RELATIVISTIC EQUATIONS OF MOTION OF CELESTIAL BODIES

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Abstract. The problem of relativistic equations of motion for extended celestial bodies in the first post-Newtonian approximation is reviewed. It is argued that the problems dealing with kinematical aspects have been solved in a satisfactory way, but more work has to be done on the dynamical side. Concepts like angular velocity, moments of inertia, Tisserand axes etc. still have to be introduced in a rigorous manner at the 1PN level.

Usually, relativistic equations of motion (EOM) for massive celestial bodies are treated within the so-called post-Newtonian framework, based on a formal expansion of the form

$$\mathrm{EOM} = (\mathrm{EOM})_{\mathrm{Newton}} + \epsilon^2(\mathrm{EOM})_{1\mathrm{PN}} + \epsilon^4(\mathrm{EOM})_{2\mathrm{PN}} + \ldots, \tag{1}$$

where

$$\epsilon^2 \sim \left(\frac{v}{c}\right)^2 \sim \frac{GM}{c^2R}. \tag{2}$$

Here, v denotes a typical orbital velocity of celestial bodies, M a typical mass and R some characteristic distance between the bodies. In the solar system, $\epsilon^2 < 10^{-5}$ and for most purposes the first post-Newtonian approximation will be sufficient. Now, the problem of celestial mechanical equations of motion can be divided into two parts: a kinematical part and a dynamical one. The kinematical problems are related with the theory of relativistic astronomical reference frames. These problems have been solved in a satisfactory manner by Brumberg, Kopejkin and Klioner (see,*e.g.*, Brumberg 1991 and references cited therein) and by Damour, Soffel and Xu (1991, 1992, 1993, 1994; DSX I-IV). The dynamical part deals with the physical

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interaction of the gravitating bodies. Some basic problems have been solved by Damour, Soffel and Xu in the first post-Newtonian approximation. However, some problems still remain unsolved. These will be mentioned below.

In relativity, the *local* equations of motion that can be written as

$$0 = \nabla_\nu T^{\mu\nu} \tag{3}$$

form the basis for any derivations of *global* equations of motion, *e.g.*, for the center-of-mass of a body. Here, ∇_ν denotes the covariant derivative and $T^{\mu\nu}$ is the energy-momentum tensor of matter. To Newtonian order, in inertial Cartesian coordinates, (ct, x^i) equations (3) take the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v^i)}{\partial x^i} = 0 \tag{4}$$

and

$$\frac{\partial(\rho v^i)}{\partial t} + \frac{\partial}{\partial x^j}\left[\rho v^i v^j + t^{ij}\right] = \rho \frac{\partial U}{\partial x^i}. \tag{5}$$

Here, ρ is the matter density, $\mathbf{v}$ the coordinate velocity of some material element, U is the Newtonian potential and t^{ij} the stress-tensor of matter. Going to higher and higher accuracies, the definition of quantities like center-of-mass, mass-moments, spin-moments and the derivation of useful equations of motion will become increasingly more complicated. For that reason, there is some chance that in the (far) future the local equations of motion (3) might be the only useful ones in which case they would have to be solved by means of techniques from numerical relativity after having specified the visco-elastic properties of matter (*i.e.*, the energy-momentum tensor) in a relativistic framework.

Damour, Soffel and Xu have demonstrated that global Laws of Motion (LOM) can be derived in the first post-Newtonian approximation without specification of the energy-momentum tensor. Let us consider a system of N weakly self-gravitating, rotating bodies of arbitrary shape and composition gravitationally interacting with each other. To describe the time evolution of such a system, we introduce $N + 1$ different coordinate systems,

- one global coordinate system $x^\mu = (ct, x^i)$ and
- one local coordinate system $X^\alpha_A = (cT, X^a)_A$, for each of the bodies, comoving with body A.

It is assumed that the global chart extends to spatial infinity. For celestial mechanical problems related with the solar system it will be a barycentric coordinate system. Among the local ones, one will be a geocentric frame, moving with the Earth. Now, in the DSX-formalism, the gravitational interaction is described by two potentials (w, w_i) in the global frame and (W, W_a) in one of the local frames. In Newton's theory of gravity, there is only one scalar potential, the Newtonian potential U. To lowest order, the scalar potential w or W agrees with U but it also has terms proportional to c^{-2}. w is also called the gravitoelectric potential describing the gravitational action of a static matter distribution. On the other hand, the vector potential w^i (or W_a) describes the gravitational effect of moving matter (*e.g.*, rotating bodies), *i.e.*, gravitomagnetic effects. It has no Newtonian analogue.

