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PROOF OF A CONJECTURE OF BANERJEE AND DASTIDAR ON ODD CRANK

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Abstract

Recently, when studying intricate connections between Ramanujan's theta functions and a class of partition functions, Banerjee and Dastidar ['Ramanujan's theta functions and parity of parts and cranks of partitions', *Ann. Comb.*, to appear] studied some arithmetic properties for $c_o(n)$, the number of partitions of n with odd crank. They conjectured a congruence modulo 4 satisfied by $c_o(n)$. We confirm the conjecture and evaluate $c_o(4n)$ modulo 8 by dissecting some q-series into even powers. Moreover, we give a conjecture on the density of divisibility of odd cranks modulo 4, 8 and 16.

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1. Introduction

A partition λ of a nonnegative integer n is a finite weakly decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i for $1 \leq i \leq r$ are called the parts of the partition λ. Let $p(n)$ denote the number of partitions of n. In 1919, Ramanujan [8] discovered three remarkable congruences enjoyed by $p(n)$, namely,

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

In 1944, Dyson [7] introduced the notion of the rank, and further conjectured that this partition statistic could provide a combinatorial interpretation for (1.1) and (1.2). Dyson's conjecture was later confirmed by Atkin and Swinnerton-Dyer [4] in 1954. Unfortunately, this partition statistic cannot interpret (1.3) combinatorially. Therefore,

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Dyson further conjectured that there exists another statistic, which he named the 'crank', providing a combinatorial interpretation of (1.3). This partition statistic was discovered by Andrews and Garvan [3] in 1988. For a partition λ, let $l(\lambda)$ denote the largest part of λ, let $\omega(\lambda)$ and $\mu(\lambda)$ denote the number of ones in λ and the number of parts of λ that are larger than $\omega(\lambda)$, respectively. The crank is defined by

$$\operatorname{crank}(\lambda) = \begin{cases} l(\lambda) & \text{if } \omega(\lambda) = 0, \\ \omega(\lambda) - \mu(\lambda) & \text{if } \omega(\lambda) > 0. \end{cases}$$

Let $c_o(n)$ denote the number of partitions of n with odd crank. The generating function of $c_o(n)$ is given by

$$\sum_{n=0}^{\infty} c_o(n) q^n = \frac{1}{2}\Big(\frac{1}{(q;q)_\infty} - \frac{(q;q)_\infty^3}{(q^2;q^2)_\infty^2}\Big).$$

Throughout the rest of this paper, we always assume that q is a complex number such that $|q| < 1$ and adopt the following customary notation:

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1 - aq^k),$$

$$(a_1, a_2, \ldots, a_m; q)_\infty := (a_1;q)_\infty (a_2;q)_\infty \cdots (a_m;q)_\infty.$$

Recently, Banerjee and Dastidar [5] considered some arithmetic properties of $c_o(n)$. By means of q-series manipulations, Banerjee and Dastidar [5, (1.10)] proved that for any $n \geq 0$,

$$c_o(5n + 4) \equiv 0 \pmod{10}.$$

Based on computer experiments, they conjectured a congruence modulo 4 satisfied by $c_o(n)$.

CONJECTURE 1.1. We have $c_o(2n) \equiv 0 \pmod 4$ for any $n \geq 0$.

Banerjee and Dastidar [5] verified that Conjecture 1.1 holds for any $1 \leq n \leq 2000$. By using some q-series techniques, we not only confirm the above congruence modulo 4, but also establish another congruence modulo 8.

THEOREM 1.2. *For any $n \geq 0$,*

$$c_o(2n) \equiv 0 \pmod 4, \tag{1.4}$$

$$c_o(4n) \equiv 0 \pmod 8. \tag{1.5}$$

2. Proof of Theorem 1.2

To prove (1.4) and (1.5), we need the following three auxiliary identities.

LEMMA 2.1 [2, Lemma 4.1]. *We have*

$$\frac{1}{(q;q)_\infty} = \frac{1}{(q^2;q^2)_\infty^2}((-q^6, -q^{10}, q^{16}; q^{16})_\infty + q(-q^2, -q^{14}, q^{16}; q^{16})_\infty). \tag{2.1}$$

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LEMMA 2.2 (Jacobi's identity [6, Theorem 1.3.9]).

