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STRONGLY REGULAR GRAPHS DERIVED FROM COMBINATORIAL DESIGNS

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1. Introduction. Several concepts in discrete mathematics such as block designs, Latin squares, Hadamard matrices, tactical configurations, error-correcting codes, geometric configurations, finite groups, and graphs are by no means independent. Combinations of these notions may serve the development of any one of them, and sometimes reveal hidden interrelations. In the present paper a central role in this respect is played by the notion of strongly regular graph, the definition of which is recalled below.

In § 2, a fibre-type construction for graphs is given which, applied to block designs with $\lambda = 1$ and Hadamard matrices, yields strongly regular graphs. The method, although still limited in its applications, may serve further developments. In § 3 we deal with block designs, first considered by Shrikhande [**22**], in which the number of points in the intersection of any pair of blocks attains only two values. Investigating the relations between these designs and the strongly regular graphs formed by their blocks, we extend work by Kalbfleisch and Stanton [**23**] who called the designs quasi-symmetric. The methods of the preceding sections are applied in § 4 to the construction of symmetric Hadamard matrices with constant diagonal. These matrices are related to special strong graphs. It was pointed out to us by Esther Seiden and by Michael Doob that symmetric Hadamard matrices with constant diagonal are involved in various current investigations. Three methods of construction and several detailed examples are given.

In § 5, tactical configurations, derived from the Steiner system (1; 24, 8, 5), by means of the methods of § 3 lead to strongly regular graphs of orders 253, 176, 120, 100, 77, and 56. Some of these are related to recently discovered graphs (Gewirtz [**4**]) and simple groups (Higman and Sims [**11**]); others seem to be new. These graphs are connected with the strongly regular graph of order 2048 which in the final section (§ 6) is derived from the extended Golay code (24, 12).

The graphs considered in the present paper are undirected, without loops, and without multiple edges. We make use of *adjacency matrices* that have elements 0 on the diagonal, -1 or $+1$ elsewhere according as the corresponding vertices are adjacent or non-adjacent, respectively (cf. [**14**; **21**; **19**; **20**]). We denote by I the unit matrix, by J the all-one matrix, by O the all-zero matrix, and by j the all-one vector, of some order.

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A non-void and non-complete graph of order v is defined to be a *strong graph* (cf. [**20**, Theorem 4; **21**]) if its adjacency matrix A satisfies the equation

$$(A - \rho_1 I)(A - \rho_2 I) = (v - 1 + \rho_1\rho_2)J, \qquad \rho_1 > \rho_2.$$

It is known that the real numbers ρ_1 and ρ_2 are odd integers unless $\rho_1 + \rho_2 = 0$. They are the only eigenvalues of A with the possible exception of one simple, integer eigenvalue ρ_0 which satisfies

$$(\rho_0 - \rho_1)(\rho_0 - \rho_2) = v(v - 1 + \rho_1\rho_2).$$

Strongly regular graphs are strong graphs that satisfy $AJ = \rho_0 J$, i.e., that are regular. This concept is due to Bose [**2**] who used a different definition (cf. [**19**]). Strong graphs with $v - 1 + \rho_1\rho_2 \neq 0$ are strongly regular [**20**, Theorem **5**].

Other combinatorial notions are introduced in the text or can be found in the books of Ryser [**18**] and Hall [**9**].

2. A construction method for graphs. Let B be the adjacency matrix of any graph of order n and let $\lambda_1, \ldots, \lambda_b, \rho_2$ be the eigenvalues of B, with $\rho_2 < 0$ of multiplicity $n - b$. From $\operatorname{tr} B = 0$ and $\operatorname{tr} B^2 = n(n-1)$ we have

$$\lambda_1 + \ldots + \lambda_b = -(n-b)\rho_2, \quad \lambda_1^2 + \ldots + \lambda_b^2 = n(n-1) - (n-b)\rho_2^2.$$

This implies that

$$(1/b)(n-b)^2\rho_2^2 \leqq n(n-1) - (n-b)\rho_2^2$$

with equality if and only if $\lambda_1 = \ldots = \lambda_b$. Therefore, we have [**14**, Lemma **6.1**] the following result.

THEOREM 2.1. *Any negative eigenvalue ρ_2, of multiplicity $n - b$, of the adjacency matrix of any graph of order n satisfies*

$$\rho_2 \geqq -(n-b)^{-1}(b(n-1)(n-b))^{1/2},$$

equality holding if and only if all other eigenvalues are mutually equal.

Let B be the adjacency matrix of any graph of order n and let

$$\lambda_1, \ldots, \lambda_{b-1}, \rho_0, \rho_2$$

be the eigenvalues of B, with $\rho_2 < 0$ of multiplicity $n - b$. From $\operatorname{tr} B = 0$ and $\operatorname{tr} B^2 = n(n-1)$ we have

$$\lambda_1 + \ldots + \lambda_{b-1} = -(n-b)\rho_2 - \rho_0,$$
$$\lambda_1^2 + \ldots + \lambda_{b-1}^2 = n(n-1) - (n-b)\rho_2^2 - \rho_0^2.$$

This implies that

$$(b-1)^{-1}[-(n-b)\rho_2 - \rho_0]^2 \leqq n(n-1) - (n-b)\rho_2^2 - \rho_0^2,$$
$$(n-b)(n-1)\rho_2^2 + 2(n-b)\rho_2\rho_0 + b\rho_0^2 \leqq n(n-1)(b-1),$$

with equality if and only if $\lambda_1 = \ldots = \lambda_{b-1}$. The last inequality, together

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with $\rho_2 < 0$, defines a domain $\mathscr{D}$ in the (ρ_2, ρ_0) plane. Its boundary consists of a *half ellipse* $\mathscr{E}$. For $(\rho_2, \rho_0) \in \mathscr{E}$, all $\lambda_1, \ldots, \lambda_{b-1}$ are denoted by ρ_1 and we have

$$\rho_0 + (b-1)\rho_1 + (n-b)\rho_2 = 0.$$

Therefore, the following theorem is proved.

THEOREM 2.2. *Graphs with $\rho_0 \neq \rho_2$ are strong graphs if and only if $(\rho_2, \rho_0) \in \mathscr{E}$.*

COROLLARY 2.3.