Let us consider the environment of a body (A) such as the Earth in its own local frame. In a special representation, one finds that the equations satisfied by the potentials $W_\alpha = (W, W_a)$ are *linear* and there is a unique split of the potentials into two pieces:

$$W_\alpha = W_\alpha^+ + \overline{W}_\alpha. \tag{6}$$

Here, the self-part of the potentials, W_α^+, describes the gravitational action of body (A) itself. It can be characterized by two sets of multipole moments, called M_L and S_L. Here, L denotes a Cartesian multi-index, $L = i_1 i_2 \dots i_l$ and each index i_j takes the values $(1,2,3) = (x,y,z)$, so we face components like M_{xx}, or M_{xxyyz} etc. The M_L's are called mass-multipole moments, S_L are the spin-moments of body (A). Both, M_L and S_L are so-called STF (symmetric and trace-free) tensors which are the Cartesian analogues of coefficients in an expansion in terms of spherical harmonics. M_L generalizes the usual Newtonian potential coefficients C_{lm} and S_{lm} to the first post-Newtonian level. S_i represents the total angular momentum (spin) of body (A). M_L and S_L, as functions of local coordinate time T, are fully determined by integrals over the densities

$$\Sigma = \frac{T^{00} + T^{ss}}{c^2} = \rho + O(c^{-2}); \qquad \Sigma^a = \frac{T^{0a}}{c} = \rho v^a + O(c^{-2}), \tag{7}$$

i.e., they can formally be obtained without specification of the energy-momentum tensor of matter.

The external part of the potential, $\overline{W}_\alpha$ describes the inertial forces in the local frame and the gravitational effects from the other bodies, *i.e.*, the tidal forces. In the DSX-formalism this external part of the potentials is available either in closed form as function of the mass- and current-moments of the other bodies and of the position and velocity of the origin of the local system in the global frame or as expansion in terms of tidal moments, G_L and H_L. In Newton's theory, the magnetic-type moments H_L play no role and

$$G_i^{\rm Newton} = \partial U_A^{\rm ext}(\mathbf{z}_A) - \frac{d^2 z_A^i}{dt^2} \qquad (l = 1) \tag{8a}$$

$$G_{i_1 \dots i_l}^{\rm Newton} = \frac{\partial^l U_A^{\rm ext}(\mathbf{z}_A)}{\partial X^{i_1} \cdots \partial X^{i_l}} \qquad (l \geq 2). \tag{8b}$$

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To derive Laws of Motion we first have to relate the origins of the local frames with the matter distribution of the bodies. Usually, one chooses the origin of the local A-frame to coincide with the center-of mass of body A. This can be achieved by means of

$$M_a = 0. \tag{9}$$

With that condition the potentials (W, W_a) are completely fixed and the trajectory, *e.g.*, of a satellite can be considered as a geodesic in the metric that is determined by the potentials. It is equivalently determined by a Lagrangian of the form

$$\mathcal{L} = \frac{1}{2}\mathbf{v}^2 + W + \frac{1}{c^2}\left[\frac{1}{8}\mathbf{v}^4 - \frac{1}{2}W^2 + \frac{3}{2}W\mathbf{v}^2 - 4W^a v^a\right], \tag{10}$$

where $\mathbf{v}$ is the satellite's coordinate velocity. This leads to an equation of the form

$$\ddot{X}^a = f^a_{\text{self}} + f^a_{\text{tidal}} + f^a_{\text{mixed}}. \tag{11}$$

Here, f^a_{self} is an acceleration resulting from the action of body (A) itself and f^a_{tidal} results from the action of the other bodies. In the DSX-formalism,

$$\begin{aligned} f^a_{\text{self}} &= W^+_{,a} + (c^{-2} - \text{terms}) \\ &= G\textstyle\sum_{l\geq 0} \frac{(-)^l}{l!} M_L(\partial_{La} R^{-1}) + (c^{-2} - \text{terms}) \end{aligned} \tag{12}$$

has been obtained fully to post-Newtonian order in terms of M_L and S_L of body (A),

$$f^a_{\text{tidal}} = \sum_{l\geq 0} \frac{X_L}{l!} G_{La} + (c^{-2} - \text{terms}) \tag{13}$$

completely to PN-order in terms of G_L and H_L or in closed, *i.e.*, non-expanded form. Similarly, the mixed acceleration

$$f^a_{\text{mixed}} = -\frac{4}{c^2}\left(\overline{W}W^+{}_{,a} + W^+\overline{W}_{,a}\right) \tag{14}$$

is completely available to post-Newtonian accuracy. In DSX IV, we have also derived post-Newtonian satellite Laws of Motion in the global barycentric frame which might be useful for high flying satellites or for comparisons with previous results from other authors.

