On some Legendre Function Formulae.

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The various formulae for the Legendre Functions, and the relations between these formulae, have been studied by Kummer, Riemann, Olbricht, Hobson, Barnes, Whipple,[1] and others. Hobson[2] obtained some of the relations directly, by expressing the functions as Pochhammer integrals, and expanding in a number of series each with its own region of convergence. To obtain some of the other formulae, such as (i) below, he transformed the differential equation, and then expressed the functions in terms of the solutions of the transformed equation. Barnes[3] succeeded, by means of his well-known integrals involving Gamma Functions, in deducing all the formulae directly from the formulae which define the functions. Notes on the history of the subject and references to previous work will be found in the papers by Hobson and Barnes.

The main object of the present paper is to show that the transformations to the asymptotic formulae such as (i) can be obtained comparatively simply by basing them on the transformations—such as (v) and (vi) below—of the hypergeometric function. In introducing the subject to a class of honours students it is suggested that, in the first place, the analytical continuation of the hypergeometric function should be discussed, the method employed being that given by the author in a previous paper in these *Proceedings* (Vol. XXXVII., pp. 80-84). The earlier part of the theory of the Legendre Functions should then be introduced by Hobson's methods, which are available in the author's *Functions of a Complex Variable,* pages 259-265. The other formulae can then be deduced with the help of the hypergeometric function formulae. Some are given as examples on pages 275, 276 of the *Complex Variable* : the method of obtaining the others is indicated in this paper.

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[1] *Proc. Lond. Math. Soc.*, **16** (2), (1917), 301–314.

[2] *Phil. Trans.*, **187** A, (1896), 443–531.

[3] *Quart. Journ. of Maths.*, **39**, (1908), 97–204.

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The formula

$$Q_n^m(z) = e^{m\pi i}\sqrt{\left(\frac{\pi}{2\sqrt{z^2-1}}\right)}\,(z-\sqrt{z^2-1})^{n+\frac{1}{2}}\,\frac{\Gamma(n+m+1)}{\Gamma(n+\frac{3}{2})}$$

$$\times F\left(\tfrac{1}{2}-m,\ \tfrac{1}{2}+m;\ n+\tfrac{3}{2};\ -\frac{z-\sqrt{z^2-1}}{2\sqrt{z^2-1}}\right), \quad \ldots\ldots\ldots\ldots(\text{i})$$

and the corresponding expression for $P_n^m(z)$ involve hypergeometric series which are convergent in certain regions of the z-plane, and in other parts of the plane are asymptotic in n. They have been discussed by the author in a paper in these *Proceedings* (Vol. XLI, pp. 82-90). The definitions of the Associated Legendre Functions employed are those of Hobson ; namely

$$P_n^{-m}(z) = \frac{1}{\Gamma(m+1)}\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}m} F\left(-n,\ n+1;\ m+1;\ \frac{1-z}{2}\right), \ \ldots(\text{ii})$$

$$Q_n^m(z) = \frac{e^{m\pi i}}{2^{n+1}}\ \frac{\Gamma(n+m+1)\,\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})}\ \frac{(z^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}$$

$$\times F\left(\frac{n+m+2}{2},\ \frac{n+m+1}{2};\ n+\tfrac{3}{2};\ \frac{1}{z^2}\right). \quad \ldots\ldots\ldots(\text{iii})$$

These functions are connected by the important relation

$$P_n^{-m}(z) = \frac{e^{-m\pi i}}{\cos n\pi\,\Gamma(m+n+1)\,\Gamma(m-n)}\,\{Q_{-n-1}^m(z) - Q_n^m(z)\}. \ (\text{iv})$$

The formulae for the Hypergeometric Function required in what follows are

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma-\alpha-\beta)\,\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}\,F(\alpha,\beta;\ \alpha+\beta-\gamma+1;\ 1-z)$$

$$+\frac{\Gamma(\alpha+\beta-\gamma)\,\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}\,(1-z)^{\gamma-\alpha-\beta}\,F(\gamma-\alpha,\gamma-\beta;\ \gamma-\alpha-\beta+1;\ 1-z), \ (\text{v})$$

and

$$F(\alpha,\beta;\gamma;z) = (1-z)^{-\beta}\,F\left(\beta,\ \gamma-\alpha;\ \gamma;\ \frac{z}{z-1}\right). \ \ldots\ldots(\text{vi})$$

The duplication formula for the Gamma Function

$$\Gamma(2p) = \frac{2^{2p-1}}{\sqrt{\pi}}\,\Gamma(p)\,\Gamma(p+\tfrac{1}{2}) \qquad \ldots\ldots(\text{vii})$$

is also given here for convenience in reference.