(i) $$\rho_1 = \rho_0 \Leftrightarrow \left(-\frac{1}{n-b}(b(n-1)(n-b))^{1/2}, \frac{1}{b}(b(n-1)(n-b))^{1/2}\right) \in \mathscr{E},$$

(ii) $$\rho_1 = \frac{b-2}{b(b-1)}((b-1)n(n-b))^{1/2}$$
$$\Leftrightarrow \left(-\frac{1}{n-b}((b-1)n(n-b))^{1/2}, \frac{2}{b}((b-1)n(n-b))^{1/2}\right) \in \mathscr{E},$$

(iii) $$\rho_1 = \frac{1}{b-1}((b-1)n(n-b))^{1/2}$$
$$\Leftrightarrow \left(-\frac{1}{n-b}((b-1)n(n-b))^{1/2}, 0\right) \in \mathscr{E}.$$

This is proved by elementary calculations.

The *construction method* which we are about to describe derives a graph from a given balanced incomplete block design (=block design) with parameters $v, b, k, r, \lambda = 1$; in some cases the graph obtained by this method is a strongly regular graph. Let N denote the $v \times b$ incidence matrix of the points and the blocks of a block design in which $\lambda = 1$, and let L denote any $x \times r$ matrix all of whose elements are ± 1 such that

$$LL^{\mathrm{T}} = rI + B,$$

where B is the adjacency matrix of some graph of order x. We construct a $vx \times b$ matrix P by transforming each row of N into an $x \times b$ matrix; this is done by replacing the r successive ones in the row by the r successive columns of L, and by replacing the $b - r$ zeros in the row by zero columns of size x. Then P satisfies

$$PP^{\mathrm{T}} = rI + A$$

for some symmetric A of order vx with zero on the diagonal and ± 1 elsewhere. If the vx row vectors of P span a vector space of dimension b, then the smallest eigenvalue ρ_2 of A has multiplicity $vx - b$, and $\rho_2 = -r$. If PP^{T} is regular with $PP^{\mathrm{T}}J = (\rho_0 + r)J$, then A is the adjacency matrix of a regular graph with $AJ = \rho_0 J$.

THEOREM 2.4. *If there exists a block design with parameters* $v, b, k, r, \lambda = 1$ *and if there exists a Hadamard matrix of order* $r + 1$, *then there exists a strongly regular graph of order* $v(r + 1)$ *whose adjacency matrix* A *satisfies*

$$[A + rI][A - (v + k - 1)I] = 0, \qquad AJ = -rJ.$$

Proof. Normalizing the first column of the Hadamard matrix H we take L such that $H = [j\, L]$; then

$$LL^{\mathrm{T}} = rI + I - J, \qquad JL = 0.$$

Application of our construction to this L, with $x = r + 1$, and to the given block design yields an adjacency matrix A of order $n = v(r + 1)$. By use of the relations $bk = vr$ and $v - 1 = r(k - 1)$ we observe that the smallest eigenvalue $\rho_2 = -r$ satisfies

$$\rho_2 = -r = \frac{-1}{n - b}(b(n - 1)(n - b))^{1/2}.$$

By Theorem 2.1 this implies that all other eigenvalues of A are equal to ρ_1, say. From

$$\operatorname{tr} A = \rho_1 b - r[v(r + 1) - b] = 0$$

we conclude that $\rho_1 = v + k - 1$. Finally, $JL = 0$ implies $JP = 0$ and $JA = -rJ$. Hence, the theorem is proved.

COROLLARY 2.5. *The columns of* P *constitute an orthogonal basis for the eigenspace of* A *belonging to* ρ_1.

Proof. As to the $v(r + 1) \times b$ matrix P, we know that PP^{T} has the eigenvalues $v + k - 1 + r$ and 0, of multiplicities b and $v(r + 1) - b$, respectively. This implies that

$$P^{\mathrm{T}}P = (v + k - 1 + r)I = (r + 1)kI, \qquad AP = (v - 1 + k)P,$$

from which the assertion follows.

THEOREM 2.6. *If there exists a finite projective plane* $\mathrm{PG}(2, r - 1)$, *and if there exists a Hadamard matrix of order* $r + 1$, *then there exists a strongly regular graph of order* $r(r^2 - r + 1)$ *with eigenvalues* $\rho_0 = 0$, $\rho_1 = r^2 - r + 1$, $\rho_2 = -r$.

Proof. Normalizing the first row and column of the Hadamard matrix H we take L such that

$$H = \begin{bmatrix} -1 & j^{\mathrm{T}} \\ j & L \end{bmatrix};$$

then

$$LL^{\mathrm{T}} = rI + I - J, \qquad JL = LJ = J.$$

Application of our construction to this L, with $x = r$, and to the block design with parameters $v = b = r^2 - r + 1$, $k = r$, $\lambda = 1$, yields an adjacency matrix A of order $n = r(r^2 - r + 1)$. Since $JPP^{\mathrm{T}} = rJP^{\mathrm{T}} = rJ$, A has the eigenvalues $\rho_0 = 0$, and $\rho_2 = -r$ of multiplicity $(r - 1)(r^2 - r + 1)$. The

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proof is completed by observing that the point (ρ_2, ρ_0) is on the ellipse $\mathscr{E}$ and by applying Theorem 2.2.

THEOREM 2.7. *If there exists a finite projective plane* PG$(2, r-1)$ *and if there exists a square matrix L of order r all of whose elements are ± 1 satisfying*

$$LL^{\mathrm{T}} = rI - I + J, \qquad JL = LJ = (2r-1)^{1/2}J,$$

then there exists a strongly regular graph of order $r(r^2 - r + 1)$ with eigenvalues $\rho_0 = 2r(r-1)$, $\rho_1 = r^2 - r - 1$, $\rho_2 = -r$.

The proof is analogous to that of Theorem 2.6, the difference being that now we have $JPP^{\mathrm{T}} = r(2r-1)J$, whence $\rho_0 = 2r(r-1)$.

