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ON RADICALS OF FINITE NEAR-RINGS

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In this paper the study of radicals of finite near-rings is initiated. The main result (Theorem 4.3) gives a description of hereditary radicals having hereditary semisimple classes too. Also it is shown that there exist non-hereditary radicals having hereditary semisimple classes.

1. Introduction

In what follows all radicals are Kurosh–Amitsur. It is well known that any radical $\mathscr{R}$ in the class of associative rings has a hereditary semisimple class $\mathscr{S}\mathscr{R}$, i.e. the class $\mathscr{S}\mathscr{R}$ is closed under taking ideals. On the other hand, Gardner [7] proved that in the variety of not necessarily associative rings, hereditary semisimple classes are quite rare. Betsch and Wiegandt [3] initiated the study of general radical theory of near-rings and they paid special attention to the hereditariness of semisimple classes. In [3] they obtain some conditions on a radical class which imply that the corresponding semisimple class is not hereditary. Our work has been inspired by the latter paper. We consider finite near-rings because we wish to apply the structure theory of near-rings with DCC on right N-subgroups. Note that the radical theory of finite rings and of some other classes of rings with finiteness conditions was considered in [5, 6, 14]. Also note that the main results of the present paper remain true for the larger class of semiprimary near-rings (for the definition see [8]).

We shall use the notions and notations of the book [11] with one exception: our near-rings satisfy the left distributive law $x(y+z)=xy+xz$, not the right one as in [11]. All near-rings will be zero-symmetric.

2. On the structure of finite near-rings

Our main tool will be the characterization of minimal ideals of near-rings with DCC on right N-subgroups obtained in [8]. For the reader's sake we recall here the necessary notions and results from [8, 9].

Definition. A set S with a fixed element $0\in S$ is called a $G,0$-*act* if the group G acts on S and $g0=0$ for all $g\in G$.

The concepts of $G,0$-*congruence* and $G,0$-*homomorphism* are defined as is usual in universal algebra. A subset $F\subseteq S$ is a set of *free generators* for a $G,0$-*act* S if for any $s\in S$, $s\neq 0$, there exist uniquely determined elements $g\in G$ and $f\in F$ such that $s=gf$.

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Definition. Let Φ be an additively written group and let a group G act on Φ by automorphisms. Then Φ turns into a $G,0$-*act* where 0 is the neutral element of the group Φ. Let ρ be a $G,0$-*congruence* of Φ and consider the set M of all transformations m on Φ satisfying the following conditions:

(i) $0m=0$;

(ii) $(g\varphi)m=g(\varphi m)$ for any $g\in G$, $\varphi\in\Phi$;

(iii) $\varphi_1\rho\varphi_2\Rightarrow\varphi_1 m=\varphi_2 m$.

This set M is closed under pointwise addition and composition of mappings so it is a near-ring. Any element $m\in M$ can be identified in an obvious way with a uniquely determined $G,0$-*homomorphism* from Φ/ρ into Φ. Therefore the near-ring M is in fact $\mathrm{Hom}_{G,0}(\Phi/\rho,\Phi)$. This construction goes back to Polin [**12**]. Note that if ρ is the equality relation then the near-ring M coincides with the so called centralizer near-ring $M_G(\Phi)$ [**11**].

Definition. A near-ring M is said to be a *matrix near-ring* on Φ if it is isomorphic to the ring of all linear transformations of a finite-dimensional vector space Φ over some division ring or to the near-ring $\mathrm{Hom}_{G,0}(\Phi/\rho,\Phi)$ where Φ/ρ is a finitely generated free $G,0$-*act* (an empty set of free generators is not allowed). Obviously, if M is a matrix near-ring on Φ then Φ can be considered as an M-group.

For any N-group Σ, we denote

$$\Sigma_N^0=\{\sigma\in\Sigma\,|\,\sigma N=0\};$$

$$\Sigma_N^1=\{\sigma\in\Sigma\,|\,\sigma N=\Sigma\}.$$

Recall that a non-zero N-group Σ is said to be *monogenic* if $\Sigma_N^1\neq\emptyset$, and strongly *monogenic* if an addition $\Sigma=\Sigma_N^1\cup\Sigma_N^0$.

Lemma 2.1. ([**8**], Lemma 4 and Theorem 2) *For any near-ring N and any strongly monogenic N-group Σ we have*

(i) *if Γ is any proper N-ideal of Σ then Σ_N^0 is a union of full cosets by Γ;*

(ii) *Σ has a largest proper N-ideal.*

Lemma 2.2. ([**8**], Proposition 2) *Let M be a matrix near-ring on Φ. Then*

(i) *Φ is a strongly monogenic M-group;*

(ii) *the M-group M is a finite direct power of Φ;*

(iii) *M has a left identity.*

Lemma 2.3. ([**8**], Section 3) *Suppose that the matrix near-ring M on Φ is contained as an ideal in some near-ring N. Then*

(i) *Φ can be considered as an N-group;*

(ii) *M is a direct summand of the N-group N;*

(iii) *if* $M=\mathrm{Hom}_{G,0}(\Phi/\rho,\Phi)$ *then*

$$\Sigma\to(\Sigma:\Phi)_M \tag{2.1}$$

induces an inclusion preserving one-to-one correspondence between all G-invariant N-ideals Σ *of* Φ *and all ideals of N contained in M*

Lemma 2.4. *Let M be a matrix near-ring on* Φ. *M is* J_2*-semisimple if and only if* Φ *is an M-group of type* 2.

