THE SOLUTION OF AN INTEGRAL EQUATION

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1. Introduction. This paper deals with the problem of finding the solutions of the integral equation

$$\int_0^\infty u^\alpha f_1(u) f_1(ux)\, du = g_1(x), \tag{1}$$

where the constant α and the function $g_1(x)$ are both given and both assumed to be real, while the function $f_1(x)$ is to be determined.

On writing

$$x^{\alpha/2} f_1(x) = f(x) \quad \text{and} \quad x^{\alpha/2} g_1(x) = g(x), \tag{2}$$

equation (1) simplifies to

$$\int_0^\infty f(u) f(ux)\, du = g(x), \tag{3}$$

and this is the equation which we shall study here.

When $\alpha = 1$, equation (1) can be associated with certain statistical problems studied by several authors **(2, 3, 4, 5)**. Let ξ and η be independent random variables drawn from the same distribution with frequency function $f_1(x)$, where $f_1(-x) = f_1(x)$. Then, when $\alpha = 1$, the left-hand side of (1) is half the frequency function for the distribution of the random variable ξ/η. Hence the problem of finding $f_1(x)$ when the distribution of ξ/η has the given frequency function $2g_1(x)$ can be reduced to the problem of solving (1). For details, see the references cited above.

Here I prove that (1) either has no solutions or that it has an infinite number of solutions. In the latter case I show how to obtain one solution by a fairly simple method and then show how all the other solutions can be deduced from this one already found.

No general solutions of (1) and (3) appear to have been given so far. Goodspeed **(1)** considers (1) in two special cases, namely when

$$\alpha = c - 1, \qquad g_1(x) = \Gamma(c)(1 + x)^{-c}$$

and when

$$\alpha = 0, \qquad g_1(x) = x^{(c-1)/2}\, \Gamma(c)(1 + x)^{-c},$$

with $c > 0$ in both cases. Goodspeed does not find general solutions of (1) in either of these cases; as the title of his paper indicates, he is mainly concerned with finding relations between the solutions and certain Watson transforms.

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For simplicity, I confine myself to L_2-space and base several of the proofs given here upon the Mellin transform. Comprehensive tables of this transform and its inverse are given in the Bateman Project (**6**, Vol. 1, Chapters 6 and 7).

2. The Mellin transform. For convenience, the Mellin transform theorems needed here are stated below. $F(s)$ is the Mellin transform of $f(x)$ if

$$(4)\qquad F(s) = \int_0^\infty f(u)u^{s-1}du \quad \text{and} \quad f(u) = \frac{1}{2\pi i}\int_C F(s)u^{-s}ds,$$

where C denotes the contour of integration for the second integral. In the complex s plane we write $s = \sigma + it$, σ and t real, and throughout this paper the contour C will be the straight line $\sigma = \frac{1}{2}$, i.e. $s = \frac{1}{2} + it$, where $-\infty < t < \infty$.

In general, the integrals of (4) converge only if σ satisfies certain inequalities. These inequalities will be stated in the text whenever necessary.

The following two well-known theorems are found in (**7**, pp. 94 and 95).

A. *If $f(x) \in L_2(0, \infty)$, then the first integral of* (4) *converges in mean square to $F(s)$ and $F(s) \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$.*

If $F(s) \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$, then the second integral of (4) *converges in mean square to $f(x)$ and $f(x) \in L_2(0, \infty)$.*

B. *The Parseval theorem. If $f(x)$ and $g(x)$ both belong to $L_2(0, \infty)$ and $F(s)$, $G(s)$ are their respective Mellin transforms, according to* (4), *then*

$$(5)\qquad \int_0^\infty f(x)g(x)dx = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(s)G(1-s)ds.$$

3. Theorems concerning (3).

THEOREM 1. *If $g(1/x) \neq xg(x)$ when $x > 0$ and the integral of* (3) *exists, then* (3) *has no solutions for $f(x)$.*

Proof. This result is almost obvious. Assuming that $f(u)$ is a solution of (3), then, on using the substitution $u = vx$, we have

$$(6)\qquad g(1/x) = \int_0^\infty f(u)f(u/x)du = \int_0^\infty f(vx)f(v)xdv = xg(x).$$

Hence, if the condition of the theorem is satisfied and (3) has a solution, we obtain a contradiction. This completes the proof of the theorem.

If $G(s)$ is the Mellin transform of $g(x)$ and $g(1/x) = xg(x)$, then it easily follows from (4) that $G(s) = G(1 - s)$. We make use of this fact in Assumption (ii) of the next theorem.

THEOREM 2.