General translational post-Newtonian Laws of Motion for a system of massive bodies have been derived from relation (9). More specifically, using

$$\frac{d^2 M_a}{dT^2} = 0 \tag{15}$$

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and the local equations of motion one finds

$$0 = \frac{d^2 M_a}{dT^2} = \sum_{l \geq 0} \frac{1}{l!} M_L G_{La} + (c^{-2} - \text{terms}). \tag{16}$$

Since

$$G_a = -\frac{d^2 z_A^a}{dt^2} + \overline{W}_{,a}|_{X^a=0} + (c^{-2} - \text{terms})$$

equation (16) can be solved for the desired acceleration of the center-of-mass of body (A). In this way, Laws of Motion in the form

$$\frac{d^2 z^a}{dt^2} = \overline{W}_{,a}|_{X^a=0} + (c^{-2} - \text{terms}) \tag{17}$$

have been derived to PN accuracy. For more details, the reader is referred to DSX II. In the case of pure mass-monopoles (*i.e.*, only the masses of bodies different from zero), one recovers the usual Einstein-Infeld-Hoffmann equations of motion.

The problem of rotational motion is even more complex than the translational case. The reason for that being that the spin of a body does not enter the Newtonian potential so only the Newtonian expression for the spin enters the post-Newtonian potentials. A post-Newtonian theory of the spin of a body therefore has to deal with gravitational potentials to 2PN-order. In DSX III, we gave a definition of a post-Newtonian spin of body (A), $S_i^{(A)}$, in the N-body case such that the resulting torque can be expressed entirely in terms of the moments $(M_L, S_L; G_L, H_L)$ in the A-frame. The resulting Law of rotational motion is of the form

$$\begin{aligned} \frac{dS_i^{(A)}}{dT} = \quad & \epsilon_{iab} \textstyle\sum_{l \geq 0} \frac{1}{l!} \left(M_{aL} G_{bL} + \frac{1}{c^2} \frac{l+1}{l+2} S_{aL} H_{bL} \right) \\ & + \text{further}(\ c^{-2} - \text{terms}). \end{aligned} \tag{18}$$

For a body with only mass-monopole and spin-dipole this equation reduces to

$$\frac{dS^{(A)}}{dT} = \frac{1}{2c^2} S_a H_b. \tag{19}$$

Now, the local frame can be oriented in space such that $H_b = 0$ and the torque in (19) vanishes. In that case, the local frame is called dynamically non-rotating and the axes to the spatial coordinate line are Fermi-transported along the central worldline (DSX III).

So far, we were just talking about Laws of Motion rather than about Equations of Motion, the reason being that the various quantities M_L and

S_L have not been specified as functions of time in the general case. Note that these multipole moments physically are determined by the visco-elastic properties of matter. To close the whole system of equations, one has to introduce specific models for the time behaviour of the various multipole moments. One might, *e.g.*, start with "rigid models" with

$$\frac{dM_{ab}}{dT} = \epsilon_{adc}\Omega^c M_{db} + \epsilon_{bcd}\Omega^c M_{ad} \qquad \text{etc.} \tag{20}$$

and

$$S_c = I_{cd}\Omega^d, \tag{21}$$

where Ω^c is some time dependent vectorial function, but there is indeed the problem if such an ansatz is consistent and compatible with Einstein's field equations. It is known (Thorne and Gürsel 1983) that for one isolated body everything is fine if one restricts to first-order terms in Ω. In that case, problems related with Lorentz-contraction effects will not show up. To higher order in Ω, equations (20) and (21) are only compatible with the field equations for stationary bodies which do not precess and show no nutation.

One can say that concepts like angular velocity, figure axes, Tisserand axes, etc., still present basic problems in the gravitational N-body problem. A good way to proceed has been outlined by Klioner (1996): he introduces some rotating local frame by the action of some vectorial function $\Omega^i(T)$ and then formally specifies the rotating frame by requiring that the resulting rotational equations of motion take a particularly simple form. More work is needed, however, to put these ideas into a more rigorous form.

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