Proof. Sufficiency being trivial, let us prove necessity. Let Δ be a proper M-subgroup of Φ and Γ an arbitrary M-group of type 2. By Lemma 2.2, $\Delta M=0$ and there exists a right M-subgroup $R\subseteq M$ such that $R\simeq_M\Delta$. Since Γ is of type 2, $\Gamma R\neq 0$ yields $\gamma R=\Gamma$ for some $\gamma\in\Gamma$. But then $\Gamma M=\gamma(RM)=0$, a contradiction. Hence $\Gamma R=0$, $R\subseteq J_2(M)=0$.

Lemma 2.5. *Let a matrix near-ring M on* Φ *be a minimal ideal of a near-ring N and let* $N=M\oplus T$, $T\lhd_r N$. *Then*

(i) $\Phi^1_M T=0$,

(ii) $\Phi^0_M T=0\Rightarrow T\lhd N$.

Proof.

(i) Since M is a minimal ideal, by Lemmas 2.2 and 2.3 Φ is an N-group of type 0. Hence for $\varphi\in\Phi^1_M$ we have either $\varphi T=0$ or $\varphi T=\Phi$. If $\varphi T=\Phi$ then $\Phi=\Phi M=\Phi TM=0$, a contradiction.

(ii) If $\Phi^0_M\ T=0$ then, according to (i), $\Phi T=0$ and $T\subseteq(0:\Phi)_N$. Now $T\neq(0:\Phi)_N$ implies $(0:\Phi)_M\neq 0$, a contradiction. Thus, $T=(0:\Phi)_N\lhd N$. Conversely, if $T\lhd N$ then $\Phi T=\Phi MT=0$.

Theorem 2.6. ([8], Theorem 6) *If N is a finite near-ring and I is a minimal ideal of N such that* $I^2\neq 0$, *then I is a matrix near-ring.*

Lemma 2.7. *Every finite* 2*-primitive near-ring is a direct summand in any near-ring in which it is contained as an ideal.*

Proof. Let M be a finite 2-primitive near-ring. Then M is simple by [1] and, by Theorem 2.6, it is a matrix near-ring on some group Φ. By Lemma 2.3, $M\lhd N$ implies $N=M\oplus T$, $T\lhd_r N$. If $T\lhd N$ does not hold then, by Lemma 2.5, there exists a $\varphi\in\Phi^0_M$ such that $\varphi T\neq 0$. But then $(\varphi T)M=0$ and Φ_M is not of type 2, contrary to Lemma 2.4.

Theorem 2.8. ([9], Theorem 2) *If I is an ideal of a finite near-ring N and* Φ *is an I-group of type* 0 *then there exists an N-group* Φ' *of type* 0 *such that* Φ *is an I-homomorphic image of* Φ'.

Theorem 2.9. ([9], Corollary 15) *If N is any near-ring, $J \lhd I \lhd N$ and I/J is a J_2-semisimple near-ring, then $J \lhd N$.*

3. Three constructions

Our results on radicals actually follow from the existence of certain near-ring extensions. First we give the construction of the so called standard lift for the near-ring $\mathrm{Hom}_{G,0}(\Phi/\rho, \Phi)$. The other two constructions are based on this one.

Consider the near-ring $\mathrm{Hom}_{G,0}(\Phi/\rho, \Phi)$ for a finitely generated free G, 0-*act* Φ/ρ. Let $\Psi = \Phi \oplus \Phi$ and define

$$g(\varphi_1, \varphi_2) = (g\varphi_1, g\varphi_2) \quad g \in G.$$

So Ψ turns into a $G,0$-*act*. Now extend the equivalence relation ρ to Ψ:

$$(\varphi_1, \varphi_2)\,\rho\,(\varphi_3, \varphi_4) \Leftrightarrow \varphi_1 \rho \varphi_3.$$

Clearly ρ is a $G,0$-*congruence* of Ψ and so we can consider the near-ring $\mathrm{Hom}_{G,0}(\Psi/\rho, \Psi)$. The following proposition gives some simple but useful properties of the triple (G, Ψ, ρ) and of the corresponding near-ring.

Proposition 3.1. *Let $M = \mathrm{Hom}_{G,0}(\Phi/\rho, \Phi)$ where Φ/ρ is a finitely generated free $G,0$-act and let $N = \mathrm{Hom}_{G,0}(\Psi/\rho, \Psi)$. Then*

(i) *Ψ/ρ is a finitely generated free $G,0$-act;*
(ii) *the subset $\Gamma = \{(0, \varphi) \mid \varphi \in \Phi\}$ is an ideal of the N-group Ψ;*
(iii) *$N/(\Gamma : \Psi)_N \simeq M$;*
(iv) *M is simple if and only if Γ is the largest proper N-ideal of Ψ.*

Proof.

(i) If $\bar{\varphi}_1, \ldots, \bar{\varphi}_n$ is a set of free generators for Φ/ρ then $\overline{(\varphi_1, 0)}, \ldots, \overline{(\varphi_n, 0)}$ is a set of free generators for Ψ/ρ ($\bar{\varphi}$ denotes the ρ-class of φ).

(ii) For any $(0, \varphi) \in \Gamma$, $(\varphi_1, \varphi_2) \in \Psi$ and $n \in N$ we have

$$((0, \varphi) + (\varphi_1, \varphi_2))n - (\varphi_1, \varphi_2)n = (\varphi_1, \varphi + \varphi_2)n$$

$$-(\varphi_1, \varphi_2)n = (\varphi_1, 0)n - (\varphi_1, 0)n = 0,$$

so $\Gamma \lhd_N \Psi$

(iii) Given an element $n \in N$, there exist mappings $s, t : \Phi \to \Phi$ such that

$$(\varphi_1, \varphi_2)n = (\varphi_1 s, \varphi_1 t), \quad (\varphi_1, \varphi_2) \in \Psi.$$