Remark. Finite projective planes of orders $\equiv 2 \pmod 4$ and $2s(s+1)$ are only known to exist for orders 2 and 4, respectively. In these cases, Theorem 2.6 for $r = 3$ leads to the triangular graph $T(7)$, and Theorem 2.7 for $r = 5$ leads to a strongly regular graph with

$$L = J_5 - 2I_5, \quad v = 105; \qquad \rho_0 = 40, \quad \rho_1 = 19, \quad \rho_2 = -5.$$

It would be interesting to know whether Theorems 2.6 and 2.7 have any further consequences, for instance whether strongly regular graphs exist of order 1221 having the eigenvalues and multiplicities

$$\rho_0 = 0, \quad \rho_1 = 111, \quad \rho_2 = -11, \quad \mu_0 = 1, \quad \mu_1 = 110, \quad \mu_2 = 1110.$$

THEOREM 2.8. *Let there exist a strongly regular graph obtained by our construction from some block design with parameters $v, b, k, r, \lambda = 1$ and from some square L of order r, all of whose elements are ± 1, with*

$$LL^{\mathrm{T}} = rI + B, \qquad JL = LJ = lJ,$$

where B has zero diagonal and ± 1 elsewhere. Then the block design is a projective geometry PG$(2, r-1)$ *and either $B = I - J$, $l^2 = 1$ or $B = J - I$, $l^2 = 2r - 1$.*

Proof. From the hypotheses it is seen that L is non-singular, hence $l \neq 0$. Furthermore, the strongly regular graph has the eigenvalues $\rho_0 = kl^2 - r$ and $\rho_2 = -r$ (the latter of multiplicity $vr - b$). By Theorem 2.2, it follows that $(-r, kl^2 - r) \in \mathscr{E}$, whence, by use of $vr = bk$ and $v - 1 = r(k-1)$, we have

$$(bk - b)(vr - 1)r^2 - 2(bk - b)r(kl^2 - r) + b(kl^2 - r)^2 = bk(vr - 1)(b - 1),$$
$$k^2l^4 - 2rk^2l^2 + k^2r^2 = k^2r^2 - 2r^2k + r^2 + (r + r^2k - r^2 - 1)(r - k),$$
$$k^2(l^2 - r)^2 = (r-1)^2(kr + k - r).$$

From $(l^2 - r)J = BJ$ we have $(l^2 - r)^2 \leqq (r-1)^2$, whence $k^2 \geqq kr + k - r$, $k \geqq r$, $b \leqq v$. Therefore, by Fisher's inequality $v \leqq b$, we have $v = b$, and hence the block design is a projective geometry. This implies that

$$l^2 - r = r - 1, \quad B = J - I \quad \text{or} \quad l^2 - r = 1 - r, \quad B = I - J,$$

which proves the theorem.

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Remark. Theorem 2.8 explains the restriction in the hypotheses of Theorems 2.6 and 2.7. It would be interesting to extend the construction method by using several distinct matrices L.

3. Quasi-symmetric block designs. Let N be the $(0, 1)$ point-block incidence matrix of a block design with parameters v, b, k, r, λ. The following relations hold:

$$JN = kJ, \quad NJ = rJ, \quad NN^{\mathrm{T}} = (r - \lambda)I + \lambda J,$$
$$bk = vr, \quad r(k-1) = \lambda(v-1).$$

In this section we deal with special block designs having the property that the number of points in the intersection of any pair of blocks attains only two values, $x + y$ and $x - y$, say,

Definition. A *quasi-symmetric* block design with matrix A is a block design whose point-block incidence matrix N satisfies

$$N^{\mathrm{T}}N = kI + x(J - I) - yA, \qquad 0 < y \leqq x < k,$$

for some symmetric A with elements 0 on the diagonal and ± 1 elsewhere.

THEOREM 3.1. *The matrix belonging to a quasi-symmetric block design is the adjacency matrix of a strongly regular graph.*

Proof. Since NN^{T} has the eigenvalues rk and $r - \lambda$, the matrix $N^{\mathrm{T}}N$ has the eigenvalues rk, $r - \lambda$, 0, with multiplicity 1, $v - 1$, $b - v$, respectively. Therefore, the eigenvalues ρ_0, ρ_1, ρ_2 of A are the numbers given by

$$\rho_0 y = (b-1)x - k(r-1), \qquad \rho_1 y = k - x, \qquad \rho_2 y = k - x - r + \lambda,$$

with multiplicity 1, $b - v$, $v - 1$, respectively. Since j is the eigenvector of ρ_0, we have

$$b(A - \rho_1 I)(A - \rho_2 I) = (\rho_0 - \rho_1)(\rho_0 - \rho_2)J, \qquad AJ = \rho_0 J;$$

hence A is the adjacency matrix of a strongly regular graph.

Remark. From $\operatorname{tr} A^2 = b(b-1)$, the following necessary condition for the parameters of a quasi-symmetric block design is obtained:

$$(b-1)(x^2 - y^2) - 2xk(r-1) + k(r-k) + k(k-1)\lambda = 0.$$

THEOREM 3.2. *The incidence matrix N and the adjacency matrix A of a quasi-symmetric block design are related by*

$$N(A - \rho_2 I) = (k/v)(\rho_0 - \rho_2)J.$$

Proof.

$$(r-\lambda)N + \lambda kJ = NN^{\mathrm{T}}N = kN + xrJ - xN - yNA,$$
$$0 = (k - x - r + \lambda)N + (xr - \lambda k)J - yNA,$$

from which the desired relation follows.

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In the following theorems, quasi-symmetric block designs which belong to some special strongly regular graphs are discussed. Theorem 3.4 is an extension of a result by Stanton and Kalbfleisch [**23**], who also obtained Theorem 3.3. In Theorem 3.5 it is seen that there exist strongly regular graphs which do not belong to any quasi-symmetric block design. The special graphs and designs are introduced in the proofs.

THEOREM 3.3. *The only quasi-symmetric block designs whose graph is a ladder graph are the designs consisting of two copies of a symmetric block design.*

Proof. The ladder graph [**20**] is the graph on $2n$ vertices whose adjacency matrix is the matrix

$$A = \begin{bmatrix} J - I & J - 2I \\ J - 2I & J - I \end{bmatrix}$$

with square blocks of size n. Its eigenvalues are $\rho_0 = 2n - 3$, $\rho_1 = 1$, $\rho_2 = -3$. For the parameters of the quasi-symmetric block design we have $b = 2n$ and (from $\operatorname{tr} A = 0$) $v = n$. Writing $N = [N_1\ N_2]$ with square N_1 and N_2 of order n we conclude from Theorem 3.2 that

$$N_1 = N_2, \quad N_1 J = kJ = JN_1, \qquad N_1 N_1^{\mathrm{T}} = (k - \tfrac{1}{2}\lambda)I + \tfrac{1}{2}\lambda J,$$

hence that N_1 is the incidence matrix of a symmetric block design with parameters v, k, $\frac{1}{2}\lambda$. Conversely, any design consisting of two copies of a symmetric block design is quasi-symmetric.

