)$ such that $0 \leq h \leq 1$ and $h = 1$ on $\tau(K)$. Note that $gh \in \mathrm{Lip}_\Sigma(X)$. For any y in K, we have

$$T(gh\otimes e)(y) = J_y\big(gh(\tau(y))e\big) = (gh)\big(\tau(y)\big)\,J_y(e) = f\big(\tau(y)\big)\,T(1\otimes e)(y).$$

Moreover, for any y, y' in K, we can derive that

$$\begin{aligned}
&\big|\, f\circ\tau(y)\|T(1\otimes e)(y)\| - f\circ\tau(y')\|T(1\otimes e)(y')\|\,\big| \\
&\quad \leq \|T(gh\otimes e)(y) - T(gh\otimes e)(y')\| \\
&\quad \leq \omega_{T(gh\otimes e)}\big(d_Y(y,y')\big) \leq M\sigma\big(d_Y(y,y')\big) \leq ML_\sigma d_Y(y,y'),
\end{aligned}$$

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for some σ in Σ' and some finite constants $M, L_\sigma > 0$. That is,

$$(f \circ \tau)(\cdot)\|T(1 \otimes e)(\cdot)\|$$

is Lipschitz on K. It then follows from the similar argument of Theorem 3.5 that $f \circ \tau$ is Lipschitz on K. Hence, τ is a locally Lipschitz homeomorphism. ■

Corollary 3.7 *Suppose that $T\colon \operatorname{Lip}(X, E) \to \operatorname{Lip}(Y, F)$ is a linear bijection preserving zero-set containments, or nonvanishing functions when E or F is of finite dimension, then X and Y are locally Lipschitz homwomorphic and T carries the form* (3.1).

Next we consider some special classes of generalized Lipschitz function spaces.

Theorem 3.8 *Assume that $T\colon \operatorname{Lip}^b(X, E) \to \operatorname{Lip}^b(Y, F)$ is a bijective weighted composition operator*

$$T(f)(y) = J_y\big(f(\tau(y))\big).$$

Then $\tau\colon Y \to X$ is a locally Lipschitz homeomorphism. Moreover, T is continuous if and only if all fiber linear bijections J_y are bounded.

Proof By Theorem 3.6, τ is a locally Lipschitz homeomorphism.

Now, suppose that T is continuous. For any y in Y and e in E, we have

$$\|J_y(e)\| = \|T(1 \otimes e)(y)\| \leq \|T\|\|e\|.$$

This tells us that all J_y are bounded linear bijections. On the other hand, the continuity of T will follow if it has a closed graph. Suppose that $f_n \in \operatorname{Lip}^b(X, E)$ with $f_n \to 0$ and $Tf_n \to g_0 \in \operatorname{Lip}^b(Y, F)$. As

$$\|(Tf_n)(y)\| = \big\| J_y\big(f_n(\tau(y))\big)\big\| \leq \|J_y\|\|f_n\| \to 0,$$

we see that $g_0(y) = 0$ for all y in Y, and hence $g_0 = 0$. This implies that T is bounded. ■

Let (X, d_X) be a metric space, α a real number in $(0, 1)$, and E a real or complex Banach space. Let X^α denote the same set X together with the new metric $d_X^\alpha(x, y) := d_X(x, y)^\alpha$. Denote by $\operatorname{Lip}(X^\alpha, E)$ the Banach space of all vector-valued functions $f\colon X \to E$ such that

$$p_\alpha(f) = \sup\Big\{ \frac{\|f(x_1) - f(x_2)\|}{d_X(x_1, x_2)^\alpha} : x_1, x_2 \in X,\ x_1 \neq x_2 \Big\}$$

and

$$\|f\|_\infty = \sup\{\|f(x)\| : x \in X\}$$

are finite, endowed with the sum norm

$$\|f\|_\alpha = p_\alpha(f) + \|f\|_\infty.$$

The *little Lipschitz function space* $\operatorname{lip}^\alpha(X, E)$ denotes the closed subspace of $\operatorname{Lip}(X^\alpha, E)$ consisting of functions f with

$$\lim_{d_X(x_1,x_2)\to 0} \frac{\|f(x_1) - f(x_2)\|}{d_X(x_1, x_2)^\alpha} = 0.$$

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Theorem 3.9 *Assume that* $T\colon \operatorname{lip}^\alpha(X,E) \to \operatorname{lip}^\alpha(Y,F)$ *is a linear bijection preserving zero-set containments, or preserving nonvanishing functions when* E *or* F *is of finite dimension. Then* T *carries the form* (3.1),

$$T(f)(y) = J_y\big(f(\tau(y))\big),$$

such that $\tau\colon Y \to X$ *is a locally little Lipschitz homeomorphism. Moreover,* T *is continuous if and only if all fiber linear bijections* J_y *is continuous.*

Proof Since both $\operatorname{lip}^\alpha(X,E)$ and $\operatorname{lip}^\alpha(Y,F)$ are nicely regular, it follows from Theorem 3.1 that T carries the stated weighted composition form. The rest of the proof is basically the same as for Theorem 3.5. For any compact subset K of Y containing at least two points, we can define a bounded linear map $S\colon \operatorname{lip}^\alpha(\tau(K)) \to \operatorname{lip}^\alpha(K)$ by

$$(Sf)(y) = f(\tau(y)), \quad \forall\, y \in K.$$

For any fixed y_1, y_2 in K, define a function f_1 by

$$f_1(x) = \min\{\gamma, d_X(x, \tau(y_1))\}, \quad \forall\, x \in \tau(K),$$

where $\gamma > 0$ is the finite diameter of the compact metric space $\tau(K)$. Then f_1 is in $\operatorname{lip}^\alpha(X)$, since

$$\frac{|d_X(x,\tau(y_1)) - d_X(x',\tau(y_1))|}{d_X(x,x')^\alpha} \le d_X(x,x')^{1-\alpha}.$$

Since Sf_1 is little Lipschitz, we can derive that

$$\begin{aligned}\frac{d_X(\tau(y_1),\tau(y_2))}{d_Y(y_1,y_2)^\alpha} &= \frac{|f_1(\tau(y_1)) - f_1(\tau(y_2))|}{d_Y(y_1,y_2)^\alpha}\\ &= \frac{|(Sf_1)(y_1) - (Sf_1)(y_2)|}{d_Y(y_1,y_2)^\alpha} \to 0\end{aligned}$$

as $d_Y(y_1,y_2) \to 0$. Hence τ is little Lipschitz on K.

The "moreover" part follows the same way as in proving Theorem 3.8, since little Lipschitz function spaces are Banach spaces. ■

Remark 3.10 (i) In the above theorems, when T is continuous, the map $y \mapsto J_y$ is continuous from Y into $(\mathcal{L}(E,F), SOT)$.

(ii) It is plausible that Theorems 3.8 and 3.9 might provide us a Lipschitz or little Lipschitz homeomorphism τ between the metric spaces X and Y. However, as indicated in [9] it is not always possible. For example, set $X = \mathbb{R}$ with the usual Euclidean metric d and $Y = (\mathbb{R}, \widetilde{d})$ with the bounded metric $\widetilde{d}(x,y) = \min\{1, d(x,y)\}$. Then $\operatorname{Lip}^b(X) = \operatorname{Lip}^b(Y)$ and $\operatorname{lip}^\alpha(X) = \operatorname{lip}^\alpha(Y)$ for all α in $(0,1)$, but X and Y are not Lipschitz or little Lipschitz homeomorphic. Note however that X and Y are locally (little) Lipschitz homeomorphic.

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