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Now, from (iv) and (iii)

$$P_n^{-m}(z)\,(z^2-1)^{-\frac{1}{2}m}$$

$$=\left\{\begin{array}{l}\dfrac{\Gamma(\frac{1}{2})\sec n\pi}{2^{-n}\,\Gamma(-n+\frac{1}{2})\,\Gamma(m+n+1)\,z^{-n+m}}\,F\left(\dfrac{-n+m+1}{2},\ \dfrac{-n+m}{2};\ -n+\frac{1}{2};\ \dfrac{1}{z^2}\right)\\ -\ \dfrac{\Gamma(\frac{1}{2})\sec n\pi}{2^{n+1}\,\Gamma(n+\frac{3}{2})\,\Gamma(m-n)\,z^{n+m+1}}\,F\left(\dfrac{n+m+2}{2},\ \dfrac{n+m+1}{2};\ n+\frac{3}{2};\ \dfrac{1}{z^2}\right)\end{array}\right\}$$

$$=\left\{\begin{array}{l}\dfrac{\Gamma(n+\frac{1}{2})}{\Gamma(m+n+1)}\,\dfrac{\Gamma(\frac{1}{2})\,\pi^{-1}}{2^{-n}\,z^{-n+m}}\,F\left(\dfrac{-n+m+1}{2},\ \dfrac{-n+m}{2};\ -n+\frac{1}{2};\ \dfrac{1}{z^2}\right)\\ +\ \dfrac{\Gamma(-n-\frac{1}{2})}{\Gamma(m-n)}\,\dfrac{\Gamma(\frac{1}{2})\,\pi^{-1}}{2^{n+1}\,z^{n+m+1}}\,F\left(\dfrac{n+m+2}{2},\ \dfrac{n+m+1}{2};\ n+\frac{3}{2};\ \dfrac{1}{z^2}\right)\end{array}\right\}$$

$$=\left\{\begin{array}{l}\dfrac{\Gamma(n+\frac{1}{2})\,2^{-m}\,z^{n-m}}{\Gamma\left(\dfrac{m+n+1}{2}\right)\Gamma\left(\dfrac{m+n+2}{2}\right)}\,F\left(\dfrac{-n+m+1}{2},\ \dfrac{-n+m}{2};\ -n+\frac{1}{2};\ \dfrac{1}{z^2}\right)\\ +\ \dfrac{\Gamma(-n-\frac{1}{2})\,2^{-m}\,z^{-n-m-1}}{\Gamma\left(\dfrac{m-n}{2}\right)\Gamma\left(\dfrac{m-n+1}{2}\right)}\,F\left(\dfrac{n+m+2}{2},\ \dfrac{n+m+1}{2};\ n+\frac{3}{2};\ \dfrac{1}{z^2}\right)\end{array}\right\}$$

by (vii). Hence

$$P_n^{-m}(z)=\frac{2^{-m}(z^2-1)^{\frac{1}{2}m}\,z^{n-m}}{\Gamma(m+1)}$$

$$\times\left\{\begin{array}{l}\dfrac{\Gamma(n+\frac{1}{2})\,\Gamma(m+1)}{\Gamma\left(\dfrac{m+n+1}{2}\right)\Gamma\left(\dfrac{m+n+2}{2}\right)}\,F\left(\dfrac{m-n+1}{2},\ \dfrac{m-n}{2};\ \frac{1}{2}-n;\ \dfrac{1}{z^2}\right)\\ +\ \dfrac{\Gamma(-n-\frac{1}{2})\,\Gamma(m+1)}{\Gamma\left(\dfrac{m-n+1}{2}\right)\Gamma\left(\dfrac{m-n}{2}\right)}\left(\dfrac{1}{z^2}\right)^{n+\frac{1}{2}}F\left(\dfrac{m+n+1}{2},\ \dfrac{m+n+2}{2};\ n+\frac{3}{2};\ \dfrac{1}{z^2}\right)\end{array}\right\}$$

$$=\frac{2^{-m}(z^2-1)^{\frac{1}{2}m}\,z^{n-m}}{\Gamma(m+1)}\,F\left(\frac{m-n+1}{2},\ \frac{m-n}{2};\ m+1;\ 1-\frac{1}{z^2}\right),\qquad\text{(viii)}$$

by (v). Thus, from (vi),

$$P_n^{-m}(z)=\frac{2^{-m}(z^2-1)^{\frac{1}{2}m}}{\Gamma(m+1)}\,F\left(\frac{m-n}{2},\ \frac{m+n+1}{2};\ m+1;\ 1-z^2\right)\qquad\text{(ix)}$$

