BULL. AUSTRAL. MATH. SOC. 18A40, 18C15, 18D99
VOL. 16 (1977), 427-448.

Density presentations of functors

B.J. Day

The article contains the basic theory resulting from the presentation of a dense functor $N : A \to C$ by means of an expansion $K(k, C) \circ NJk \cong C$, the term dense functor being used instead of the equivalent term left-adequate functor. Results by various authors on the density type of a functor are formulated in the V-context for V symmetric monoidal closed, and elementary proofs are given. In particular a characterisation theorem containing the well-known results of Beck and Ulmer is established.

Introduction

The concept of density presentation is significant in multilinear algebra as treated from the categorical viewpoint (*cf.* Day [7] and Day [8]). At the same time it is implicit in providing a uniform treatment of some of the basic characterisation theorems of categorical algebra (*cf.* Diers [9] and Diers [10]).

By introducing density presentations one attempts to utilise existing analogies between categorical algebra and elementary linear algebra. The precise relationship with elementary analysis is not known because, as yet, there exists no satisfactory method of "measuring" categorical colimits.

The first step is the introduction of density presentations of functors $N : A \to C$ as expansions $K(k, C) \circ NJk \cong C$ indexed by $k \in K$. In Section 2 the relationship of presentations to Kan extensions is discussed. The basic characterisation theorem is established in Section 3.

Received 2 February 1977. The author gratefully acknowledges the support of a Postdoctoral Fellowship from the Australian Research Grants Committee.

https://doi.org/10.1017/S0004972700023509 Published online by Cambridge University Press

The remainder of the article is devoted to examples and Section 5 deals with the important example of linear monads.

Because one aim of the article is to describe density presentations in the general context of V-category theory we shall assume that, unless otherwise stated, the categorical algebra is *relative* to a fixed and suitably complete symmetric monoidal closed ground category $V = (V, \otimes, I, \ldots)$. We mainly adhere to the notation and terminology of Eilenberg and Kelly [11] and Mac Lane [21].

We note that the term dense functor (Ulmer [22]) is used here instead of the term left-adequate functor (Isbell [16]) without intending to justify one or discard the other. The closely related (but much stronger) condition which reduces to topological density when $V = \mathrm{R}^+$ (the positive real numbers as described by Lawvere [18]) is referred to here as Cauchy density.

1. Density presentations

Given a category C we denote the discrete category associated with C by $|C|$. A presentation of a functor $N : A \to C$ consists of an *index* functor $J : K \to A$, a *coefficient* functor $K : K^{\mathrm{op}} \otimes |C| \to V$, and a *structure* transformation $K(k, C) \to C(NJk, C)$ which is natural in $k \in K$ and $C \in |C|$. In practice the presentation may be "partially natural" in C .

If $S : K^{\mathrm{op}} \to V$ and $T : K \to C$ are functors then we denote their "mean" tensor product (in the sense of Borceux and Kelly [5]) by $Sk \circ Tk$. Thus, when it exists, $Sk \circ Tk$ has the defining property

$$C(Sk \circ Tk, C) \cong \int_k [Sk, C(Tk, C)]$$

naturally in $C \in C$.

A presentation is called *strong* if the tensor product $K(k, C) \circ NJk$ exists in C and the resultant transformation

$$\xi_C : K(k, C) \circ NJk \to C$$

is a strong epimorphism in C .

A presentation is called a *density* presentation if the tensor product $K(k,\ C) \circ NJk$ exists in $\mathcal{C}$ and the resultant transformation

$$\xi_C : K(k,\ C) \circ NJk \to C$$

is a regular epimorphism in $\mathcal{C}$ such that $\mathcal{C}(NA,\ \xi)$ is an epimorphism for all $A \in \mathcal{A}$, $C \in \mathcal{C}$, and $\bar{\xi} : K(k,\ C) \circ \mathcal{A}(A,\ Hk) \to \mathcal{C}\big(NA,\ K(k,\ C) \circ NJk\big)$ is an epimorphism for all $A \in \mathcal{A}$, $C \in \mathcal{C}$. Thus, if N is full and faithful then the conditions for a density presentation state that the objects NA are required to be "projective" with respect to the colimits involved in the presentation.

