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# WEAK SEQUENTIAL COMPACTNESS AND COMPLETENESS IN RIESZ SPACES

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**1. Introduction.** If $L$ is an Archimedean Riesz space and $M$ an ideal in the order dual of $L$, the subset $A$ of $L$ is called *M-equicontinuous* if and only if each monotone decreasing sequence of positive elements of $M$ is uniformly Cauchy on $A$. It was shown by Luxemburg and Zaanen [**7**] that if $L_\rho$ is a Banach function space with associate space $L_\rho'$ and if $M$ is a closed ideal in $L_\rho'$, then $L_\rho$ is sequentially $\sigma(L_\rho, M)$ complete and a subset $A \subset L_\rho$ is relatively sequentially $\sigma(L_\rho, M)$ compact if and only if $A$ is $M$-equicontinuous. One interesting consequence of these results is the fact that each norm bounded subset of a Banach function space $L_\rho$ is relatively sequentially $\sigma(L_\rho, L_\rho'^a)$ compact, where $L_\rho'^a$ denotes the ideal in $L_\rho'$ of elements of absolutely continuous norm. It is our intention in this note to derive these results from abstract theorems in the theory of Riesz spaces. The basic tools necessary for our work are, in essence, the well known results of H. Nakano [**4**] that if $L$ is a Dedekind complete Riesz space with a separating family $L_n^\sim$ of normal integrals then $L_n^\sim$ is $\sigma(L_n^\sim, L)$ sequentially complete and a subset $A$ of $L_n^\sim$ is $\sigma(L_n^\sim, L)$ relatively compact if and only if $A$ is $L$-equicontinuous. These results and some of their various extensions are treated in [**1**] and [**2**].

A further motivation for the present work is provided by the paper [**3**] where several results are proved concerning weak sequential completeness and compactness in Banach lattices. Some of the most interesting of these results are proved under the assumption that the Banach lattice norm is order continuous. While this assumption in general cannot be omitted, it does preclude a direct discussion of related results in Banach function spaces such as those indicated above. Our more general approach does in fact yield simultaneously extensions of a number of the main results of [**3**] without excluding general Banach function spaces from consideration. As pointed out above, our techniques are based directly on two well known order theoretic results of Nakano, whereas those of [**3**] are based on a direct application of the theorem of Dunford-Pettis on weak compactness in spaces of type $L^1(\mu)$ and the classical Hahn-Saks theorem via the representation theorem of Kakutani for abstract $L$-spaces.

We use without further explanation the basic terminology and results from the theory of Riesz spaces as set out in [**6, 8**]. Throughout the paper $L$ shall denote an Archimedean Riesz space and $M$ shall denote an ideal in the order

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dual $L^{\sim}$. We shall use also the following terminology which differs slightly from that used in [7]. The subset $A$ of $L$ is called *conditionally* sequentially $\sigma(L, M)$ compact if and only if each sequence in $A$ contains a subsequence which is $\sigma(L, M)$ Cauchy. The subset $A$ of $L$ will be called *relatively* sequentially $\sigma(L, M)$ compact if and only if each sequence in $A$ contains a subsequence which is $\sigma(L, M)$ convergent to some element of $L$.

## 2. Equicontinuity.

*Definition* 2.1. Let $M$ be an ideal in the order dual $L^{\sim}$. The subset $A$ of $L$ will be called *$M$-equicontinuous* if and only if whenever the sequence $\{\phi_n\} \subset M$ satisfies $\phi_n \downarrow_n 0$ it follows that

$$\sup\{|\phi_n(x)| : x \in A\} \to 0 \quad \text{as } n \to \infty.$$

As far as the authors are aware, the above notion of equicontinuity was introduced by H. Nakano [**4**]. Our main purpose in this section is to gather a number of essentially known results concerning $M$-equicontinuity which go back to Nakano [**4**]. The proofs are omitted as they involve only simple modifications of corresponding results given in [**1**] and Theorem 45.6 of [**5**]. Recall that if $M$ is an ideal in $L^{\sim}$, the topology $|\phi|(L, M)$ is the locally solid topology on $L$ defined by the Riesz semi-norms $x \mapsto |\phi|(|x|)$, $x \in L$, $\phi \in M$.