Assumptions: (i) *$g(x) \in L_2(0, \infty)$ and $G(s)$ is its Mellin transform according to* (4);

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(ii) $G(s) = G(1-s) = H(s)\,H(1-s)$, *where* $H(s) \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$;
(iii) $h(x)$ *and* $H(s)$ *are Mellin transforms of each other according to* (4).
Conclusion: $f(x) = h(x)$ *is a solution of* (3).

Proof. The proof is by verification. On writing $u = vx$ in (4), $x > 0$, it follows that if $h(u)$ and $H(s)$ are Mellin transforms of each other, so are $h(ux)$, $x > 0$, considered as a function of u, and $H(s)x^{-s}$. Also $h(u) \in L_2(0, \infty)$ implies that, considered as a function of u, $h(ux) \in L_2(0, \infty)$, when $x > 0$. Hence, from the Parseval theorem, we have

$$\int_0^\infty h(u)h(ux)du = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} H(s)x^{-s}H(1-s)ds \tag{7}$$

and so, by Assumption (ii),

$$\int_0^\infty h(u)h(ux)du = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} G(s)x^{-s}ds = g(x). \tag{8}$$

The theorem is thus verified.

A simple illustration, which will be useful later, occurs when (3) takes the form

$$\int_0^\infty f(u)f(ux)du = \frac{1}{1+x}. \tag{9}$$

Here $g(x) = (1+x)^{-1}$ and so by **(7**, p. 192**)**, $G(s) = \Gamma(s)\Gamma(1-s)$. Assumption (ii) of Theorem 2 is satisfied and we may take $H(s) = \Gamma(s)$. From the tables, or by computing the second integral of (4), it is found that $h(x)$, the inverse Mellin transform of $H(s)$, is given by $h(x) = e^{-x}$. The verification that e^{-x} is a solution of (9) is immediate.

An example with statistical interest occurs when (1) takes the form

$$\int_0^\infty uf_1(u)f_1(ux)du = \frac{1}{1+x^2} \tag{10}$$

and (3) takes the form

$$\int_0^\infty f(u)f(ux)du = \frac{x^{\frac{1}{2}}}{1+x^2}. \tag{11}$$

Here $g(x) = x^{\frac{1}{2}}(1+x^2)^{-1}$ and so, by **(7**, p. 192**)**,

$$G(s) = 2^{-1}\Gamma\left(\frac{2s+1}{4}\right)\Gamma\left(\frac{3-2s}{4}\right). \tag{12}$$

Evidently Assumption (ii) of Theorem 2 is satisfied and we may take

$$H(s) = 2^{(2s-3)/4}\Gamma\left(\frac{2s+1}{4}\right). \tag{13}$$

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We can obtain $h(x)$, the inverse Mellin transform of $H(s)$, either from tables or by computing the second integral of (4). We then find that

$$h(x) = x^{\frac{1}{2}} \exp(-x^2/2)$$

and it is easily verified that this is a solution of (11). From (2) we then find that $f_1(x) = \exp(-x^2/2)$ is a solution of (10). A possible statistical interpretation of (10) is then as follows: If ξ and η are independent random variables, both normally distributed, then the random variable ξ/η has the distribution whose frequency function is $1/\{\pi(1 + x^2)\}$, i.e. the Cauchy distribution.

It is natural to ask whether the normal distribution is the only distribution with the above property or whether there are other distributions with this property; see (**2**, **3**, **4**, and **5**). This question is answered in Theorems 3 and 4 below.

4. Fourier images. We now discuss uniqueness problems in connection with (3). For this purpose the generalized Fourier transform, given by (14) and (15) below, is most useful,

$$\phi(x) = \int_0^\infty k(ux)f(u)du, \tag{14}$$

$$f(x) = \int_0^\infty k(ux)\phi(u)du. \tag{15}$$

The function $k(x)$ is known as the kernel, or better the Fourier kernel, of the transform. Not every function can be a Fourier kernel.

We shall call $k(x)$ a Fourier kernel if the following two conditions are satisfied. Writing $s = \sigma + it$ (σ and t real), let $K(s)$ denote the Mellin transform of $k(x)$. Then we must have (i) $K(s)$ is bounded on the line $\sigma = 1/2$ and (ii)

$$K(s)\,K(1 - s) = 1. \tag{16}$$

In particular, when $s = \frac{1}{2} + it$, (16) becomes $|K(\frac{1}{2} + it)| = 1$. If these conditions are satisfied, then the reciprocity (14), (15) can be established in L_1 space or, in a modified form, in L_2 space, see (**7**, Chapter 8) for a full discussion.

For our purposes we need the following results in generalized Fourier transform theory.