THEOREM 3.4. *The only quasi-symmetric block design with graph $H(n)$ is the double Hadamard design of order $n + 1$.*

Proof. The graph $H(n)$ is the complement of the ladder graph on $2n$ vertices [**20**] and has the adjacency matrix

$$A = \begin{bmatrix} I - J & 2I - J \\ 2I - J & I - J \end{bmatrix}$$

with square blocks of size n. This matrix has the eigenvalues $\rho_0 = 3 - 2n$, $\rho_1 = 3$, $\rho_2 = -1$. For the parameters of the quasi-symmetric block design we have $b = 2n$ and (from $\operatorname{tr} A = 0$) $v = n + 1$, whence $2nk = (n + 1)r$. Since v divides $2k$ and $v > k$, it follows that $v = 2k$, $r = n$, $\lambda = k - 1$. Since $\rho_1 = 3$, $\rho_2 = -1$ implies $x = y = \frac{1}{4}k$, we have

$$b = 2n, \quad v = n + 1, \quad r = n, \quad k = \tfrac{1}{2}(n + 1),$$
$$\lambda = \tfrac{1}{2}(n - 1), \quad x + y = \tfrac{1}{4}(n + 1), \quad x - y = 0.$$

Writing $N = [N_1\ N_2]$ with $(n + 1) \times n$ matrices N_1 and N_2 we conclude from Theorem 3.2 that

$$N_1 + N_2 = J, \qquad JN_1 = \tfrac{1}{2}(n + 1)J,$$

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and

$$N_1N_1^{\mathrm{T}} + (J - N_1)(J - N_1)^{\mathrm{T}} = \tfrac{1}{2}(n+1)I + \tfrac{1}{2}(n-1)J,$$
$$(2N_1 - J)(2N_1 - J)^{\mathrm{T}} = (n+1)I - J,$$
$$[2N_1 - J \quad j][2N_1 - J \quad j]^{\mathrm{T}} = (n+1)I.$$

Hence $[2N_1 - J \quad j]$ is a normalized Hadamard matrix of order $n + 1$. Conversely, if $H = [L \quad j]$ is any normalized Hadamard matrix of order $n + 1$, then the matrix $N = \frac{1}{2}[J + L \quad J - L]$ is the incidence matrix of a quasi-symmetric block design with graph $H(n)$ and with the parameters indicated above. We call this block design the *double Hadamard design* of order $n + 1$.

THEOREM 3.5. *There is no quasi-symmetric block design whose graph is the lattice graph $L_2(n)$ or its complement.*

Proof. Suppose that a quasi-symmetric block design exists whose graph is the lattice graph $L_2(n)$ on $b = n^2$ vertices. The eigenvalues are (cf. [**19**])

$$\rho_0 = (n-1)(n-3), \qquad \rho_1 = 3, \qquad \rho_2 = 3 - 2n.$$

From $\mathrm{tr}\, A = 0$ we have $v = 2n - 1$, so that $n^2k = (2n-1)r$, but this would imply that $b = n^2$ divides r, which is impossible.

Suppose that a quasi-symmetric block design exists whose graph is the complement of $L_2(n)$ on $b = n^2$ vertices. The eigenvalues are

$$\rho_0 = -(n-1)(n-3), \qquad \rho_1 = 2n - 3, \qquad \rho_2 = -3.$$

From $\mathrm{tr}\, A = 0$ we have $v = (n-1)^2 + 1$, so that $n^2k = (n-1)^2r + r$, but this would imply that $b = n^2$ divides $2r$, whence

$$n^2 = 2r, \qquad k - 1 = \tfrac{1}{2}n^2 - n, \qquad 4\lambda(n-1)^2 = n^3(n-2),$$

which is impossible. Therefore, these designs do not exist.

4. Symmetric Hadamard matrices with constant diagonal. A Hadamard matrix H is called *symmetric with constant diagonal* if $H = A \pm I$, A symmetric. Let the order of such a matrix be x, then we have

$$H^2 = xI, \qquad (A \pm I - x^{1/2}I)(A \pm I + x^{1/2}I) = 0.$$

Therefore, A is the adjacency matrix of a strong graph, and its eigenvalues are odd integers whose sum equals ∓ 2. This implies the following theorem.

THEOREM 4.1. *Symmetric Hadamard matrices with constant diagonal have order $4s^2$, s an integer. They exist if and only if strong graphs of that order exist with eigenvalues*

$$\rho_1 = 2s \mp 1, \qquad \rho_2 = -2s \mp 1.$$

For the remaining eigenvalue ρ_0 of A we have $\rho_0 = \rho_1$ or $\rho_0 = \rho_2$. If in addition $AJ = \rho_0 J$, then A, and also the Hadamard matrix $H = A \pm I$, is

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regular with $HJ = 2sJ$ or $HJ = -2sJ$. For the eigenvalues of these strongly regular graphs on $4s^2$ vertices there are the following possibilities:

$$\rho_0 = \rho_1 = 2s - 1, \quad \rho_2 = -2s - 1 \quad \text{and} \quad \rho_1 = 2s + 1, \quad \rho_0 = \rho_2 = -2s + 1,$$
$$\rho_1 = 2s - 1, \rho_0 = \rho_2 = -2s - 1 \quad \text{and} \quad \rho_0 = \rho_1 = 2s + 1, \quad \rho_2 = -2s + 1.$$

The first case corresponds to the pseudo Latin-square graphs $\mathrm{L}_s(2s)$ and their complements [**2**; **15**]. The second case corresponds to the negative-Latin-square graphs $\mathrm{NL}_s(2s)$ and their complements. Negative-Latin-square graphs $\mathrm{NL}_s(n)$ have been introduced by Mesner [**15**] who gave a construction for prime power n. The Clebsch graph, for $n = 4$, $s = 2$, and its complement, for $n = 4$, $s = 3$, belong to this class [**20**; **21**]. Summarizing we have the following theorem.

THEOREM 4.2. *Regular symmetric Hadamard matrices with constant diagonal of order* $4s^2$ *exist if and only if pseudo Latin-square graphs* $\mathrm{L}_s(2s)$ *or negative Latin-square graphs* $\mathrm{NL}_s(2s)$ *exist.*

We now present three methods to construct symmetric Hadamard matrices with constant diagonal. The first method consists in changing somewhat the argument used in [**7**, Theorem 4.2], which had its origin in a result of Ehlich [**3**]. We recall that a symmetric or skew C-matrix C_v of order v and the corresponding S-matrix S_{v-1} of order $v - 1$ are matrices with diagonal elements 0 and other elements $+1$ and -1 which satisfy

$$C = \begin{bmatrix} 0 & j^{\mathrm{T}} \\ \pm j & S \end{bmatrix},$$

$$CC^{\mathrm{T}} = (v-1)I, \quad C^{\mathrm{T}} = \pm C, \quad SS^{\mathrm{T}} = (v-1)I - J, \quad SJ = JS = 0, \quad S^{\mathrm{T}} = \pm S.$$

These matrices are symmetric for $v \equiv 2 \pmod 4$ and skew for $v \equiv 0 \pmod 4$.