PROPOSITION 2.2. *Let $M$ be an ideal in $L^{\sim}$.*

(a) *The subset $A$ of $L$ is $M$-equicontinuous if and only if its convex solid hull is $M$-equicontinuous.*

(b) *If the subset $A$ of $L$ is $M$-equicontinuous, then $A$ is $|\sigma|(L, M)$ bounded.*

PROPOSITION 2.3. *Let $M$ be an ideal in $L^{\sim}$. If the subset $A$ of $L$ is conditionally sequentially $\sigma(L, M)$ compact, then $A$ is $M$-equicontinuous.*

We remark that the above Proposition 2.3 is essentially proved in [**5**, Theorem 45.6] as a consequence of the well-known lemma of Phillips. It is to be noted that the converse to Proposition 2.3 is not, in general, valid. By way of example, let $L$ be the space of bounded sequences $l^{\infty}$ and let $M$ be the Banach dual space $(l^{\infty})^*$. It is easily seen that the unit ball of $L$ is $M$-equicontinuous. On the other hand, it is well known that there exists an isometric (though not order-preserving) mapping from the sequence space $l^1$ into $L$. It follows that the unit ball of $L$ is not conditionally sequentially $\sigma(L, M)$ compact.

The notion of $M$-equicontinuity occurs also in the work [**7**] of Luxemburg and Zaanen in the setting of Banach function spaces. Since the formulation in [**7**] differs slightly from that given in Definition 2.1 above, we insert the following result for ease of comparison of the results of this paper and those of [**7**]. The proof is again a simple modification of Proposition 2.11 of [**1**] and is accordingly omitted.

PROPOSITION 2.4. *Let $M$ be an ideal in $L^\sim$. The following statements are equivalent for a subset $A$ of $L$.*

(i) *$A$ is $M$-equicontinuous.*

(ii) *$A$ is $|\sigma|(L, M)$ bounded and for each $0 \leqq \phi \in M$ and sequence $\{\phi_n\}$ of principal components of $\phi$, it follows from $\phi_n \downarrow_n 0$ in $M$ that*

$\sup\{\phi_n(|x|) : x \in A\} \downarrow_n 0.$

(iii) *$A$ is $|\sigma|(L, M)$ bounded and $\sup\{\phi_n(|x|) : x \in A\} \to 0$ as $n \to \infty$ for each disjoint order bounded sequence $\{\phi_n\}$ in $M^+$.*

**3. Conditional $\sigma(L, M)$ sequential compactness.** It was observed in the preceding section that $M$-equicontinuity does not in general characterize the conditionally sequentially $\sigma(L, M)$ compact subsets of $L$. In this section some natural conditions on $L$ and $M$ are given for which the converse to Proposition 2.3 is valid.

PROPOSITION 3.1. *Let $M \subset L^\sim$ be an ideal with an (at most) countable order basis. The following statements are equivalent for a subset $A$ of $L$.*

(i) *$A$ is $M$-equicontinuous.*

(ii) *$A$ is conditionally sequentially $\sigma(L, M)$ compact.*

*Proof.* By Proposition 2.3, it is only necessary to prove the implication (i) $\Rightarrow$ (ii). Assume then that $A \subset L$ is $M$-equicontinuous and let $0 \leqq \phi_n \uparrow_n \subset M$ be a countable order basis for $M$. Consider $A$ as a subset of $M_n^\sim$. Since $A$ is $M$-equicontinuous, it follows from Proposition 4.6 of [**1**] that given $0 \leqq \phi \in M$, each sequence in $A$ contains a subsequence which converges pointwise on the principal ideal generated by $\phi$ in $M$. Let $\{x_n\} \subset A$ be a sequence. By a diagonal argument, it follows that there is a subsequence $\{z_n\} \subset \{x_n\}$ such that $\lim \psi(z_n)$ exists whenever $\psi$ belongs to the ideal in $M$ generated by the order basis $\{\phi_n\}$. Suppose now that $0 \leqq \phi \in M$ and let $\epsilon > 0$ be given. Since the sequence $\{\phi_n\}$ is an order basis for $M$, and since $A$ is $M$-equicontinuous, it follows that there exists an integer $k > 0$ such that

$$\sup\{(\phi - k\phi_k \wedge \phi)(|x|) : x \in A\} < \epsilon.$$

Since $k\phi_k \wedge \phi$ is a member of the ideal generated by $\{\phi_n\}$, it follows that there exists $n_0$ such that $n, m \geqq n_0$ imply $|k\phi_k \wedge \phi(z_n - z_m)| < \epsilon$. It follows that, for $n, m \geqq n_0$,

$$|\phi(z_n) - \phi(z_m)| \leqq (\phi - k\phi_k \wedge \phi)(|z_n| + |z_m|) + |k\phi_k \wedge \phi(z_n - z_m)| \leqq 3\epsilon.$$

Thus, the sequence $\{z_n\}$ is $\sigma(L, M)$ Cauchy and the proof of the implication (i) $\Rightarrow$ (ii) is complete.