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \tag{2.2}$$

LEMMA 2.3 (Jacobi's triple product identity [1, Lemma 1.2.2]).

$$\sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2} = (-a,-b,ab;ab)_\infty, \quad |ab|<1. \tag{2.3}$$

Now we are in a position to prove Theorem 1.2.

PROOF OF THEOREM 1.2. Define the sequence $\{A(n)\}_{n\geq 0}$ by

$$\sum_{n=0}^{\infty} A(n)q^n = \frac{1}{(q;q)_\infty} - \frac{(q;q)_\infty^3}{(q^2;q^2)_\infty^2}. \tag{2.4}$$

Therefore, (1.4) and (1.5) are equivalent respectively to

$$A(2n) \equiv 0 \pmod 8, \tag{2.5}$$

and

$$A(4n) \equiv 0 \pmod{16}. \tag{2.6}$$

However, from (2.2),

$$\begin{aligned}
(q;q)_\infty^3 &= \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}\\
&= \sum_{n=0}^{\infty}(-1)^{8n}(16n+1)q^{4n(8n+1)} + \sum_{n=0}^{\infty}(-1)^{8n+1}(16n+3)q^{(4n+1)(8n+1)}\\
&\quad + \sum_{n=0}^{\infty}(-1)^{8n+2}(16n+5)q^{(4n+1)(8n+3)} + \sum_{n=0}^{\infty}(-1)^{8n+3}(16n+7)q^{(4n+2)(8n+3)}\\
&\quad + \sum_{n=0}^{\infty}(-1)^{8n+4}(16n+9)q^{(4n+2)(8n+5)} + \sum_{n=0}^{\infty}(-1)^{8n+5}(16n+11)q^{(4n+3)(8n+5)}\\
&\quad + \sum_{n=0}^{\infty}(-1)^{8n+6}(16n+13)q^{(4n+3)(8n+7)} + \sum_{n=0}^{\infty}(-1)^{8n+7}(16n+15)q^{(4n+4)(8n+7)},
\end{aligned}$$

from which we further obtain that

$$\begin{aligned}
(q;q)_\infty^3 \equiv{}& \sum_{n=0}^{\infty} q^{4n(8n+1)} - 3\sum_{n=0}^{\infty} q^{(4n+1)(8n+1)}\\
&+ 5\sum_{n=0}^{\infty} q^{(4n+1)(8n+3)} - 7\sum_{n=0}^{\infty} q^{(4n+2)(8n+3)}
\end{aligned}$$

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$$-7\sum_{n=0}^{\infty}q^{(4n+2)(8n+5)}+5\sum_{n=0}^{\infty}q^{(4n+3)(8n+5)}$$
$$-3\sum_{n=0}^{\infty}q^{(4n+3)(8n+7)}+\sum_{n=0}^{\infty}q^{(4n+4)(8n+7)} \pmod{16}. \tag{2.7}$$

Replacing n by $-n-1$ in the last four infinite sums in (2.7),

$$\sum_{n=0}^{\infty}q^{(4n+2)(8n+5)}=\sum_{n=-\infty}^{-1}q^{(4n+2)(8n+3)}, \tag{2.8}$$

$$\sum_{n=0}^{\infty}q^{(4n+3)(8n+5)}=\sum_{n=-\infty}^{-1}q^{(4n+1)(8n+3)}, \tag{2.9}$$

$$\sum_{n=0}^{\infty}q^{(4n+3)(8n+7)}=\sum_{n=-\infty}^{-1}q^{(4n+1)(8n+1)}, \tag{2.10}$$

$$\sum_{n=0}^{\infty}q^{(4n+4)(8n+7)}=\sum_{n=-\infty}^{-1}q^{4n(8n+1)}. \tag{2.11}$$

Substituting (2.8)–(2.11) into (2.9),

$$(q;q)_\infty^3\equiv\sum_{n=-\infty}^{\infty}q^{4n(8n+1)}-3\sum_{n=-\infty}^{\infty}q^{(4n+1)(8n+1)}$$
$$+5\sum_{n=-\infty}^{\infty}q^{(4n+1)(8n+3)}-7\sum_{n=-\infty}^{\infty}q^{(4n+2)(8n+3)} \pmod{16}.$$