PROPOSITION 1.1. *(a) If* $(J,\ K,\ \xi)$ *is a strong presentation of* $N : \mathcal{A} \to \mathcal{C}$ *then* N *strongly generates* $\mathcal{C}$ *.*

(b) If $(J,\ K,\ \xi)$ *is a density presentation of* $N : \mathcal{A} \to \mathcal{C}$ *then* N *is dense.*

Proof. *(a)* The following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}(C,D) \arrow[r, "\mathcal{C}(\xi{,}1)"] \arrow[d] & \mathcal{C}\big(K(k,C) \circ NJk,D\big) \arrow[d, "\wr"] \\
\int_A [\mathcal{C}(NA,C),\mathcal{C}(NA,D)] \arrow[r] & \int_A [K(k,C) \circ \mathcal{A}(A,Jk),\mathcal{C}(NA,D)] \ .
\end{tikzcd}$$

Thus the transformation $\mathcal{C}(C,\ D) \to \int_A [\mathcal{C}(NA,\ C),\ \mathcal{C}(NA,\ D)]$ is a strong monomorphism because $\mathcal{C}(\xi,\ 1)$ is a strong monomorphism.

(b) The proof is similar to that for *(a)*. Let

$$E \rightrightarrows K(k,\ C) \circ NJk \xrightarrow{\xi} C$$

be a coequaliser diagram for ξ_C and consider the resultant coequaliser diagram

$$K(k,\ E) \circ NJk \to E \rightrightarrows K(k,\ C) \circ NJk \to C \ .$$

By naturality we obtain a factorisation

https://doi.org/10.1017/S0004972700023509 Published online by Cambridge University Press

$$\begin{tikzcd}
\mathcal{C}(C,D) \arrow[rr] \arrow[dd, shift right] & & \mathcal{C}\big(K(k,C)\circ NJk, D\big) \arrow[d] \\
& & \int_A [K(k,C)\circ \mathrm{A}(A,Jk), \mathcal{C}(NA,D)] \arrow[d, shift right] \\
\int_A [\mathcal{C}(NA,C), \mathcal{C}(NA,D)] \arrow[rr] \arrow[uu, shift right] \arrow[uurr] & & \int_A \big[\mathcal{C}\big(NA, K(k,C)\circ NJk\big), \mathcal{C}(NA,D)\big] \arrow[u, shift right]
\end{tikzcd} .$$

The transformation $\mathcal{C}(C, D) \to \int_A [\mathcal{C}(NA, C), \mathcal{C}(NA, D)]$ is a monomorphism by *(a)* and is a retraction because the lower arrow is a monomorphism since $\mathcal{C}(NA, \xi)$ is an epimorphism. Thus the transformation is an isomorphism, as required. //

A density presentation is called a *strict* presentation if both ξ and $\bar{\xi}$ are isomorphisms. Indeed, more complex density presentations than those introduced here do arise but we shall primarily be concerned with strict presentations.

PROPOSITION 1.2 (The Representation Theorem). *If* (J, K, ξ) *is a strict presentation of* $N : \mathrm{A} \to \mathcal{C}$ *then the induced transformation*

$$\int_A [\mathcal{C}(NA, C), GA] \to \int_k [K(k, C), GJk]$$

is an isomorphism for all $N : \mathrm{A}^{\mathrm{op}} \to V$ *and* $C \in \mathcal{C}$.

Proof. We have

$$\begin{aligned}
\int_A [\mathcal{C}(NA, C), GA] &\cong \int_A \big[\mathcal{C}\big(NA, K(k, C)\circ NJk\big), GA\big] \\
&\cong \int_A [K(k, C)\circ \mathrm{A}(A, Jk), GA] \\
&\cong \int_k \Big[K(k, C), \int_A [\mathrm{A}(A, Jk), GA]\Big]
\end{aligned}$$

on interchanging limits

$$\cong \int_k [K(k, C), GJk]$$

by the standard representation theorem. //

PROPOSITION 1.3. *If* (J, K, ξ) *are such that*

https://doi.org/10.1017/S0004972700023509 Published online by Cambridge University Press

$$K(k,\ C) \circ \mathrm{A}(A,\ Jk) \cong \mathrm{C}(NA,\ C)$$

for all $A \in \mathrm{A}$ *and* $C \in \mathrm{C}$ *then* $\xi : K(k,\ C) \circ NJk \to C$ *is an isomorphism iff* N *strongly generates* C *and* $K(k,\ C) \circ \mathrm{A}(A,\ Jk) \cong \mathrm{C}\big(NA,\ K(k,\ C) \circ NJk\big)$ *for all* $A \in \mathrm{A}$ *and* $C \in \mathrm{C}$.

The proof is straightforward.