THEOREM 4.3. *If there exists a symmetric S-matrix of order* $n + 1$, *and a skew S-matrix of order* $n - 1$ *which is symmetric with respect to its antidiagonal, then there exists a regular symmetric Hadamard matrix with constant diagonal of order* n^2.

Proof. Let $n \equiv 0 \pmod 4$ and suppose that there exist a symmetric S_{n+1}, a skew S_{n-1}, and a permutation matrix U_{n-1} such that

$$U_{n-1}^2 = I_{n-1}, \qquad U_{n-1}S_{n-1} = (U_{n-1}S_{n-1})^{\mathrm{T}}.$$

The matrix of order $n^2 - 1$,

$$K = U_{n-1}S_{n-1} \otimes S_{n+1} + U_{n-1} \otimes (J_{n+1} - I_{n+1}) - J_{n-1} \otimes I_{n+1},$$

is symmetric and satisfies

$$KJ = JK = J, \qquad KK^{\mathrm{T}} = n^2 I - J.$$

With the aid of the diagonal matrix F_{n+1}, whose diagonal elements are $+1$

and -1 alternatingly, the following matrices of order n^2 are defined:

$$H = \begin{bmatrix} -1 & j^{\mathrm{T}} \\ j & K \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0^{\mathrm{T}} \\ 0 & I_{n-1} \otimes F_{n+1} \end{bmatrix}, \qquad \bar{H} = GHG.$$

Then H and $\bar{H}$ are symmetric Hadamard matrices with constant diagonal and $\bar{H}$ is regular with $\bar{H}J = -nJ$. The theorem just proved applies for $n \equiv 0 \pmod 4$, $n + 1$ and $n - 1$ prime powers, for instance $n = 12$, since in this case the Paley construction may be used, cf. [**9**, p. 211; **7**].

The second method of construction is an application of § 2 and yields regular symmetric Hadamard matrices with constant diagonal of orders which are divisible by 16.

THEOREM 4.4. *If there exists a Hadamard matrix of order n, then there exists a regular symmetric Hadamard matrix with constant diagonal of order n^2.*

Proof. The pair design with parameters

$$v = n, \quad b = \tfrac{1}{2}n(n-1), \quad k = 2, \quad r = n - 1, \quad \lambda = 1$$

exists for all n. If a Hadamard matrix of order n exists, then Theorem 2.4 may be applied so as to obtain a strongly regular graph of order n^2 with adjacency matrix A satisfying

$$(A + nI - I)(A - nI - I) = 0, \qquad AJ = -(n-1)J.$$

Therefore, $H = A - I$ is a Hadamard matrix which meets the required conditions.

As a third method of construction, in order to obtain symmetric Hadamard matrices with constant diagonal of other orders, we investigate block designs with $\lambda = 1$. Since any two blocks have one or no points of intersection, these designs are quasi-symmetric in the sense of § 3 with

$$x + y = 1, \quad x - y = 0, \quad r = \frac{v-1}{k-1}, \qquad b = \frac{v(v-1)}{k(k-1)},$$

$$\rho_0 = 2k + 1 - 2v + \frac{v(v-1)}{k(k-1)} - \frac{2(v-1)}{k-1},$$

$$\rho_1 = 2k - 1, \quad \rho_2 = 2k + 1 - \frac{2(v-1)}{k-1}.$$

Some special cases are considered.

(i) If $v = k(k-1) + 1$, then the design is a symmetric block design with parameters $(k-1)^2 + (k-1) + 1, k, 1$, that is, a projective geometry $\mathrm{PG}(2, k-1)$, and its graph is the complete graph.

(ii) If $v = k(k-1) + k$, then we have

$$v = k^2, \quad b = k(k+1), \quad r = k + 1, \quad k = k, \quad \lambda = 1,$$
$$\rho_0 = -k^2 + k - 1, \qquad \rho_1 = 2k - 1, \quad \rho_2 = -1.$$

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The graph is the complement of the graph consisting of $k+1$ mutually non-adjacent complete subgraphs of order k. In the case of prime k, this design exists. Indeed, its incidence matrix N is written in terms of the $k \times k$ permutation matrix P, defined by its elements $p_{i,i+1} = p_{k,1} = 1$ $(i = 1, \ldots, k-1)$ and $p_{i,j} = 0$ otherwise, as follows:

$$N = \begin{bmatrix} P & P^2 & \ldots & P^k & [j & 0 & \ldots & 0] \\ P^2 & P^4 & \ldots & P^{2k} & [0 & j & \ldots & 0] \\ . & . & \ldots & . & & & \ldots & \\ I & I & \ldots & I & [0 & 0 & \ldots & j] \end{bmatrix}.$$

(iii) If $v = 2k(k-1)+1$, then we have

$$v = 2k^2 - 2k + 1, \qquad b = (2k-1)^2 + 1, \qquad k = k,$$
$$r = 2k, \qquad \lambda = 1, \qquad \rho_1 = -\rho_0 - \rho_2 = 2k - 1.$$

The adjacency matrix of these designs satisfies

$$A^2 = (2k-1)^2 I, \qquad AJ = (1-2k)J,$$

hence is a regular orthogonal matrix with zero diagonal. These matrices exist for orders which exceed prime powers by 1 and for some other orders, for instance 226, cf. [**7**]. It would be interesting to know which of these matrices correspond to a quasi-symmetric design. For $k = 2$ we have the adjacency matrix of the triangular graph $T(5)$; this case was discussed in Theorem 3.6. For $k = 3$ we have $v = 13$, $b = 26$, and the design is one of the two existing mutually non-isomorphic Steiner triple systems on thirteen symbols. The two corresponding graphs of order 26 satisfy

$$A^2 = 25I, \qquad AJ = -5J.$$

These are the only graphs satisfying the equations and belonging to any quasi-symmetric design; this follows easily from the relations between the parameters. However, two further graphs satisfying the equations exist. These are obtained by suitable *complementation* (cf. [**14**; **19**; **20**; **21**]) of the graphs consisting of any one of the two existing non-isomorphic Latin-square graphs $L_3(5)$ together with an isolated vertex. They do not belong to any quasi-symmetric design since they are not equivalent to one of the Steiner graphs [**7**; **21**].