Thanks to (2.3),

$$(q;q)_\infty^3\equiv(-q^{28},-q^{36},q^{64};q^{64})_\infty-3q(-q^{20},-q^{44},q^{64};q^{64})_\infty$$
$$+5q^3(-q^{12},-q^{52},q^{64};q^{64})_\infty-7q^6(-q^4,-q^{60},q^{64};q^{64})_\infty \pmod{16}. \tag{2.12}$$

Substituting (2.1) and (2.12) into (2.4) yields

$$\sum_{n=0}^{\infty}A(n)q^n\equiv\frac{1}{(q^2;q^2)_\infty^2}((-q^6,-q^{10},q^{16};q^{16})_\infty+q(-q^2,-q^{14},q^{16};q^{16})_\infty)$$
$$-\frac{1}{(q^2;q^2)_\infty^2}((-q^{28},-q^{36},q^{64};q^{64})_\infty-3q(-q^{20},-q^{44},q^{64};q^{64})_\infty$$
$$+5q^3(-q^{12},-q^{52},q^{64};q^{64})_\infty-7q^6(-q^4,-q^{60},q^{64};q^{64})_\infty) \pmod{16}. \tag{2.13}$$

Taking all terms of the form q^{2n} in (2.13), after simplification,

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$$\sum_{n=0}^{\infty} A(2n)q^n \equiv \frac{1}{(q;q)_\infty^2}((-q^3,-q^5,q^8;q^8)_\infty - (-q^{14},-q^{18},q^{32};q^{32})_\infty + 7q^3(-q^2,-q^{30},q^{32};q^{32})_\infty) \pmod{16}. \tag{2.14}$$

According to (2.3),

$$\begin{aligned}(-q^3,-q^5,q^8;q^8)_\infty &= \sum_{n=-\infty}^{\infty} q^{4n^2+n} \\ &= \sum_{n=-\infty}^{\infty} q^{4(2n)^2+2n} + \sum_{n=-\infty}^{\infty} q^{4(2n-1)^2+(2n-1)} \\ &= \sum_{n=-\infty}^{\infty} q^{16n^2+2n} + \sum_{n=-\infty}^{\infty} q^{16n^2-14n+3} \\ &= (-q^{14},-q^{18},q^{32};q^{32})_\infty + q^3(-q^2,-q^{30},q^{32};q^{32})_\infty, \end{aligned} \tag{2.15}$$

where we have used (2.3) in the last step. Combining (2.14) and (2.15) gives

$$\sum_{n=0}^{\infty} A(2n)q^n \equiv 8q^3 \frac{(-q^2,-q^{30},q^{32};q^{32})_\infty}{(q;q)_\infty^2} \pmod{16}. \tag{2.16}$$

The congruence (2.5) follows immediately from (2.16).

Moreover, from the congruence $(q;q)_\infty^2 \equiv (q^2;q^2)_\infty \pmod 2$,

$$\sum_{n=0}^{\infty} A(2n)q^n \equiv 8q^3 \frac{(-q^2,-q^{30},q^{32};q^{32})_\infty}{(q^2;q^2)_\infty} \pmod{16}. \tag{2.17}$$

The congruence (2.6) follows immediately from (2.17).

This completes the proof of Theorem 1.2. □

3. Concluding remarks

We conclude this paper with two remarks.

First, the numerical evidence suggests the following conjecture.

CONJECTURE 3.1. We have

$$\lim_{n\to\infty} \frac{\#\{m \mid c_o(2m+1) \equiv 0 \pmod 4,\ \ 1 \le m \le n\}}{n} = \frac{1}{2},$$

$$\lim_{n\to\infty} \frac{\#\{m \mid c_o(2m) \equiv 0 \pmod 8,\ \ 1 \le m \le n\}}{n} = \frac{1}{4},$$

$$\lim_{n\to\infty} \frac{\#\{m \mid c_o(4m+2) \equiv 0 \pmod 8,\ \ 1 \le m \le n\}}{n} = \frac{1}{2},$$

$$\lim_{n\to\infty} \frac{\#\{m \mid c_o(4m) \equiv 0 \pmod{16},\ \ 1 \le m \le n\}}{n} = \frac{1}{2}.$$

Second, it would be interesting find a combinatorial proof of (1.4) and (1.5).

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