If $K(s)$ satisfies (i) above, then

$$K(s)/(1 - s) \in L_2(\tfrac{1}{2} - i\infty, \tfrac{1}{2} + i\infty)$$

and, since the integral below converges in mean square, we may then write

$$\frac{k_1(u)}{u} = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{K(s)}{1 - s}\, u^{-s} ds. \tag{17}$$

Further, if $k_1(u)$ is differentiable, we may write $k(u) = d\{k_1(u)\}/du$. In

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L_2-space theory, the Fourier transform theory takes the following form **(7**, p. 222**)**:

If Conditions (i) and (ii) above are satisfied and $f(x) \in L_2(0, \infty)$, then

$$\phi(x) = \frac{d}{dx} \int_0^\infty k_1(ux) f(u) \frac{du}{u}, \tag{18}$$

where $\phi(x) \in L_2(0, \infty)$ and

$$f(x) = \frac{d}{dx} \int_0^\infty k_1(ux) \phi(u) \frac{du}{u}. \tag{19}$$

If, as frequently happens, differentiation through each integral sign is justified, then (18) and (19) reduce to (14) and (15) respectively.

If $\phi(x)$ and $f(x)$ are related to each other either by (14), (15) or by (18), (19) we shall say that each is the Fourier image of the other by the kernel $k(x)$.

The Parseval theorem takes the following form **(7**, pp. 223 and 225**)**: If $f(x)$ and $m(x)$ both belong to $L_2(0, \infty)$ and $\phi(x)$, $\mu(x)$ are their respective Fourier images by the same kernel $k(x)$, then

$$\int_0^\infty f(u) m(u) du = \int_0^\infty \phi(u) \mu(u) du. \tag{20}$$

5. Relations among solutions of (3).

THEOREM 3. *If $f(x) \in L_2(0, \infty)$ is a solution of* (3), *then every Fourier image of $f(x)$ by any Fourier kernel is also a solution of* (3).

Proof. Let $f(x)$ and $\phi(x)$ be Fourier images of each other by any Fourier kernel. In (14) or (18) replace x by x/y and u by vy. It follows that $f(xy)$ and $y^{-1}\phi(x/y)$, considered as functions of x, are also Fourier images of each other by the same kernel. Hence, from the Parseval theorem (20), we have

$$\int_0^\infty f(uy) f(u) du = \int_0^\infty \frac{1}{y} \phi\left(\frac{u}{y}\right) \phi(u) du. \tag{21}$$

On writing $u = vy$ in the right-hand integral of (21) and using the fact that $f(u)$ is a solution of (3), it follows that

$$\int_0^\infty \phi(v) \phi(vx) dv = g(x). \tag{22}$$

Hence, since $\phi(x)$ is a Fourier image of $f(x)$ by any kernel, the theorem is proved.

We illustrate by means of equation (9), of which a known solution is given by $f(x) = e^{-x}$. The Fourier images of e^{-x} by the Fourier kernels $(2/\pi)^{\frac{1}{2}} \sin x$ and $(2/\pi)^{\frac{1}{2}} \cos x$ are, respectively, $(2/\pi)^{\frac{1}{2}} x/(1 + x^2)$ and $(2/\pi)^{\frac{1}{2}}/(1 + x^2)$. It is easily verified that these are also solutions of (9).

In the case of equation (11) there are, of course, as we have just shown, an infinite number of solutions. But the Fourier images of $x^{\frac{1}{2}} \exp(-x^2/2)$ are

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somewhat complicated, even with the comparatively simple Fourier kernels above. For this reason we shall not consider other solutions of (11) except to state that the answer to the question at the end of §3 is that there may be an infinite number of distributions with the property required by the question.

6. The converse of Theorem 3. In §5 it was shown that if $f(x)$ is a solution of (3), then so are all Fourier images of $f(x)$ by any Fourier kernel. Suppose that two solutions of (3) are known: Is one the Fourier image of the other by some Fourier kernel? This question is answered in the affirmative by Theorem 4 below, a theorem which is not as easy to prove as the previous ones.

THEOREM 4.

Assumptions: (i) *In* (3), $g(x) \in L_2(0, \infty)$ *and* $G(s)$ *is its Mellin transform;*

(ii) *Equation* (3) *has two solutions given by* $f(x) = h(x)$ *and* $f(x) = m(x)$, *both of which belong to* $L_2(0, \infty)$;

(iii) $H(s)$ *and* $M(s)$, *the respective Mellin transforms of* $h(x)$ *and* $m(x)$, *are both bounded and zero-free on the line* $s = \frac{1}{2} + it$, *where* $-\infty < t < \infty$.