Associated with a presentation $P = (J,\ K,\ \xi)$ of $N : \mathrm{A} \to \mathrm{C}$ for which

$$K(k,\ C) \circ \mathrm{A}(A,\ Jk) \cong \mathrm{C}(NA,\ C)$$

for *all* $A \in \mathrm{A}$ and $C \in \mathrm{C}$, is the category $P\mathrm{C} \subset \mathrm{C}$ of P-objects defined by $P\mathrm{C} = \left\{C \in \mathrm{C};\ \mathrm{C}(\dot{C},\ D) \cong \int_k [K(k,\ C),\ \mathrm{C}(NJk,\ D)] \text{ for all } D \in \mathrm{C}\right\}$. If $NA \in P\mathrm{C}$ for all $A \in \mathrm{A}$ then $N : \mathrm{A} \to P\mathrm{C}$ is dense because

$$\begin{aligned}\int_A [\mathrm{C}(NA,\ C),\ \mathrm{C}(NA,\ D)] &\cong \int_A [K(k,\ C) \circ \mathrm{A}(A,\ Jk),\ \mathrm{C}(NA,\ D)] \\ &\cong \int_k [K(k,\ C),\ \mathrm{C}(NJk,\ D)] \\ &\cong \mathrm{C}(C,\ D)\ .\end{aligned}$$

In particular, given a strict presentation $(J,\ K,\ \xi)$ of N , each full subcategory of A through which J factors has such a "closure" in C .

2. Kan extensions and adjoints

PROPOSITION 2.1. *Let* $(J,\ K,\ \xi)$ *be a strict presentation of* $N : \mathrm{A} \to \mathrm{C}$. *Then* $F : \mathrm{A} \to \mathrm{B}$ *has a left Kan extension along* N *if* $K(k,\ C) \circ FJk$ *exists in* B *for all* $C \in \mathrm{C}$.

Proof.

$$\begin{aligned}K(k,\ C) \circ FJk &\cong \left(\int^k K(k,\ C) \otimes \mathrm{A}(A,\ Jk)\right) \circ FA \quad \text{by the representation theorem} \\ &\cong \mathrm{C}(NA,\ C) \circ FA\ . \qquad //\end{aligned}$$

PROPOSITION 2.2. *Let* $(J,\ K,\ \xi)$ *be a strict presentation of* $N : \mathrm{A} \to \mathrm{C}$. *Then* $F : \mathrm{C} \to \mathrm{B}$ *is a left Kan extension along* N *of its restriction by* N *iff* F *preserves* $K(k,\ C) \circ NJk$ *for all* $C \in \mathrm{C}$.

Proof. $FC \cong F\big(K(k,\ C) \circ NJk\big)$ so we have

https://doi.org/10.1017/S0004972700023509 Published online by Cambridge University Press

$$FC \cong K(k, C) \circ FNJk$$
$$\cong \left[\int^k K(k, C) \otimes A(A, Jk)\right] \circ FA \quad \text{by the representation theorem}$$
$$\cong C(NA, C) \circ FNA$$

and the result follows. //

Let $N : A \to C$ be a functor and let $N' : C \to [A^{op}, V]$ be the induced Yoneda functor.

PROPOSITION 2.3. *Let* $N : A \to C$ *be dense by strict presentation* (J, K, ξ) .

(a) If N *is fully faithful then* $F : C \to B$ *has a right* N'*-adjoint iff* $F\big(K(k, C) \circ NJk\big) \cong K(k, C) \circ FNJk$ *for all* $C \in C$.

(b) $F : C \to B$ *has a right adjoint iff it commutes with* $K(k, C) \circ NJk$ *for all* $C \in C$ *and* FN *has a right* N*-adjoint.*

(c) If $F \underset{N}{\dashv} R : B \to C$ *then* R *has a left adjoint* $\overline{F}C = K(k, C) \circ FJk$ *if this colimit exists in* B .

Proof. *(a)* Suppose $F \underset{N'}{\dashv} R$. Then

$$B(FC, B) \cong [A^{op}, V](N'C, RB)$$
$$\cong [A^{op}, V]\big(N'C, B(FN-, B)\big) ,$$

because

$$B(FNA, B) \cong [A^{op}, V]\big(N'(NA), RB\big) \cong RB(A)$$

by the representation theorem and the full faithfulness of N . Thus

$$B(FC, B) = \int_A [C(NA, C), B(FNA, B)]$$
$$\cong B\big(C(NA, C) \circ FNA, B\big) \quad \text{for all } C \in C \text{ and } B \in B .$$

Thus $FC \cong C(NA, C) \circ FNA$ for all $C \in C$ and Proposition 2.2 applies. Conversely, if $F\big(K(k, C) \circ NJk\big) \cong K(k, C) \circ FNJk$ for all $C \in C$ then we define R by $R(B)(A) = B(FNA, B)$ and Proposition 2.2 applies.