For $k = 4$ we have $v = 25$, $b = 50$, $\rho_1 = -\rho_0 = -\rho_2 = 7$. Apart from the quasi-symmetric designs no. 22 [**9**, p. 291], we have a further such design given by the incidence matrix

$$\begin{bmatrix} 1 & j^T & 0^T & 0^T & 0^T & 0^T & 0^T & 0^T \\ 1 & 0^T & j^T & 0^T & 0^T & 0^T & 0^T & 0^T \\ 1 & 0^T & 0^T & j^T & 0^T & 0^T & 0^T & 0^T \\ 1 & 0^T & 0^T & 0^T & j^T & 0^T & 0^T & 0^T \\ 0 & I & I & I & I & Q & 0 & I \\ 0 & I & P & P^2 & P^4 & I & Q & 0 \\ 0 & I & P^3 & P^6 & P^5 & 0 & I & Q \end{bmatrix}$$

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with I, P, Q circulant matrices of order 7, $P = (0, 1, 0, 0, 0, 0, 0)$, $Q = (0, 1, 1, 0, 1, 0, 0)$. The corresponding graph is not equivalent to the Latin-square graph $L_4(7)$ together with an isolated vertex, since it contains only one set of parallel cliques [private communication by Ph. Delsarte].

(iv) If $v = 2k(k-1) + k$, then we have

$$v = k(2k-1), \qquad b = (2k+1)(2k-1), \qquad r = 2k+1,$$
$$\rho_0 = -2, \qquad \rho_1 = 2k-1, \qquad \rho_2 = -2k-1.$$

The adjacency matrix of these designs satisfies

$$(A+I)^2 = 2k^2 I - J, \qquad AJ = -2J.$$

For the bordered matrix B of order $4k^2$ we have

$$B = \begin{bmatrix} 0 & j^{\mathrm{T}} \\ j & A \end{bmatrix}, \qquad (B+I)^2 = 4k^2 I.$$

Therefore, the following theorem† is proved.

THEOREM 4.5. *If there exists a block design with parameters*

$$v = k(2k-1), \quad b = 4k^2 - 1, \quad k = k, \quad r = 2k+1, \quad \lambda = 1,$$

then there exists a symmetric Hadamard matrix with constant diagonal of order $(2k)^2$.

If the blocks of the block design can be partitioned into two sets of orders $k(2k-1)$ and $(k+1)(2k-1)$, respectively, in such a way that each point appears exactly k and $k+1$ times, respectively, in each set, then there exists a regular symmetric Hadamard matrix with constant diagonal of order $4k^2$. Indeed, this partition induces the following partitioning of the adjacency matrix A of the design, from which the following bordered B is obtained:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_2^{\mathrm{T}} & A_3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -j^{\mathrm{T}} & j^{\mathrm{T}} \\ -j & A_1 & -A_2 \\ j & -A_2^{\mathrm{T}} & A_3 \end{bmatrix}.$$

Here A_1 and A_3 are square matrices of orders $k(2k-1)$ and $(k+1)(2k-1)$, respectively, and

$$A_1 J = (k-1)J, \quad A_2 J = (-k-1)J, \quad A_2^{\mathrm{T}} J = -kJ, \quad A_3 J = (k-2)J.$$

Hence $BJ = JB = (2k-1)J$, and therefore the Hadamard matrix $B + I$ is regular. In addition, if $[N_1 \;\; N_2]$ is the partitioned incidence matrix of the design, then it follows by Theorem 3.2 that the matrix $[j \;\; N_1 \;\; -N_2]$ represents the eigenspace of B belonging to the eigenvalue $\rho_2 = -1 - 2k$.

†S. S. Shrikhande informed us that this theorem is contained in a paper by S. S. Shrikhande and N. K. Singh [*On a method of constructing symmetrical balanced incomplete block designs*, Sankhyā A *24* (1962), 25–32].

We now mention some examples to Theorem 4.2.

For $k = 3$ we have $v = 15$, $b = 35$, and the design is one of the eighty existing non-isomorphic Steiner triple systems on fifteen symbols [**10**]. This yields eighty symmetric Hadamard matrices with constant diagonal of order 36. There are twelve further such matrices of order 36, which are obtained by use of Theorem 4.1 from the twelve non-isomorphic Latin-square graphs $L_3(6)$. Among these 92 matrices, only two are equivalent, both originating from Latin squares [private communication by F. C. Bussemaker].

For $k = 5$ we have $v = 45$, $b = 99$, $k = 5$, $r = 11$, $\lambda = 1$. Since this design exists [**9**, p. 294, no. 51], we have a symmetric Hadamard matrix with constant diagonal of order 100. Explicitly, this amounts to the following construction. With the aid of the square matrices C and D of order 9, defined by

$$P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad P = \begin{bmatrix} I_3 & 0_3 & 0_3 \\ 0_3 & P_3 & 0_3 \\ 0_3 & 0_3 & P_3^{\,2} \end{bmatrix},$$

$$C = \begin{bmatrix} I_3 - J_3 & J_3 - 2I_3 & J_3 - 2I_3 \\ J_3 - 2I_3 & I_3 - J_3 & J_3 - 2I_3 \\ J_3 - 2I_3 & J_3 - 2I_3 & I_3 - J_3 \end{bmatrix}, \qquad D = P^{\mathrm{T}}CP,$$

we form the following matrix A of order 99:

$$\begin{bmatrix}
J-I & C-I & C-I & C-I & C-I & C-I & -C-I & -C-I & -C-I & -C-I & -C-I \\
C-I & C & I+C & I-C & I-C & I+C & -J & I-D & I+D & I-D & I+D \\
C-I & I+C & C & I+C & I-C & I-C & I+D & -J & I-D & I+D & I-D \\
C-I & I-C & I+C & C & I+C & I-C & I-D & I+D & -J & I-D & I+D \\
C-I & I-C & I-C & I+C & C & I+C & I+D & I-D & I+D & -J & I-D \\
C-I & I+C & I-C & I-C & I+C & C & I-D & I+D & I-D & I+D & -J \\
-C-I & -J & I+D & I-D & I+D & I-D & -C & I-C & I+C & I+C & I-C \\
-C-I & I-D & -J & I+D & I-D & I+D & I-C & -C & I-C & I+C & I+C \\
-C-I & I+D & I-D & -J & I+D & I-D & I+C & I-C & -C & I-C & I+C \\
-C-I & I-D & I+D & I-D & -J & I+D & I+C & I+C & I-C & -C & I-C \\
-C-I & I+D & I-D & I+D & I-D & -J & I-C & I+C & I+C & I-C & -C
\end{bmatrix}$$

This matrix A is the adjacency matrix of the graph obtained from the design with parameters 45, 99, 5, 11, 1. The desired symmetric Hadamard matrix with constant diagonal of order 100 is obtained by bordering $A + I$. By inspection, it is seen that a second such matrix is obtained by changing any D into C in the above matrix A. Both A and this second matrix can be made regular by suitable complementation, by use of the method explained after Theorem 4.5. This implies the existence of two non-trivial symmetric block designs with parameters $v = 100$, $k = 45$, $\lambda = 20$. If complementations could be found that would make the graph with adjacency matrix

$$\begin{bmatrix} 0 & j^{\mathrm{T}} \\ j & A \end{bmatrix}$$

regular in such a way that it consisted of five parallel classes of 10-cliques, then three orthogonal Latin squares of order 10 would have been found. The

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obvious complementations, by which the graph is made regular, do not possess this property.