Conclusion: $h(x)$ *and* $m(x)$ *are the Fourier images of each other by some Fourier kernel.*

Proof. First we establish a relation between $H(s)$ and $M(s)$ and then show that this relation is equivalent to an equation of the form (18).

By (4), if $H(s)$ is the Mellin transform of $h(u)$, then $H(s)x^{-s}$ is the Mellin transform of $h(ux)$, considered as a function of u. Also, by Assumption (ii), $h(u) \in L_2(0, \infty)$ and so, considered as a function of u, it follows that $h(ux) \in L_2(0, \infty)$, $x > 0$. Hence, by the Parseval theorem B of §2, we have

$$\frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} H(s)x^{-s}H(1-s)ds = \int_0^\infty h(u)h(ux)du \tag{23}$$

$$= g(x) \tag{24}$$

since $h(u)$ is a solution of (3), Assumption (ii).

From A §2 and (ii) above, $H(s) \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ and the same must therefore be true for $H(s)x^{-s}$, $x > 0$, considered as a function of s and also for $H(1 - s)$. Hence, by Schwarz's inequality (8, p. 381), the integral on the left of (23) is absolutely convergent and so $H(s)H(1 - s)$ must be the Mellin transform of $g(x)$. Consequently,

$$H(s)H(1-s) = G(s) \tag{25}$$

and, similarly, we must also have

$$M(s)M(1-s) = G(s). \tag{26}$$

From (25) and (26) we may write

$$M(s) = K(s)H(1-s), \tag{27}$$

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where

$$K(s) = \frac{H(s)}{M(1-s)}. \tag{28}$$

We now show that (27) is equivalent to an equation of type (18). If $M(\frac{1}{2}+it)$, $-\infty < t < \infty$, is bounded and zero-free, so is $M(\frac{1}{2}-it)$, $-\infty < t < \infty$. Hence, from Assumption (iii) and (28), $K(s)$ must be bounded on the line $s = \frac{1}{2}+it$. From (26), (27), and (28) we have

$$K(s)\,K(1-s) = 1. \tag{29}$$

Hence, $K(s)$ is the Mellin transform of a Fourier kernel which can be used with either of the reciprocities (14), (15) or (18), (19).

Since $K(s)$ is bounded on the line $s = \frac{1}{2}+it$, it follows that $K(s)/(1-s) \in L_2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$. Hence, by A §2, the integral on the right of (17) converges in mean square and defines the function $k_1(u)/u$ on the left-hand side of (17). Further, if $k_1(u)$ is differentiable, we have $k(u) = d\{k_1(u)\}/du$.

On using (17) to define $k_1(u)$, and then $k(u)$, we see that $k_1(u)/u$ and $K(s)/(1-s)$ are Mellin transforms of each other. Hence, on writing ux for u in (17), it follows that $k_1(ux)/u$, as a function of u, and $K(s)x^{1-s}/(1-s)$ are also Mellin transforms of each other. Since the s-function belongs to $L_2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$, by A §2, the u-function belongs to $L_2(0,\infty)$.

Again $h(u) \in L_2(0,\infty)$ by Assumption (ii), and so we may use the Parseval theorem, B §2, to establish

$$\int_0^\infty k_1(ux)h(u)\frac{du}{u} = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{K(s)x^{1-s}}{1-s}H(1-s)ds \tag{30}$$

$$= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M(s)\frac{x^{1-s}}{1-s}ds, \tag{31}$$

by (27).

Again, from the elementary integral

$$\int_0^x x^{s-1}dx = \frac{x^s}{s}, \tag{32}$$

it follows that the function $\lambda(u)$, defined by $\lambda(u) = 1$ when $u < x$ and $\lambda(u) = 0$ when $u > x$, belongs to $L_2(0,\infty)$ and has x^s/s as its Mellin transform. Hence, by the Parseval theorem B §2, we have

$$\int_0^x m(u)du = \int_0^\infty m(u)\lambda(u)du = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M(s)\frac{x^{1-s}}{1-s}ds. \tag{33}$$

From (31) and (33) we finally obtain

$$\int_0^x m(u)du = \int_0^\infty k_1(ux)h(u)\frac{du}{u}. \tag{34}$$

On differentiating (34) with respect to x and comparing the result with (18), we see that $m(x)$ is the Fourier image of $h(x)$ by the Fourier kernel $k(x)$ whose

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Mellin transform $K(s)$ is given by (28). This completes the proof of the theorem.

To illustrate this result, consider the three solutions of (9) found in §5. These functions are tabulated below together with their Mellin transforms, found in **(6** or **7)**, and the inequalities which must be satisfied by σ, where $s = \sigma + it$ (σ and t real).