Before stating the second main result of this section, we need two preliminary results which are in part extensions to Archimedean Riesz spaces of Theorem 27.20 and Lemma 25.9 of [**6**]. We recall that the carrier $L_1$ in $L$ of $L_n^\sim$ is the band $\{x \in L : x \perp {}^\circ L_n^\sim\}$ where ${}^\circ L_n^\sim = \{x \in L : \phi(x) = 0 \text{ for all } \phi \in L_n^\sim\}$.

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PROPOSITION 3.2. *If the carrier* $L_1$ *in* $L$ *of* $L_n^\sim$ *has an (at most) countable order basis, then*

(a) $L_n^\sim$ *is Dedekind super complete.*

(b) *The following statements are equivalent.*

(i) $L_1$ *has the countable* sup *property.*

(ii) $L_n^\sim$ *has an (at most) countable order basis.*

*Proof.* (a) This is contained in Theorem 31.13 of [**6**].

(b) The implication (ii) $\Rightarrow$ (i) is Corollary 31.14 of [**6**]. We prove here the implication (i) $\Rightarrow$ (ii). Recall first that if $0 \leqq \phi \in L_n^\sim$, the Riesz subspace $C_\phi \oplus N_\phi$ is order dense in $L$, where $C_\phi$, $N_\phi$ denote respectively the carrier band and the null band in $L$ of the normal integral $\phi$. If $0 \leqq u \in L$, it follows that $\phi(u) \neq 0$ if and only if there exists $v \in C_\phi$ with $0 \leqq v \leqq u$ and $\phi(v) \neq 0$. Suppose now that $\{\phi_\alpha\} \subset L_n^{\sim +}$ is a maximal pairwise disjoint system and let $\{u_n\} \subset L^+$ be an (at most) countable order basis for the carrier $L_1$ of $L_n^\sim$. Since each order bounded disjoint system in $L_1$ is at most countable by Theorem 18D of [**2**] and since $C_{\phi_\alpha} \perp C_{\phi_\beta}$ if $\alpha \neq \beta$, it follows that for each $n$, $\phi_\alpha(u_n) = 0$ with the exception of at most countably many indices $\alpha$. Since $\{u_n\}$ is an order basis for $L_1$, and since $L_1 \oplus {}^\circ L_n^\sim$ is order dense in $L$, it follows that $\phi_\alpha = 0$ with the exception of an at most countable set of indices $\alpha$ and by this the implication (i) $\Rightarrow$ (ii) is proved.

PROPOSITION 3.3. *Let* $L$ *have the countable* sup *property. If* $L$ *has an (at most) countable order basis and* $J$ *is an ideal in* $L$, *then* $J$ *has an (at most) countable order basis.*

*Proof.* It is sufficient to show that each disjoint system of positive elements in $J$ has at most countably many non-zero members, since any maximal disjoint system in $J$ (which exists by Zorn's lemma) is an order basis for $J$. Let $\{v_\lambda\}$ be a disjoint system in $J$ and let $\{u_n\}$ be an at most countable order basis for $L$. Since $L$ has the countable sup property, it follows that $v_\lambda \wedge u_n = 0$ for every $n$, with the exception of an at most countable set of indices $\lambda$. Since $\{u_n\}$ is an order basis for $L$, it follows that $v_\lambda = 0$ with the exception of an at most countable number of indices $\lambda$ and the proof is complete.

The final result of this section is proved in [**3**] for Banach lattices with order continuous norm via an appeal to the criterion of Dunford-Pettis for weak compactness in spaces of type $L^1(\mu)$. Even in this special case our proof is quite different and is similar to that of Nakano [**4**] to whom the result is due in the case that the Riesz space $L$ is Dedekind super complete and perfect and the ideal $M$ is the band of normal integrals $L_n^\sim$.

THEOREM 3.4. *Let* $M$ *be an ideal in* $L_n^\sim$ *and assume that* $L$ *has the countable* sup *property. The following statements are equivalent for a subset* $A$ *of* $L$.