A further application of (v) gives

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$$P_n^{-m}(z) = 2^{-m}(z^2-1)^{\frac{1}{2}m}$$

$$\times\left\{\frac{\Gamma(\frac{1}{2})}{\Gamma\left(\frac{m+n+2}{2}\right)\Gamma\left(\frac{m-n+1}{2}\right)} F\left(\frac{m-n}{2},\ \frac{m+n+1}{2};\ \tfrac{1}{2};\ z^2\right) + \frac{\Gamma(-\frac{1}{2})}{\Gamma\left(\frac{m-n}{2}\right)\Gamma\left(\frac{m+n+1}{2}\right)} zF\left(\frac{m+n+2}{2},\ \frac{m-n+1}{2};\ \tfrac{3}{2};\ z^2\right)\right\} \quad \text{(x)}$$

Again, from (ii) and (viii) we have, near $z=1$,

$$F\left(-n,\ n+1;\ m+1;\ \frac{1-z}{2}\right)$$

$$= \left(\frac{z+1}{2}\right)^m z^{n-m} F\left(\frac{m-n+1}{2},\ \frac{m-n}{2};\ m+1;\ 1-\frac{1}{z^2}\right).$$

Here put $z = \dfrac{x}{\sqrt{(x^2-1)}}$, so that

$$\frac{1+z}{2} = \frac{x+\sqrt{(x^2-1)}}{2\sqrt{(x^2-1)}},\quad \frac{1-z}{2} = \frac{\sqrt{(x^2-1)}-x}{2\sqrt{(x^2-1)}},\quad 1-\frac{1}{z^2} = \frac{1}{x^2},$$

and replace m by $n+\frac{1}{2}$ and n by $-m-\frac{1}{2}$. Then

$$F\left(\tfrac{1}{2}+m,\ \tfrac{1}{2}-m;\ n+\tfrac{3}{2};\ \frac{-x+\sqrt{x^2-1}}{2\sqrt{x^2-1}}\right)$$

$$= \left\{\frac{x+\sqrt{(x^2-1)}}{2\sqrt{(x^2-1)}}\right\}^{n+\frac{1}{2}} \left\{\frac{x}{\sqrt{(x^2-1)}}\right\}^{-n-m-1}$$

$$\times F\left(\frac{m+n+2}{2},\ \frac{m+n+1}{2};\ n+\tfrac{3}{2};\ \frac{1}{x^2}\right).$$

Applying this formula to (iii), we deduce that

$$Q_n^m(z) = \frac{e^{m\pi i}}{2^{n+1}} \frac{\Gamma(n+m+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})} \frac{(z^2-1)^{\frac{1}{2}m}}{z^{n+m+1}} \frac{2^{n+\frac{1}{2}}(z^2-1)^{-\frac{1}{2}(m+\frac{1}{2})} z^{n+m+1}}{(z+\sqrt{z^2-1})^{n+\frac{1}{2}}}$$

$$\times F\left(\tfrac{1}{2}+m,\ \tfrac{1}{2}-m;\ n+\tfrac{3}{2};\ \frac{-z+\sqrt{z^2-1}}{2\sqrt{z^2-1}}\right),$$

from which, on simplifying, (i) is obtained. From this the corresponding formula for $P_n^m(z)$ can be derived.[1]

In conclusion, a formula will be established which is a generalisation of the expansion of a Legendre Coefficient in the form of a

[1] *Cf. Proc. Edin. Math. Soc.*, **41** (1923), 88.