Put $s = \xi(n)$. If g is an arbitrary element from G then

$$((g\varphi)s,(g\varphi)t)=(g\varphi,0)n=(g(\varphi,0))n=g((\varphi,0)n)$$

$$=g(\varphi s,\varphi t)=(g(\varphi s),g(\varphi t)),$$

hence $(g\varphi)s=g(\varphi s)$. If $\varphi_1\rho\varphi_2$ then

$$(\varphi_1 s,\varphi_1 t)=(\varphi_1,0)n=(\varphi_2,0)n=(\varphi_2 s,\varphi_2 t),$$

hence $\varphi_1 s=\varphi_2 s$. Therefore $s\in M$ and ξ is a mapping from N into M. Moreover, if m is an arbitrary element from M then the mapping n given by the rule

$$(\varphi_1,\varphi_2)n=(\varphi_1 m,0)$$

belongs to N and $\xi(n)=m$. Thus the mapping ξ is onto. By straightforward arguments one can prove that ξ is a near-ring homomorphism with kernel $(\Gamma:\Psi)_N$.

(iv) If Γ is the largest N-ideal of Ψ then M is simple by Lemma 2.3(iii). Conversely, let M be simple. Then $(\Gamma:\Psi)_N$ is a maximal ideal of N. Since Γ is G-invariant, Lemma 2.3(iii) yields maximality of Γ as an N-ideal of Ψ. Applying Lemma 2.2(i) and Lemma 2.1(ii) we see that Γ is the largest N-ideal of Ψ.

Definition. We call the near-ring N constructed in Proposition 3.1 the *standard lift* of the near-ring M.

It is a well-known fact in associative ring theory that any minimal ideal I with $I^2\neq 0$ is a simple ring. It was noticed in **[8]** that this result is not true for near-rings and a counter-example was published in **[10]**, Example 5.4. If I is a minimal ideal of a near-ring N, $I^2\neq 0$, and J is a maximal ideal of I, $J\neq 0$, then I/J is a J_2-radical near-ring by Theorem 2.9. Next we show that any finite simple J_2-radical near-ring M with $M^2\neq 0$ can occur in the place of I/J above.

Theorem 3.2. *Let M be any finite simple J_2-radical near-ring, $M^2\neq 0$. Then there exists a finite near-ring N having a unique minimal ideal I which has a non-zero ideal J such that $J^2=0$, $I/J\simeq M$ and $N^2\subseteq I$.*

Proof. By Theorem 2.6 we have $M=\mathrm{Hom}_{G,0}(\Phi/\rho,\Phi)$ where Φ/ρ is a finitely generated free $G,0$-*act* (since M is J_2-radical, it is not a ring). Also, J_2-radicality yields the existence of a non-zero subgroup $\Delta\subseteq\Phi$ such that $\Delta M=0$ (see Lemma 2.4). In what follows Ψ and Γ have the same meaning as in Proposition 3.1. Let α be a fixed element of Φ_M^1.

To start our construction we consider the set U of all transformations $u:\Psi\to\Psi$ satisfying the following conditions:

(i) $g(\psi u)=(g\psi)u$ for any $g\in G$ and $\psi=\Psi$;

(ii) if $\varphi_1,\varphi_3\in\Phi_M^1,\varphi_2\in\Phi$ and $\varphi_1\rho\varphi_3$ then $(\varphi_1,\varphi_2)u=(\varphi_3,0)u$;

(iii) $(0,\alpha)u=(\delta,0)$ for some $\delta\in\Delta$;

(iv) $\varphi_1\in\Phi_M^0$ and $\varphi_2\notin G\alpha$ imply $(\varphi_1,\varphi_2)u=0$.

By straightforward computation one can check that the set U is closed under addition and multiplication, so it is a near-ring. The group Ψ can be considered as a U-group.

Represent Ψ as the union of two disjoint subsets A and B:

$$A=\{(\varphi_1,\varphi_2)\,|\,\varphi_1\in\Phi_M^0\},$$

$$B=\{(\varphi_1,\varphi_2)\,|\,\varphi_1\in\Phi_M^1\}.$$

Then every transformation t on Ψ can be represented as a sum $t=v+w$ where $Av=0$ and $Bw=0$. By the definition of U, $t\in U$ implies $v,w\in U$. Thus U is a direct sum of its right ideals $V=(0:A)_U$ and $W=(0:B)_U$. Moreover, since $AU\subseteq A$, V is an ideal of U. Comparing the definitions of V and of the standard lift of M we see that they actually consist of the same transformations on Ψ. So we can identify $V=\mathrm{Hom}_{G,0}(\Psi/\rho,\Psi)$.

Since M is simple, Γ is the largest proper V-ideal of Ψ (Proposition 3.1). Hence the U-group Ψ has a largest proper ideal, say Π, which must be contained in Γ. We are going to show that $\Pi\neq\Gamma$. To do this it is enough to find an element $u\in U$ such that $(0,\alpha)u=(\delta,0)\neq 0$. Define $u:\Psi\to\Psi$ as follows

$$(0,g\alpha)u=(g\delta,0)\quad\text{for any}\quad g\in G,$$

$$\psi u=0\quad\text{if}\quad\psi\notin(0,G\alpha).$$

This definition is correct for $g\alpha=\alpha$ implies $g=1$ since Ψ/ρ is a free $G,0$-*act*. Obviously, the element u defined above satisfies conditions (i)–(iv).