(i) $A$ *is* $M$-*equicontinuous.*

(ii) $A$ *is conditionally sequentially* $\sigma(L, M)$ *compact.*

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*Proof*. It is only necessary to prove the implication (i) $\Rightarrow$ (ii). Let $\{x_n\} \subset A$ be a sequence and let $I$ denote the ideal in $L$ generated by the sequence $\{x_n\}$. It is clear that $I$ has the countable sup property and an at most countable order basis and so by Propositions 3.2, 3.3 above it follows that $I_n^{\sim}$ has an at most countable order basis, and is Dedekind super complete. Denote by $[M]$ the set of restrictions to $I$ of the elements of $M$. We show that $[M]$ is an ideal in $I_n^{\sim}$ and that the sequence $\{x_n\}$ is $[M]$-equicontinuous. It will then follow from Proposition 3.3 that $[M]$ has an at most countable order basis and the assertion of the implication (i) $\Rightarrow$ (ii) will follow from an appeal to Proposition 3.1.

It is clear that $[M]$ is a linear subspace of $I_n^{\sim}$. To show that $[M]$ is an ideal in $I_n^{\sim}$, suppose that $0 \leqq \psi \in I_n^{\sim}$ satisfies $0 \leqq \psi \leqq [\phi]$ for some $\phi \in M$. Denote by $\rho$ the Riesz semi-norm $y \mapsto |\phi|(|y|)$, $y \in L$. Observe that $|\psi(u)| \leqq \rho(u)$ holds for $u \in I$ and so it follows from Theorem 19.2 of [**6**] that there exists $\psi' \in L_n^{\sim}$ such that $\psi' = \psi$ on $I$ and $0 \leqq \psi' \leqq |\phi|$ on $L$. Since $M$ is an ideal in $L_n^{\sim}$ it follows that $\psi' \in M$ and $[\psi'] = [\psi]$. It follows that $[M]$ is an ideal in $I_n^{\sim}$. To show that the sequence $\{x_n\}$ is $[M]$-equicontinuous, it is sufficient to show that if $[\phi_n] \downarrow_n \geqq 0$ holds in $[M]$ there exists $\{\phi_n'\} \subset M$ such that $[\phi_n] = [\phi_n']$ holds for each $n$ and $\phi_n' \downarrow_n \geqq 0$ holds in $M$. To this end, choose $0 \leqq \phi_1' \in M$ such that $[\phi_1'] = [\phi_1]$ and suppose $\phi_1', \phi_2', \ldots, \phi_{n-1}'$ have been defined such that $[\phi_i'] = [\phi_i]$, $\phi_i' \in M$ for $1 \leqq i \leqq n-1$ and that $\phi_1' \geqq \phi_2' \geqq \ldots \geqq \phi_{n-1}' \geqq 0$. By Theorem 19.2 of [**6**], there exists $0 \leqq \psi_n \in M$ such that $[\psi_n] = [\phi_{n-1}' - \phi_n]$ and $0 \leqq \psi_n \leqq \phi_{n-1}'$. It follows that $[\phi_n] = [\phi_{n-1}' - \psi_n]$ and so defining $\phi_n' = \phi_{n-1}' - \psi_n$ the induction step is complete. Since $\{x_n\}$ is $[M]$-equicontinuous and since $[M]$ has an (at most) countable order basis, it follows from Proposition 3.1 that $\{x_n\}$ contains a subsequence which is $\sigma(I, [M])$-Cauchy and so evidently $\sigma(L, M)$ Cauchy and the proof is complete.

We conclude the section with some examples which show that in general neither of the assumptions of Theorem 3.4 can be dropped.

*Example* 3.5. If $L = l^\infty$, $M = (l^\infty)^*$, the unit ball of $L$ is $M$-equicontinuous but not conditionally sequentially $\sigma(L, M)$ compact. Thus the assumption that $M \subset L_n^{\sim}$ cannot be entirely omitted, even if $L$ is Dedekind super complete.

*Example* 3.6. Denote by $\omega$ the set of natural numbers, $Q$ the set of rational numbers and let $X = \{\kappa : \kappa \text{ is a subsequence of } \omega\}$. Let $L$ be $l^\infty(X)$. For each $\kappa \in X$, let $\{r_\kappa^p : p \in \kappa\}$ be an enumeration of $Q \cap [0, 1]$. For $n \in \omega$, define $x_n \in L$ as follows.

$$x_n(\kappa) = \begin{cases} r_\kappa^n & \text{if } n \in \kappa \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{x_p : p \in \kappa_0\}$ be a subsequence of $\{x_n\}$. It follows that $\{x_p(\kappa_0) : p \in \kappa_0\} = Q \cap [0, 1]$. Thus if $M$ denotes $l^1(X)$, the unit ball of $L$ is not conditionally sequentially $\sigma(L, M)$ compact, although it is $M$-equicontinuous. Thus even if $M \subset L_n^\sim$ and $L$ is Dedekind complete, it is not possible to drop entirely the assumption that $L$ has the countable sup property.