5. Tactical configurations. A *tactical configuration* $(b_t; v, k, t)$ is a set V of v elements and a collection of k-subsets of V, called blocks, such that every t-subset of V is contained in exactly b_t blocks. Tactical configurations with $t \geqq 2$ are block designs with parameters b, v, r, k, λ. For $i = 0, 1, \ldots, t$ the number b_i of blocks containing any i-subset of V is a constant and we have

$$b_i(k-i) = b_{i+1}(v-i), \quad b_i\binom{v}{i} = b_0\binom{k}{i}, \quad b_i\binom{k-i}{t-i} = b_t\binom{v-i}{t-i},$$

where $b_0 = b$, $b_1 = r$, $b_2 = \lambda$. This well-known result [**1**; **12**] may be proved by consideration of the *derived* tactical configuration $(b_t; v-1, k-1, t-1)$ which is defined by the blocks of $(b_t; v, k, t)$ containing any given element of V. The remaining blocks constitute the so-called *residual* tactical configuration $(b_{t-1} - b_t; v-1, k, t-1)$. Tactical configurations $(1; v, k, t)$, with $b_t = 1$, are called *Steiner systems*.

Witt [**24**; **25**] has proved the existence and uniqueness of the Steiner system $(1; 24, 8, 5)$ which has the Mathieu group M_{24} as its automorphism group. This Steiner system and its derived and residual tactical configurations have parameters as follows

no.	v	k	t	b_0	b_1	b_2	b_3	b_4	b_5	s
1	24	8	5	759	253	77	21	5	1	4, 2, 0
2	23	7	4	253	77	21	5	1		3, 1
3	22	6	3	77	21	5	1			2, 0
4	21	5	2	21	5	1				1
5	23	8	4	506	176	56	16	4		4, 2, 0
6	22	7	3	176	56	16	4			3, 1
7	21	6	2	56	16	4				2, 0
8	22	8	3	330	120	40	12			4, 2, 0
9	21	7	2	120	40	12				3, 1
10	21	8	2	210	80	28				4, 2, 0

The last column gives the numbers s of the elements that are common to pairs of blocks. Indeed, from the following Lemma 5.1 it follows that in the design no. 1, any two blocks have either 4 or 2 or 0 points of intersection. Then for the derived and the residual tactical configurations we arrive at the other values of s. Note that no. 4 is the symmetric block design PG(2, 4).

LEMMA 5.1. *In the Steiner system* $(1; 24, 8, 5)$, *any block intersects* 280 *blocks in exactly four elements*, 448 *blocks in exactly two elements*, *and* 30 *blocks in no elements*.

Proof. Let β be any fixed block. The number of its 4-subsets equals $\binom{8}{4} = 70$ and each of these 4-subsets is contained in four other blocks. Hence β intersects 280 other blocks in four elements. The number of 3-subsets of β, each of

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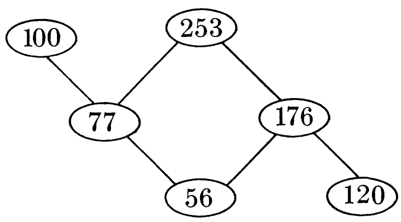
which is contained in 20 other blocks, equals 56. However, the 1120 not necessarily distinct blocks which intersect β in three elements have been counted already. Hence β intersects no blocks in exactly three elements. The number of 2-subsets, each contained in 76 other blocks, equals 28, hence β intersects $28 \times 76 - 280 \times 6 = 448$ blocks in exactly two elements. Since $8 \times 252 = 4 \times 280 + 2 \times 448$ the block β intersects no blocks in one element, hence it intersects the remaining thirty blocks in no element.

THEOREM 5.2. *There exist strongly regular graphs of order b, eigenvalues ρ_i, multiplicities μ_i, $i = 0, 1, 2$, as follows:*

no.	b	ρ_0	ρ_1	ρ_2	μ_0	μ_1	μ_2
2	253	-28	5	-51	1	230	22
3	77	-44	5	-11	1	55	21
6	176	-35	5	-35	1	154	21
7	56	-35	5	-7	1	35	20
9	120	-35	5	-23	1	99	20

Proof. We observe that the tactical configurations no. 2, 3, 6, 7, 9, derived from (1; 24, 8, 5), are all quasi-symmetric block designs. By application of Theorem 3.1, they yield strongly regular graphs whose eigenvalues are easily calculated, since their parameters, $x + y$ and $x - y$, are known.

The graphs of Theorem 5.2 and the forthcoming graph of Theorem 5.3, indicated by their number of vertices, are partially ordered by containment of one graph by another as a subgraph according to the following diagram:



The existence and the uniqueness of the strongly regular graph on 56 vertices was first proved by Gewirtz [**4**; **5**], who discovered its complement as a graph of diameter 2, girth 4, valency 10, order 56. Gewirtz also proved the uniqueness of the graph of order 100 which, together with its subgraph of order 77, was first constructed by Higman and Sims [**11**]. The other graphs and the relation between the graphs of orders 77 and 56 seem to be new, although in the setting of coding theory they are implicitly anticipated in [**6**]. In the following theorem a construction is given for the graph of order 100 from the graph of order 77 in a way analogous to the construction of the complement of the Clebsch graph from the Petersen graph, cf. [**4**; **21**, Example 1.3].