Now we are able to conclude our proof. Let $X=(\Pi:\Psi)_U$ and $Y=(\Gamma:\Psi)_U$. From the definition of U it follows easily that $X=(\Pi:\Psi)_V$ and similarly $Y=(\Gamma:\Psi)_V$. Observe that $N=U/X$, $I=V/X$ and $J=Y/X$ satisfy the conditions we need.

a) From the definition of U we conclude $AU^2=0$. Thus $U^2\subseteq V$ and $N/I\simeq U/V$ implies $N^2\subseteq I$.

b) By Proposition 3.1 we have $I/J\simeq V/Y\simeq M$.

c) Since $\Gamma\subseteq\Psi_V^0$, $\Psi Y^2\subseteq\Gamma V=0$, implying $Y^2=0$, $J^2=0$.

d) By Lemma 2.3(iii), X is the largest ideal of U properly contained in V.

e) Obviously $\Psi=\Psi/\Pi$ is an N-group of type 0 so N is a 0-primitive near-ring, therefore N is prime and its minimal ideal I is unique. The theorem is proven.

Now we turn to our third construction. It will show that a finite near-ring N may have a minimal ideal I which has a proper homomorphic image isomorphic to N/I.

Theorem 3.3. *Let $M=\mathrm{Hom}_{G,0}(\Phi/\rho,\Phi)$ be a finite simple J_2-radical near-ring such that there exists a non-zero group homomorphism $\xi:\Phi\to\Phi$ satisfying the following conditions:*

(i) $\xi(g\varphi)=g\xi(\varphi)$ *for any* $g\in G$, $\varphi\in\Phi$,

(ii) $\xi(\Phi)M=0$.

Then there exists a finite near-ring N having a unique minimal ideal I which has a non-zero ideal J such that $J^2=0$, $N/I\simeq I/J\simeq M$.

The proof of this theorem is similar to that of Theorem 3.2 so we omit the details.

Let Ψ be the $G, 0$-*act* considered in Proposition 3.1 and let $\Delta=\{(\xi(\varphi),\varphi)\mid\varphi\in\Phi\}$. Then Δ is isomorphic to Φ as a group and as a $G, 0$-*act*, too. Consider the set U of all transformations $u:\Psi\to\Psi$ such that

(i) $g(\psi u)=(g\psi)u$ for any $g\in G$, $\psi\in\Psi$;

(ii) if $\varphi_1,\varphi_3\in\Phi_M^1,\varphi_2\in\Phi$ and $\varphi_1\rho\varphi_3$ then $(\varphi_1,\varphi_2)u=(\varphi_3,0)u$;

(iii) $\Delta u\subseteq\Delta$;

(iv) there exists an $m\in M$ such that $(\xi(\varphi),\varphi)u=(\xi(\varphi m),\varphi m)$ for any $\varphi\in\Phi$;

(v) $\varphi_1\in\Phi_M^0$ and $(\varphi_1,\varphi_2)\notin\Delta\Rightarrow(\varphi_1,\varphi_2)u=0$.

Then U is a near-ring and Ψ is a U-group. Let A, B, V and W denote the same as in Theorem 3.2. Then $V\lhd U$, $W\ntriangleleft U$, and V can be identified with a standard lift of M. Thus for $Y=(\Gamma:\Psi)_V$ we have $V/Y\simeq M$. Now consider the right ideal W. From (iii), (iv) and (v) we conclude easily that any element of W is uniquely determined by some element $m\in M$. Also, it is easy to see that this correspondence is a near-ring isomorphism.

Next we show that the largest U-ideal Π of Ψ is properly contained in Γ. Since M is simple, by Proposition 3.1 we need only to show that Γ is not a U-ideal. Take $\varphi\in\Phi_M^1$ and $m\in M$ such that $\xi(\varphi m)\neq0$. Then there exists a $w\in W$ such that $(\xi(\varphi),\varphi)w=(\xi(\varphi m),\varphi m)$ and we have

$$(\xi(\varphi),\varphi)w-((\xi(\varphi),\varphi)+(0,-\varphi))w$$

$$=(\xi(\varphi m),\varphi m)-(\xi(\varphi),0)w=(\xi(\varphi m),\varphi m)\notin\Gamma.$$

Therefore $(0,-\varphi)\in\Gamma\setminus\Pi$.

Now put $X=(\Pi:\Psi)_U$ and observe that $X=(\Pi:\Psi)_V$. From the definition of U we can easily conclude that $v+w\in X$ where $v\in V$, $w\in W$ if and only if $v,w\in X$. Suppose that there exists a non-zero element $w\in W\cap X$. Then $W\cap X$ is a non-zero ideal of W and since W is isomorphic to the simple near-ring M, we have $W\subseteq X$, $\Psi W\subseteq\Pi$. By condition (iv) this gives $\xi(\Phi M)=\xi(\Phi)=0$, a contradiction.

To conclude, define $N=U/X$, $I=V/X$, and $J=Y/X$. It is easy to check (similarly to Theorem 3.2) that all the conditions we need are satisfied.

4. On hereditary radicals of finite near-rings

Obviously, any radical $\mathscr{R}$ in the class of finite near-rings determines the partition $(\mathscr{P},\mathscr{Q})$ of the class of finite simple near-rings:

$$\mathscr{P}=\{N\mid\mathscr{R}(N)=N \quad\text{and}\quad N \quad\text{is a finite simple near-ring}\},$$

$$\mathscr{Q}=\{N\mid\mathscr{R}(N)=0 \quad\text{and}\quad N \quad\text{is a finite simple near-ring}\}.$$

As in the case of rings, different radicals may determine the same partition. This follows, for example, from our Theorem 4.3. But this cannot happen in the case of hereditary radicals.

Theorem 4.1. *For any partition $(\mathscr{P},\mathscr{Q})$ of the class of finite simple near-rings there exists exactly one hereditary radical class determining this partition. This is the lower radical class $\mathscr{R}_{\mathscr{P}}$ determined by $\mathscr{P}$. In fact it coincides with the class $\mathscr{C}_{\mathscr{P}}$ of all near-rings having composition series with all factors from $\mathscr{P}$.*

Proof. Using the isomorphism theorems it is routine to check that $\mathscr{C}_{\mathscr{P}}$ is a hereditary radical class for the class of finite near-rings. Since $\mathscr{P}\subseteq\mathscr{C}_{\mathscr{P}}\subseteq\mathscr{R}_{\mathscr{P}}$, we have the equality $\mathscr{C}_{\mathscr{P}}=\mathscr{R}_{\mathscr{P}}$.