**4. Sequential $\sigma(L, M)$ completeness.** The principal theorem of this section, which yields directly our results on weak sequential completeness, is an extension of Satz 1.7 of [**3**]. As pointed out in the Introduction, the proof we give here is an intrinsic order-theoretic proof which is based on the well-known fact that if the Riesz space $L$ is Dedekind complete, then the space $L_n^\sim$ of normal integrals is $\sigma(L_n^\sim, L)$ sequentially complete.

THEOREM 4.1. *Let $L$ be an (Archimedean) Riesz space with the countable* sup *property and let $M \subset L_n^\sim$ be a separating ideal. If the sequence $\{x_n\} \subset L$ is $\sigma(L, M)$ Cauchy, there exist sequences $0 \leqq v_n \uparrow_n$, $0 \leqq w_n \uparrow_n \subset L$ such that $x_n - v_n + w_n$ is $\sigma(L, M)$ convergent to* 0.

*Proof*. Let $\{x_n\} \subset L$ be a $\sigma(L, M)$ Cauchy sequence and denote by $I$ the ideal generated in $L$ by the sequence $\{x_n\}$ and by $[M]$ the set of restrictions to $I$ of the elements of $M$. As in the proof of Theorem 3.4 above, $[M]$ is a separating ideal in $I_n^\sim$ and it is clear that $\{x_n\}$ is $\sigma(I, [M])$ Cauchy. From Proposition 3.2 it follows that $I_n^\sim$ is Dedekind super complete and has an (at most) countable order basis and so $[M]$ is Dedekind super complete. By Proposition 3.3, $[M]$ has an (at most) countable order basis from which it follows that $[M]_n^\sim$ is Dedekind super complete. By Theorem 32.11 of [**6**], the ideal generated by $I$ in $[M]_n^\sim$ is a Dedekind completion of $I$, which is moreover order dense in $[M]_n^\sim$ by 32B of [**2**]. Since $[M]_n^\sim$ is $\sigma([M]_n^\sim, [M])$ sequentially complete, there exists $x \in [M]_n^\sim$ such that $x_n \to x$, $\sigma([M]_n^\sim, [M])$. It follows that there exist sequences $\{v_n\}$, $\{w_n\} \subset L$ with $0 \leqq v_n \uparrow_n x^+$, $0 \leqq w_n \uparrow_n x^-$ holds in $[M]_n^\sim$. Since each element of $M$ defines a normal integral on $[M]_n^\sim$, it follows that $[\phi](x_n - v_n + w_n) \to 0$ for each $\phi \in M$ and the statement of the theorem follows.

We state without proof the following simple consequence of Theorem 4.1.

COROLLARY 4.2. *Let $L$ have the countable* sup *property and let $M \subset L_n^\sim$ be a separating ideal. The following statements are equivalent.*

(i) *$L$ is $\sigma(L, M)$ sequentially complete.*

(ii) *Each monotone $\sigma(L, M)$ Cauchy sequence in $L$ is $\sigma(L, M)$ convergent to some element of $L$.*

(iii) $0 \leqq u_n \uparrow_n \subset L$, $\sup_n \phi(u_n) < \infty$ *for each* $0 \leqq \phi \in M$ *implies* $\sup_n u_n$ *exists in $L$.*

In the above Theorem 4.1, if in addition to the countable sup property it is assumed that $L$ is Dedekind complete, then the assumption that the ideal $M$ be a separating ideal may be dropped. In fact, an inspection of the proof of Theorem 4.1 shows that it is sufficient to work with the carrier band of $M$ in $L$ rather than with $L$ itself. Corollary 4.2 must then be reformulated as follows.