Fourier Sine Series. Following the notation of Ferrers, we define the real function $T_n^m(\cos\theta)$ by the equation.

$$T_n^m(\cos\theta) = e^{\frac{1}{2}m\pi i} P_n^m(\cos\theta),$$

and the formula referred to at the end of the last paragraph then gives

$$T_n^m(\cos\theta) = \frac{1}{\sqrt{(2\pi\sin\theta)}} \frac{\Gamma(n+m+1)}{\Gamma(n+\frac{3}{2})}$$

$$\times\left\{\begin{array}{l} e^{-\frac{1}{2}m\pi i+\frac{1}{4}\pi i-(n+\frac{1}{2})\theta i}\, F\left(\frac{1}{2}-m,\, \frac{1}{2}+m;\, n+\frac{3}{2};\, -\frac{e^{-i\theta}}{2i\sin\theta}\right) \\ +\text{ the conjugate complex function} \end{array}\right\}$$

$$= \frac{1}{\sqrt{(2\pi\sin\theta)}} \frac{\Gamma(n+m+1)}{\Gamma(m+\frac{1}{2})\,\Gamma(n-m+1)}$$

$$\times\left\{\begin{array}{l} e^{-\frac{1}{2}m\pi i+\frac{1}{4}\pi i-(n+\frac{1}{2})\theta i} \int_0^1 \lambda^{m-\frac{1}{2}}(1-\lambda)^{n-m}\left(1+\frac{\lambda e^{-i\theta}}{2i\sin\theta}\right)^{m-\frac{1}{2}} d\lambda \\ +\text{ the conjugate complex function} \end{array}\right\}$$

$$= \frac{1}{\sqrt{\pi}\,.\,(2\sin\theta)^m} \frac{\Gamma(n+m+1)}{\Gamma(m+\frac{1}{2})\,\Gamma(n-m+1)}$$

$$\times\left\{\begin{array}{l} e^{-m\pi i+\frac{1}{2}\pi i-(n+m)\theta i} \int_0^1 \frac{\lambda^{m-\frac{1}{2}}(1-\lambda)^{n-m}}{(\lambda-1+e^{2i\theta})^{\frac{1}{2}-m}} d\lambda \\ +\text{ the conjugate complex function} \end{array}\right\}$$

$$= \frac{1}{\sqrt{\pi}\,.\,(2\sin\theta)^m} \frac{\Gamma(n+m+1)}{\Gamma(m+\frac{1}{2})\,\Gamma(n-m+1)}$$

$$\times\left[\begin{array}{l} e^{-m\pi i+\frac{1}{2}\pi i+(m-n-1)\theta i} \\ \quad\times\int_0^1 \lambda^{m-\frac{1}{2}}(1-\lambda)^{n-m}\left\{1+\frac{\frac{1}{2}-m}{1}(1-\lambda)\,e^{-2i\theta}+\ldots\right\} d\lambda \\ \quad+\text{ the conjugate complex function} \end{array}\right]$$

$$= \frac{2\,\Gamma(n+m+1)}{\sqrt{\pi}\,.\,(2\sin\theta)^m\,\Gamma(m+\frac{1}{2})\,\Gamma(n-m+1)}$$

$$\times\left[\begin{array}{l} B(m+\frac{1}{2},\, n-m+1)\sin\{m\pi+(n-m+1)\theta\} \\ +B(m+\frac{1}{2},\, n-m+2)\frac{\frac{1}{2}-m}{1}\sin\{m\pi+(n-m+3)\theta\} \\ +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right]$$

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$$= \frac{2}{\sqrt{\pi}\,.\,(2\sin\theta)^m\,\Gamma(\frac{1}{2}-m)}$$

$$\times\left[\begin{array}{l} B(\frac{1}{2}-m,\ n+m+1)\sin\{m\pi+(n-m+1)\theta\} \\ +B(\frac{3}{2}-m,\ n+m+1)\dfrac{n-m+1}{1}\sin\{m\pi+(n-m+3)\theta\} \\ +B(\frac{5}{2}-m,\ n+m+1)\dfrac{(n-m+1)(n-m+2)}{2!}\sin\{m\pi+(n-m+5)\theta\} \\ +\ \dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right], \quad \text{(xi)}$$

which converges for $0 < \theta < \pi$, provided that $R(m) > -\frac{1}{2}$. The convergence is of the same nature as that of series of the type

$$\Sigma\,\frac{\sin 2r\theta}{r^{2m+1}},\ \Sigma\,\frac{\cos 2r\theta}{r^{2m+1}}.$$

This follows from the formula

$$\operatorname*{Lim}_{z\to\infty}\frac{\Gamma(z+a)}{\Gamma(z)\,z^a} = 1,$$

where $|\operatorname{amp} z| < \pi$.

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