If $\mathscr{R}$ is an arbitrary hereditary radical class and $\mathscr{P}$ is the class of all finite simple $\mathscr{R}$-radical near-rings then $\mathscr{C}_{\mathscr{P}}\subseteq\mathscr{R}$ because $\mathscr{R}$ is closed under extensions. On the other hand, since $\mathscr{R}$ is hereditary, all of its composition factors belong to $\mathscr{P}$, so $\mathscr{R}\subseteq\mathscr{C}_{\mathscr{P}}$. To conclude, note that any simple near-ring N belongs to $\mathscr{P}$ if and only if $\mathscr{C}_{\mathscr{P}}(N)=N$.

In what follows we make use of

Theorem 4.2. ([2], Theorem 3.3) *If a non-trivial radical class $\mathscr{R}$ of near-rings has a hereditary semisimple class then $\mathscr{R}$ is supernilpotent, i.e. it contains all nilpotent near-rings.*

In [2] this result was proved for the class of all near-rings but the proof works for the class of finite 0-symmetric near-rings as well.

Now we are able to state and prove our main result. It gives a description of hereditary radical classes of finite near-rings having hereditary semisimple classes.

Theorem 4.3. *The following conditions are equivalent for a partition $(\mathscr{P},\mathscr{Q})$ of the class of finite simple near-rings:*

(i) *$\mathscr{R}_{\mathscr{P}}$ has a hereditary semisimple class;*

(ii) *$\mathscr{R}_{\mathscr{P}}=\mathscr{U}_{\mathscr{Q}}$, the upper radical class determined by $\mathscr{Q}$;*

(iii) *$\mathscr{U}_{\mathscr{Q}}$ is hereditary;*

(iv) *$\mathscr{P}$ contains all J_2-radical finite simple near-rings* or $\mathscr{P}=\emptyset$.

Proof. (i)⇒(iv) Suppose that $\mathscr{P}\neq\emptyset$, $\mathscr{R}_{\mathscr{P}}$ has a hereditary semisimple class, and there exists a finite simple J_2-radical near-ring $M\in\mathscr{Q}$. By Theorem 4.2 $M^2\neq 0$ and so $M=\mathrm{Hom}_{G,0}(\Phi/\rho,\Phi)$ where Φ/ρ is a finitely generated free $G,0$-*act*. Now take the near-ring N constructed in Theorem 3.2 for the near-ring M. If $N\in\mathscr{S}\mathscr{R}_{\mathscr{P}}$ then by the hereditariness of $\mathscr{S}\mathscr{R}_{\mathscr{P}}$ we have $J\in\mathscr{S}\mathscr{R}_{\mathscr{P}}$, which contradicts Theorem 4.2. If $N\notin\mathscr{S}\mathscr{R}_{\mathscr{P}}$ then $I\subseteq\mathscr{R}_{\mathscr{P}}(N)$ and by hereditariness of $\mathscr{R}_{\mathscr{P}}$ we conclude that $I\in\mathscr{R}_{\mathscr{P}}$. Then $M\simeq I/J\in\mathscr{R}_{\mathscr{P}}$, a contradiction. Hence $M\in\mathscr{P}$.

(iv)⇒(i) Let the condition (iv) hold and let N be a finite $\mathscr{R}_{\mathscr{P}}$-semisimple near-ring. Since all J_2-radical finite simple near-rings are in $\mathscr{P}$, $J_2(N)=0$. So N is a direct sum of finite simple near-rings (see Betsch [1] and Blackett [4]) which are obviously in $\mathscr{Q}$. Hence every ideal I of N is also a direct sum of near-rings from $\mathscr{Q}$, which gives $I\in\mathscr{S}\mathscr{R}_{\mathscr{P}}$.

(iv)⇒(ii) Let the condition (iv) hold, $\mathscr{P}\neq\emptyset$, and suppose that there exists a near-ring $N\in\mathscr{U}_{\mathscr{Q}}\backslash\mathscr{R}_{\mathscr{P}}$. Then for $L=N/\mathscr{R}_{\mathscr{P}}(N)$ we have $0\neq L\in\mathscr{U}_{\mathscr{Q}}\cap\mathscr{S}\mathscr{R}_{\mathscr{P}}$. Let I be a minimal ideal of L. From (iv) it follows that $\mathscr{R}_{\mathscr{P}}$ is supernilpotent, hence $L\in\mathscr{S}\mathscr{R}_{\mathscr{P}}$ implies $I^2\neq 0$. By Theorem 2.6, I is a matrix near-ring. If I is a ring then it is simple and $L\in\mathscr{S}\mathscr{R}_{\mathscr{P}}$ implies $I\in\mathscr{Q}$. Let now $I=\mathrm{Hom}_{G,0}(\Phi/\rho,\Phi)$ and let J be a maximal ideal of I. By Lemma 2.3(iii),

$J=(\Sigma:\Phi)_I$ where Σ is a proper I-ideal of Φ and $\Sigma\subseteq\Phi_I^0$ by Lemma 2.2. Since $\Phi J^2\subseteq\Sigma I=0$, we have $J^2=0$, which yields $J\in\mathscr{R}_{\mathscr{P}}$. Now $I\notin\mathscr{R}_{\mathscr{P}}$ implies $I/J\notin\mathscr{R}_{\mathscr{P}}$, hence $I/J\in\mathscr{Q}$. Furthermore, by (iv) I/J is J_2-semisimple and by Theorem 2.9 $J\lhd N$. Now the minimality of I yields $J=0$ and, as above, $I\in\mathscr{Q}$. Using Lemma 2.7 we obtain that I is a direct summand of the near-ring L. So L can be mapped homomorphically onto the near-ring $I\in\mathscr{Q}$, which contradicts $L\in\mathscr{U}_{\mathscr{Q}}$. Therefore condition (ii) holds.