COROLLARY 4.3. *Let $L$ be Dedekind super complete and let $M \subset L_n^{\sim}$ be an ideal. The following statements are equivalent.*

(i) *$L$ is $\sigma(L, M)$ complete.*

(ii) *Each monotone $\sigma(L, M)$ Cauchy sequence in $L$ is $\sigma(L, M)$ convergent to some element of $L$.*

(iii) $0 \leqq u_n \uparrow_n$, $\sup_n \phi(u_n) < \infty$ *for each* $0 \leqq \phi \in M$ *implies* $\sup_n P_M u_n$ *exists in $L$, where $P_M$ denotes the band projection of $L$ onto the carrier band of $M$ in $L$.*

**5. Banach function spaces.** Our concluding remarks are directed towards Banach function spaces as defined in [**7**] and in particular we indicate how the results of the preceding sections may be used to derive the corresponding results of [**7**].

PROPOSITION 5.1. *Let $L_\rho$ be a Dedekind super complete, norm perfect Banach lattice. If $M \subset L_{\rho,n}^*$ is a closed ideal, then*

(a) *$L_\rho$ is $\sigma(L_\rho, M)$ sequentially complete.*

(b) *The following statements are equivalent for a subset $A \subset L_\rho$.*

 (i) *$A$ is $M$-equicontinuous.*

 (ii) *$A$ is relatively sequentially $\sigma(L_\rho, M)$ compact.*

*Proof.* (a) Denote by $P_M$ the band projection of $L_\rho$ onto the carrier band of $M$ in $L_\rho$. It is a consequence of Corollary 31.6 of [**6**] that if $0 \leqq \psi \in L_{\rho,n}^*$ then $P_M^*\psi$ belongs to the band generated by $M$ in $L_{\rho,n}^*$ and so

$$P_M^*\psi = \sup\{\phi : 0 \leqq \phi \in M \quad \text{and} \quad 0 \leqq \phi \leqq P_M^*\psi\}.$$

Thus, if $0 \leqq x \in L_\rho$ and $0 \leqq \psi \in L_{\rho,n}^*$ then

$$\psi(P_M x) = P_M^*\psi(x) = \sup\{\phi(x) : 0 \leqq \phi \in M \quad \text{and} \quad 0 \leqq \phi \leqq P_M^*\psi\}.$$

Now since $L_\rho$ is norm perfect, there exists a constant $k(\rho) > 0$ such that

$$\rho(x) \leqq k(\rho) \sup\{\psi(x) : 0 \leqq \psi \in L_{\rho,n}^*,\ \rho^*(\psi) \leqq 1\}$$

holds for each $0 \leqq x \in L_\rho$. It follows that

$$\rho(P_M x) \leqq k(\rho) \sup\{\psi(x) : 0 \leqq \phi \in M,\ \rho^*(\phi) \leqq 1\}.$$

To see now that $L_\rho$ is sequentially $\sigma(L_\rho, M)$ complete, suppose that $0 \leqq u_n \uparrow_n \subset L_\rho$ satisfies $\sup \phi(u_n) < \infty$ for each $0 \leqq \phi \in M$. Since $M$ is closed, it is a consequence of the uniform boundedness principle that

$$\sup\{\phi(u_n) : 0 \leqq \phi \in M,\ \rho^*(\phi) \leqq 1,\ n = 1, 2, \ldots\} < \infty.$$

Thus $\sup_n \rho(P_M u_n) < \infty$ and so $\sup P_M u_n$ exists in $L_\rho$ since $L_\rho$ is perfect. The assertion of (a) now follows from Corollary 4.3.

(b) The equivalence of the statements in (b) is now an immediate consequence of part (a) and Theorem 3.4.

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The ideal $L_{\rho,n}{}^{*a}$ of elements of absolutely continuous norm in $L_{\rho,n}{}^{*}$ is of special interest and we single out the following consequence of Proposition 5.1 above (cf. Theorem 5.4 of [**7**]).

COROLLARY 5.2. *If the Banach lattice $L_\rho$ is Dedekind super complete and norm perfect, then each bounded set in $L_\rho$ is relatively $\sigma(L_\rho, L_{\rho,n}{}^{*a})$ sequentially compact.*

The examples given in Section 3 above are both norm-perfect Banach lattices. It follows then that in statements (a) and (b) of Proposition 5.1 above, the assumption that $M \subset L_{\rho,n}{}^{*}$ cannot be omitted, even if $L_\rho$ is Dedekind super complete and in statement (b) of Proposition 5.1 above, the assumption that $L_\rho$ be Dedekind super complete cannot be dropped even if $L_\rho$ is norm perfect and $M \subset L_{\rho,n}{}^{*}$.

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