THEOREM 5.3. *There exists a strongly regular graph with*

$$b = 100, \quad \rho_0 = -55, \quad \rho_1 = 5, \quad \rho_2 = -15, \quad \mu_1 = 77, \quad \mu_2 = 22.$$

Proof. Let N be the 22×77 incidence matrix of the quasi-symmetric block

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design no. 3. From Theorems 3.1 and 3.2 we have

$$NN^{T} = 16I + 5J, \qquad N^{T}N = 5I + J - A, \qquad N(A + 11I) = -9J,$$

where A is the adjacency matrix of the graph no. 3 of Theorem 5.2. The matrix

$$B = \begin{bmatrix} 0 & j^{T} & -j^{T} \\ j & I - J & 2N - J \\ -j & 2N^{T} - J & A \end{bmatrix}$$

of order 100 then satisfies

$$(B - 5I)(B + 15I) = 24J, \qquad BJ = -55J,$$

which proves the theorem.

6. The extended Golay code (24, 12)**.** Let $V(n, q)$ denote the vector space of dimension n over the Galois field GF(q). A *linear code* (n, k) is a set of vectors in $V(n, q)$ which form a linear subspace $V(k, q)$. The *distance* of any two vectors is the number of coordinates in which they differ. The *weight* of any vector is its distance to the zero vector, that is, the number of its non-zero coordinates. In a linear code the distribution of the distances is governed by the distribution of the weights. The existence of the perfect code (23, 12) over GF(2), due to Golay [**8**], is known. From this, the extended Golay code (24, 12) is obtained by adding an extra parity check bit.

Let Q be the square circulant matrix of order 11 whose first row (0, 1, 0, 1, 1, 1, 0, 0, 0, 1, 0) has ones at the quadratic residues modulo 11. Slightly varying a result due to Karlin [**13**], we can represent the linear code (24, 12) over GF(2) by the following twelve generating row vectors

$$\begin{bmatrix} 0 & 0^{T} & 1 & j^{T} \\ j & I & j & Q \end{bmatrix}$$

or, since $QQ^{T} \equiv I \pmod 2$ equivalently, by the twelve generating row vectors

$$\begin{bmatrix} 0 & 0^{T} & 1 & j^{T} \\ j & Q^{T} & j & I \end{bmatrix}.$$

The number N of vectors of weight w in this code is as follows, cf. [**17**, p. 70],

w:	0	8	12	16	24
N:	1	759	2576	759	1.

Any vector of the code having x ones at the first twelve coordinates and y ones at the last twelve coordinates is said to be of *type* $x + y$.

LEMMA 6.1. *In the subspace of* $V(24, 2)$ *of dimension* 11 *generated by the vectors* $[j \quad I \quad j \quad Q]$, *there are* 792 *pairwise complementary vectors of type* $6 + 6$.

Proof. The subspace contains the all-one vector. Hence any vector in the subspace also has its complement in it. There are $\binom{12}{6} = 924$ vectors of type $6 + y$. By use of $[j \quad Q^{T} \quad j \quad I]$, we conclude that there are equally many of type $x + 6$. There are $\binom{12}{2} = 66$ vectors of type $2 + 6$ and equally many of type $10 + 6$. Since only vectors of weights 8, 12, 16, 24 exist, the lemma is proved.

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LEMMA 6.2. *The* (24, 12) *code contains* 1584 *pairwise complementary vectors of type* $6 + 6$.

Proof. Apart from any of the vectors mentioned in Lemma 6.1, its sum with $(0 \quad 0^{\mathrm{T}} \quad 1 \quad j^{\mathrm{T}})$ is of type $6 + 6$. This proves the lemma.

The linear (24, 12) code may be interpreted as a set of 2^{12} vectors with coordinates 1 and 0 in the vector space R_{24} of dimension 24 over the reals. Taking the origin of R_{24} in the centre of the cube with sides of length 2 parallel to the coordinate axes, we write the vectors with the coordinates -1 and 1 in place of 1 and 0, respectively. The vectors then are opposite in pairs and span 2^{11} lines through the origin. These lines mutually have the angles $\frac{1}{2}\pi$ and $\arccos \frac{1}{3}$. Indeed, from the weight distribution in the (24, 12) code given above we have the following inner products of j with the other code vectors:

$$(j, x) = 8, \quad (j, y) = 0, \quad (j, z) = -8, \quad (j, -j) = -24,$$

for x of weight 8, y of weight 12, z of weight 16. Therefore, the line spanned by $-j$ and j has the angle $\arccos \frac{1}{3}$ with 759 other lines and the angle $\frac{1}{2}\pi$ with 1288 other lines. The linearity of the code yields the same property for any line of the code.

THEOREM 6.3. *There exists a strongly regular graph on* 2048 *vertices with*

$$\rho_0 = -529, \quad \rho_1 = 111, \quad \rho_2 = -17, \quad \mu_1 = 276, \quad \mu_2 = 1771.$$

Proof. The set of 2048 lines obtained from the code (24, 12) is made into a graph as follows. The vertices of the graph are the lines. Any two vertices are adjacent (non-adjacent) whenever the angle of the corresponding lines equals $\frac{1}{2}\pi$ ($\arccos \frac{1}{3}$). The resulting graph is regular of valency 1288, hence $\rho_0 = 759 - 1288 = -529$. Let l and m be any adjacent pair of lines. It is no restriction of generality to suppose that l is spanned by j and that m is spanned by $(1 \quad j^{\mathrm{T}} \quad -1 \quad -j^{\mathrm{T}})$. Indeed, the code is linear and the order of the coordinates is irrelevant. By Lemma 6.2, the number $p_{11}{}^1$ of lines adjacent to both l and m equals 792. In addition, the number $p_{22}{}^2$ of lines non-adjacent to any non-adjacent pair is a constant. Indeed, the order of the coordinates can be taken such that any given vector of weight 8 is a generator of the code. This implies that the graph is strongly regular. By use of (cf. [**19**, p. 190])

$$4p_{11}{}^1 = 2(v - 3 - \rho_0) - (\rho_1 - 1)(1 - \rho_2),$$
$$(\rho_0 - \rho_1)(\rho_0 - \rho_2) = v(v - 1 + \rho_1\rho_2),$$

the additional eigenvalues ρ_1 and ρ_2 are calculated, which proves the theorem.

Remark. Certain subgraphs of the graph explained in Theorem 6.3 again are strongly regular. We have reasons to believe that the subgraph on the 1288 vertices which are adjacent to any one vertex is strongly regular with $\rho_1 = 71$, $\rho_2 = -17$. In addition, the graphs which in Theorem 5.2 are derived from the Steiner system are related to the graph of Theorem 6.3. Indeed, as was observed by Paige [**16**] (cf. [**1**]), the 759 vectors of weight 8 in the Golay code (24, 12) constitute the blocks of the Steiner system (1; 24, 8, 5).

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