(ii)⇒(iii) This follows from Theorem 4.1.

(iii)⇒(iv) Suppose that condition (iii) holds, $\mathscr{P}\neq\emptyset$, and $\mathscr{Q}$ contains some J_2-radical near-ring. We have to consider two cases separately.

a) $\mathscr{Q}$ does not contain near-rings with zero multiplication. In this case $\mathscr{Q}$ contains a J_2-radical matrix near-ring $M=\mathrm{Hom}_{G,0}(\Phi/\rho,\Phi)$. Consider again the near-ring N constructed in Theorem 3.2. Since N is not simple and every proper homomorphic image of N has zero multiplication, $N\in\mathscr{U}_{\mathscr{Q}}$. Now, by condition (iii), $I\in\mathscr{U}_{\mathscr{Q}}$, which gives $M\simeq I/J\in\mathscr{U}_{\mathscr{Q}}$. This contradicts $M\in\mathscr{Q}$.

b) $\mathscr{Q}$ contains a simple near-ring K with $K^2=0$. We are going to prove that in this case, contrary to our assumption, $\mathscr{P}=\emptyset$. We need four steps to do this.

Claim 1. $\mathscr{Q}$ contains a simple near-ring L with $L^2=0$ and L^+ (the additive group of L) non-abelian.

If K^+ is not abelian then we are done. If K cannot be chosen to be so then we can take an $L\in\mathscr{P}$ such that L^+ is not abelian, $L^2=0$, and consider the group $W=K^+\,\mathrm{wr}\,L^+$, the wreath product of K^+ by L^+. Since W is non-solvable, there exists an integer n such that $W^{(n)}$, the nth commutator subgroup of W, equals $W^{(n+1)}$ and $W^{(n)}\neq 0$. Let V be the near-ring with zero multiplication on $W^{(n)}$. Obviously, the only simple homomorphic image of V is L, hence $V\in\mathscr{U}_{\mathscr{Q}}$. On the other hand, V contains an ideal isomorphic to a direct power of K. Since $\mathscr{U}_{\mathscr{Q}}$ is hereditary, this yields $K\in\mathscr{P}$, a contradiction.

Claim 2. $\mathscr{Q}$ contains all finite simple near-rings S with $S^2=0$.

By Claim 1 $\mathscr{Q}$ contains a simple near-ring L with $L^2=0$ and L^+ non-abelian. Suppose that $S\notin\mathscr{Q}$ and consider the near-ring V with zero multiplication of the additive group $L^+\,\mathrm{wr}\,S^+$. Then the only simple homomorphic image of V is S, implying $V\in\mathscr{U}_{\mathscr{Q}}$. But, as above, V has an ideal isomorphic to a direct power of L, which contradicts $L\in\mathscr{Q}$.

Claim 3. $\mathscr{Q}$ contains all finite simple matrix near-rings $M=\mathrm{Hom}_{G,0}(\Phi/\rho,\Phi)$.

Suppose that $M\in\mathscr{P}$ and let N be the standard lift of M. By Proposition 3.1, $I=(\Gamma:\Psi)_N$ is the largest proper ideal of N and $N/I\simeq M$. Therefore $N\in\mathscr{U}_{\mathscr{Q}}$. Since $I^2=0$ and $\mathscr{U}_{\mathscr{Q}}$ is hereditary, we conclude that $\mathscr{U}_{\mathscr{Q}}$ contains a simple near-ring with zero multiplication. This contradicts Claim 2.

Claim 4. $\mathscr{Q}$ contains all finite simple rings $M=M_n(D)$ where D is a division ring.

The idea of the proof is the same as that of Claim 3. Instead of the standard lift we take the subring $N\subseteq M_{n+1}(D)$ consisting of all matrices with zeros in the last column. It is easy to see that the only simple homomorphic image of N is M. Furthermore, N has a non-zero ideal with zero multiplication.

5. On non-hereditary radicals with hereditary semisimple classes

Now we ask the following question. Do there exist non-hereditary radicals having hereditary semisimple classes? Or equivalently, can a finite simple J_2-radical near-ring be contained in some non-trivial semisimple class? We give an affirmative answer to this question. But, on the other hand, we show that there exist finite simple near-rings M with $M^2 \neq 0$ which cannot be contained in any hereditary semisimple class.

Theorem 5.1. *Let $\mathcal{Q}$ be a class of finite simple matrix near-rings satisfying the following condition:*

If K and L are matrix near-rings on Φ and Ψ, respectively, $K, L \in \mathcal{Q}$ and $\xi: \Phi \to \Psi$ is a non-zero group homomorphism, then $\xi(\Phi)L \neq 0$. Then

(i) *$\mathcal{U}_{\mathcal{Q}}$ has a hereditary semisimple class;*

(ii) *for any finite near-ring N,*

$$\mathcal{U}_{\mathcal{Q}}(N) = (N)\mathcal{Q} = \cap \{I \mid I \lhd N, N/I \in \mathcal{Q}\};$$

(iii) *any $\mathcal{U}_{\mathcal{Q}}$-semisimple near-ring is a direct sum of near-rings from $\mathcal{Q}$.*

Proof. By Lemma 2.2 any matrix near-ring M has a strongly monogenic M-group. Therefore any simple matrix near-ring is 1-primitive. Since every finite J_1-semisimple near-ring is a direct sum of 1-primitive near-rings ([**13**], Theorem 2.3), we conclude that $N/(N)\mathcal{Q}$ is a direct sum of some near-rings $M_1, \ldots, M_n$ from $\mathcal{Q}$. Let M_i be a matrix near-ring on Φ_i, $i = 1, \ldots, n$.