Function	*Mellin transform*	*σ restrictions*
e^{-x}	$\Gamma(s)$	$0 < \sigma$
$\dfrac{(2/\pi)^{\frac{1}{2}}x}{1+x^2}$	$\dfrac{(2/\pi)^{-\frac{1}{2}}}{\cos(s\pi/2)}$	$-1 < \sigma < 1$
$\dfrac{(2/\pi)^{\frac{1}{2}}}{1+x^2}$	$\dfrac{(2/\pi)^{-\frac{1}{2}}}{\sin(s\pi/2)}$	$0 < \sigma < 2$

If we write $h(x) = e^{-x}$ and $m(x) = (2/\pi)^{\frac{1}{2}}\, x/(1 + x^2)$, then, by Theorem 4, $h(x)$ and $m(x)$ must be Fourier images of each other by the Fourier kernel $k(x)$ whose Mellin transform $K(s)$, by (28), is given by

$$K(s) = H(s)/M(1 - s) = \Gamma(s)(2/\pi)^{\frac{1}{2}} \sin(s\pi/2). \tag{35}$$

Hence, $k(x) = (2/\pi)^{\frac{1}{2}} \sin x$. It is easily verified that $h(x)$ and $m(x)$ are Fourier images by this kernel, in agreement with the way $m(x)$ was deduced from $h(x)$ in §5.

If we write $h(x) = e^{-x}$ and $m(x) = (2/\pi)^{\frac{1}{2}}/(1 + x^2)$, we find in the same way that $h(x)$ and $m(x)$ are Fourier images of one another by the Fourier kernel $(2/\pi)^{\frac{1}{2}} \cos x$. These two cases are simple because we may use the forms (14), (15) instead of (18), (19).

If we write $h(x) = (2/\pi)^{\frac{1}{2}}/(1 + x^2)$ and $m(x) = (2/\pi)^{\frac{1}{2}}x/(1 + x^2)$, we find a somewhat more difficult case to deal with. According to Theorem 4, $h(x)$ and $m(x)$ should be Fourier images of each other by the kernel $k(x)$ whose Mellin transform $K(s)$, by (28), is given by $K(s) = 1$. For this case there is no $k(x)$ and so we must work with $k_1(x)$. This means that we cannot use the forms (14), (15) but must use (18), (19) instead. On using (17) for this case, we have

$$\frac{k_1(u)}{u} = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{1}{1-s}\, u^{-s} ds. \tag{36}$$

On completing the contour by a left semicircle when $0 < u < 1$ and by a right semicircle when $u > 1$, it is easily established that $k_1(u) = 0$ when $0 < u < 1$ and $k_1(u) = 1$ when $u > 1$. Thus, $k_1(u)$ is a Heavyside unit function and its derivative, $k(u)$, may be regarded as a delta function.

Continuing with this case, according to (34) there should be a relation between the two functions as follows:

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(37) $$\frac{(2/\pi)^{\frac{1}{2}}x}{1+x^2} = \frac{d}{dx}\int_{1/x}^{\infty}\left\{\frac{(2/\pi)^{\frac{1}{2}}}{(1+u^2)}\right\}\frac{du}{u},$$

and this is easily verified.

Hence, $h(x)$ and $m(x)$, the two functions of this case, are Fourier images of each other in L_2-space; although the kernel is in the nature of a delta function rather than a conventional function.

I am indebted to the referee for pointing out that (1) can also be solved by using Fourier transforms as well as Mellin transforms such as used here. Let $P(v)$ and $Q(v)$ be, respectively, the Fourier transforms of $p(t)$ and $q(t)$, i.e.

(38) $$P(v) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{ivt}p(t)dt$$

and similarly for $Q(v)$. Then (1) is readily transformed to

(39) $$\int_{-\infty}^{\infty} p(t)p(t+y)dt = q(y)$$

and this in turn to

(40) $$P(v)\,P(-v) = Q(v).$$

The study of (40) would then give us conditions and solutions for (1), and such a study would undoubtedly be of great interest and value. Equation (1), however, occurs with $\alpha = 1$ and $g_1(x) = 1/(1+x^2)$ in certain problems of statistical theory, a very brief account of which is given in §1 and at the end of §3; detailed accounts are given in (**2**, **3**, **4**, and **5**). Because of its statistical use, I felt that a direct attack on (1) with Mellin transforms would be more useful than first transforming to (39) and then using Fourier transforms.

In conclusion, I wish to thank my colleague Professor V. Seshadri for drawing my attention to the statistical problems which led me to study equation (1).

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