All we have to prove now is the equality $\mathcal{U}_{\mathcal{Q}}(N) = (N)\mathcal{Q}$. Obviously, $\mathcal{U}_{\mathcal{Q}}(N) \subseteq (N)\mathcal{Q}$. Hence to prove the equality it suffices to show that $(N)\mathcal{Q} \in \mathcal{U}_{\mathcal{Q}}$, i.e. $(N)\mathcal{Q}$ has no homomorphic image in $\mathcal{Q}$. Put $(N)\mathcal{Q} = I$ and suppose that there exists a $J \lhd I$ such that $I/J \in \mathcal{Q}$. Let I/J be a matrix near-ring on Φ.

First consider the case of Φ being an N-group. Then $J \lhd N$ and without loss of generality we may assume $J = 0$. Hence by Lemma 2.3, $N = I \oplus L$ where $L \lhd_r N$. Since $L \simeq N/I$, L is the direct sum of $M_1, \ldots, M_n$. Then, by Lemma 2.2, for the N-group L we have $L = L_1 \oplus \ldots \oplus L_m$ where each of the N-groups L_j is isomorphic to some Φ_i, $i = 1, \ldots, n$.

Now we have to consider two subcases.

a) $\Phi_I^0 N = 0$. Then by Lemma 2.5 $L \lhd N$ and from the definition of I we conclude $I \subseteq L$, a contradiction.

b) $\Phi_I^0 N \neq 0$. Then there exist $\varphi \in \Phi_I^0$ and $j \in \{1, \ldots, m\}$ such that $\varphi L_j \neq 0$. Now φL_j is a non-zero homomorphic image of some group Φ_i in Φ and $(\varphi L_j)I \subseteq \varphi I = 0$. This contradicts our assumption.

Next consider the case where Φ is not an N-group. Then there exists an N-group Σ of type 0 which has an I-ideal Δ such that $\Phi \simeq \Sigma/\Delta$ (see Theorem 2.8). Obviously, $\Sigma I \neq 0$ and we can choose an ideal $V \lhd N$, $V \subseteq I$ which is minimal with respect to the property $\Sigma V \neq 0$. Without loss of generality we may assume $(0:\Sigma)_V = 0$ and so V is a minimal ideal of N. Since V does not annihilate the N-group Σ of type 0, $V^2 \neq 0$. Hence V is a matrix near-ring on some group Ψ and $N=V\oplus X$, $X\mathrel{\triangleleft_r} N$ (see Theorem 2.6 and Lemma 2.3).

We now prove the N-isomorphism $\Sigma\simeq_N\Psi$. By Lemma 2.2, $V=V_1\oplus\ldots\oplus V_k$ where all V_i are N-isomorphic to Ψ. Take $\sigma\in\Sigma_N^1$, then $\sigma V\neq 0$ for otherwise $\Sigma V=\sigma NV=0$. Since Σ is of type 0 and $\sigma V\neq 0$, there exists an $i\in\{1,\ldots,k\}$ such that $\sigma V_i=\Sigma$. So we have an N-homomorphism $v\to\sigma v$ from V_i onto Σ. Since V_i is also of type 0, the kernel of this N-homomorphism is zero. Hence $\Sigma\simeq_N V_i={}_N\Psi$.

Next we show that $\Delta\mathrel{\triangleleft_N}\Sigma$. Since Σ is an N-group of type 0, this will be a contradiction which will prove the theorem. We have to show that for any $\sigma\in\Sigma$, $\delta\in\Delta$ and $n\in N$,

$$(\sigma+\delta)n-\sigma n\in\Delta. \tag{5.1}$$

First consider the case $\sigma\in\Sigma_V^1$. Since $V\subseteq I$ and $\Delta\mathrel{\triangleleft_I}\Sigma$, we have $\Delta\mathrel{\triangleleft_V}\Sigma$. Now Σ being a strongly monogenic V-group, it follows from Lemma 2.1 that $\sigma+\delta\in\Sigma_V^1$. Therefore writing $n=v+x$ where $v\in V$, $x\in X$, and using Lemma 2.5(i) we get

$$(\sigma+\delta)n-\sigma n=(\sigma+\delta)(v+x)-\sigma(v+x)$$

$$=(\sigma+\delta)v+(\sigma+\delta)x-\sigma x-\sigma v=(\sigma+\delta)v-\sigma v.$$

Since $V\subseteq I$ and $\Delta\mathrel{\triangleleft_I}\Sigma$, $(\sigma+\delta)v-\sigma v\in\Delta$.

Now to prove (5.1) it suffices to exhibit

$$\Sigma_V^0 N\subseteq\Delta. \tag{5.2}$$

To do this we first observe that $\Sigma_V^0 N\subseteq\Sigma_V^0$ and $\Sigma_V^0 I\subseteq\Delta$. For any $\sigma\in\Sigma_V^0$ we have $(\sigma N)V\subseteq\sigma V=0$ so $\sigma N\subseteq\Sigma_V^0$. Thus by Lemma 2.1 $\sigma N+\Delta\subseteq\Sigma_V^0$. Particularly, $\sigma I+\Delta\subseteq\Sigma_V^0$ which yields $\sigma I\subseteq\Delta$, because Σ/Δ is a strongly monogenic I-group.

Since N/I is a direct sum of near-rings $M_1,\ldots,M_n$, by Lemma 2.2 we have $N/I=N_1/I\oplus\ldots\oplus N_m/I$ where each N_j/I is N-isomorphic to some of the Φ_i, $i=1,\ldots,n$. Now suppose that (5.2) does not hold. Then there exist $\sigma\in\Sigma_V^0$ and $j\in\{1,\ldots,m\}$ such that $\sigma N_j\nsubseteq\Delta$. Hence $\sigma N_j/\sigma I$ is a non-zero N-homomorphic image of N_j/I. Since N_j/I is of type 0, we have the isomorphism $\sigma N_j/\sigma I\simeq_N N_j/I$. Furthermore, by the isomorphism theorem

$$\sigma N_j+\Delta/\Delta\simeq_I\sigma N_j/\sigma N_j\cap\Delta.$$

Here on the left we have a non-zero subgroup of $\Sigma/\Delta\simeq_I\Phi$ and on the right a homomorphic image of $\sigma N_j/\sigma I$ (because $\sigma I\subseteq\sigma N_j\cap\Delta$). The $\sigma N_j+\Delta$ is a proper I-subgroup of Σ (it is contained in Σ_V^0). Since Φ is a strongly monogenic I-group, this yields $(\sigma N_j+\Delta)I\subseteq\Delta$. Hence the group Φ contains a non-zero homomorphic image Λ of some group Φ_i, such that $\Lambda I=0$. So our assumption is contradicted and the theorem is proved.

Remark. First note that in view of Lemma 2.4 the assumption on the class $\mathcal{Q}$ is fulfilled if $\mathcal{Q}$ contains only J_2-semisimple near-rings. It is natural to ask: do there exist

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classes $\mathcal{Q}$ satisfying the condition of Theorem 5.1 and containing some J_2-radical matrix near-ring? The simplest way to construct such a class is the following. Let Φ be a finite non-abelian simple group and let Δ be one of its proper non-zero subgroup. Let ρ be the least equivalence relation on Φ for which all elements of the subgroup Δ are in the same class, and let $G=\{1\}$. Then $M=\mathrm{Hom}_{G,0}(\Phi/\rho,\Phi)$ is a simple J_2-radical near-ring and the class $\mathcal{Q}=\{M\}$ satisfies the condition of Theorem 5.1.

Proposition 5.2. *A matrix near-ring M satisfying the assumptions of Theorem* 3.3 *cannot be contained in any non-trivial hereditary semisimple class of near-rings.*

Proof. Take the near-ring N constructed in Theorem 3.3 and let I and J be same as there. Suppose that there exists a non-trivial radical class $\mathcal{R}$ with hereditary semisimple class $\mathcal{SR}$ such that $M\in\mathcal{SR}$. If $N\in\mathcal{SR}$ then $J\in\mathcal{SR}$, contrary to Theorem 4.2. If $N\notin\mathcal{SR}$ then $I\subseteq\mathcal{R}(N)$, for I is the unique minimal ideal of N. On the other hand, $N/I\simeq M$ yields $\mathcal{R}(N)\subseteq I$, so $\mathcal{R}(N)=I$. Hence $I\in\mathcal{R}$ and $M\simeq I/J\in\mathcal{R}$, a contradiction.

Remark. Near-rings satisfying the conditions of Theorem 3.3 really exist. Take any finite group Φ which can be mapped homomorphically onto one of its proper non-zero subgroup Δ. Let ρ be the least equivalence relation on Φ for which all elements of Δ lie in the same class and let $G=\{1\}$. Then $\mathrm{Hom}_{G,0}(\Phi/\rho,\Phi)$ is the near-ring we need.

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REFERENCES

1. G. Betsch, Ein radikal für fastringe, *Math. Z.* **78** (1962), 86–90.

2. G. Betsch and K. Kaarli, Supernilpotent radicals and hereditariness of semisimple classes of near-rings, *Radical Theory* (Proc. Conf. Eger, 1982, Colloq. Math. Soc. J. Bolyai, North-Holland, Amsterdam), to appear.

3. G. Betsch and R. Wiegandt, Non-hereditary semisimple classes of near-rings, *Studia Sci. Math Hungar.* **17** (1982), 69–75.

4. D. W. Blackett, Simple and semisimple near-rings, *Proc. Amer. Math. Soc.* **4** (1953),772–785.

5. J. L. Fisher, Radicals for finite rings, *manuscript.*

6. J. L. Fisher, Radicals of rings with composition series, *manuscript.*

7. B. J. Gardner, Some degeneracy and pathology in non-associative radical theory, *Annales Univ. Sci. Budapest Sect. Math.* **22–23** (1979/80), 65–74.

8. K. Kaarli, Minimal ideals of near-rings, *Tartu Riikl. Ül. Toimetised, Vih.* **366** (1975), 105–142 (in Russian).

9. K. Kaarli, Radicals of near-rings, *Tartu Riikl. Ül. Toimetised, Vih.* **390** (1976), 134–171 (in Russian).

10. K. Kaarli, The classification of irreducible R-groups over a semiprimary near-ring, *Tartu Riikl. Ül. Toimetised, Vih.* **556** (1981), 47–63 (in Russian).

11. G. Pilz, *Near-rings* (North-Holland, Amsterdam, 1977).

12. S. V. Polin, Primitive m-near-rings over multi-operator groups, *Math. Sbornik* **13** (1971), 247–265 (in Russian).

13. S. V. Polin, Radicals in $m\Omega$-near-rings II, *Izv. Vysš. Učebn. Zaved. Mathematika* **2** (1972), 63–71 (in Russian).

14. A. Widiger and R. Wiegandt, Theory of radicals for hereditarily artinian rings, *Acta Sci. Math.* **39** (1977), 303–312.

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