(b) Suppose $FN \underset{N}{\dashv} R$ and $F\big(K(k, C) \circ NJk\big) \cong K(k, C) \circ FNJk$. By Proposition 2.2, F is the Kan extension of its restriction by N . Thus $FC \cong C(NA, C) \circ FNA$ whence $F \dashv R$ because

https://doi.org/10.1017/S0004972700023509 Published online by Cambridge University Press

$$\begin{aligned}\mathcal{B}(FC, B) &\cong \mathcal{B}\big(\mathcal{C}(NA, C)\circ FNA, B\big)\\ &\cong \int_A [\mathcal{C}(NA, C), \mathcal{B}(FNA, B)]\\ &\cong \int_A [\mathcal{C}(NA, C), \mathcal{C}(NA, RB)]\\ &\cong \mathcal{C}(C, RB) \quad \text{because } N \text{ is dense.}\end{aligned}$$

Conversely, if $F \dashv R$ then F preserves colimits and

$$\mathcal{B}(FNA, B) = \mathcal{C}(NA, RB) ,$$

as required.

(c)

$$\begin{aligned}\mathcal{B}(\overline{F}C, B) &= \mathcal{B}\big(K(k, C)\circ FJk, B\big)\\ &\cong \int_k [K(k, C), \mathcal{B}(FJk, B)]\\ &\cong \int_k [K(k, C), \mathcal{C}(NJk, RB)]\\ &\cong \mathcal{C}\big(K(k, C)\circ NJk, RB\big)\\ &\cong \mathcal{C}(C, RB) . \qquad //\end{aligned}$$

3. The characterisation theorem

We suppose that $N : \mathcal{A} \to \mathcal{C}$ is dense by a strict presentation $\mathcal{P} = (J, K, \xi)$.

THEOREM 3.1. *Suppose* $F \underset{N}{\dashv} R : \mathcal{B} \to \mathcal{C}$ *and* $\overline{F}C = K(k, C) \circ FJk$ *exists in* $\mathcal{B}$ *for all* $C \in \mathcal{C}$. *Then*

(a) $\overline{F} \dashv R$ *is a reflective embedding if* R *reflects isomorphisms and preserves* $K(k, RB) \circ FJk$ *for all* $B \in \mathcal{B}$ *and* $N \cong RF$ *: in this case* F *is dense;*

(b) $\overline{F} \dashv R$ *is an equivalence iff* R *reflects isomorphisms and preserves* $K(k, C) \circ FJk$ *for all* $C \in \mathcal{C}$ *and* $N \cong RF$ *;*

(c) $\overline{F} \dashv R$ *is an isomorphism iff the conditions of (b) hold and* R *creates isomorphisms.*

Proof. *(a)* Applying R to the counit of $\overline{F} \dashv R$ we obtain:

https://doi.org/10.1017/S0004972700023509 Published online by Cambridge University Press

$$\begin{aligned} R\bar{F}RB &= R\big(K(k,\ RB)\circ FJk\big) \\ &\cong K(k,\ RB)\circ RFJk \\ &\cong K(k,\ RB)\circ NJk \\ &\cong RB\ , \end{aligned}$$

so $\bar{F}R \cong 1$, as required. Thus F is dense by index J and coefficients $K(k,\ RB)$.

(b) Necessity of the conditions is clear. Conversely, by *(a)*, it remains to show that the unit of $\bar{F} \dashv R$ is an isomorphism. We have

$$\begin{aligned} C &\cong K(k,\ C)\circ NJk \\ &\cong K(k,\ C)\circ RFJk \\ &\cong R\big(K(k,\ C)\circ FJk\big) \\ &\cong R\bar{F}C \end{aligned}$$

as required.

(c) This follows from *(b)*.

DEFINITION 3.2. An object $D \in \mathcal{B}$ is called *P-presentable* if $\mathcal{B}(D,\ -)$ preserves $K(k,\ C)\circ FJk$ for all $C \in \mathcal{B}$.

THEOREM 3.3. *Suppose* N *is fully faithful,* $F \underset{N}{\dashv} R : \mathcal{B} \to \mathcal{C}$ *, and* $\bar{F}C = K(k,\ C)\circ FJk$ *exists in* $\mathcal{B}$ *for all* $C \in \mathcal{C}$ *. Then* $\bar{F} \dashv R$ *is an equivalence iff* $N \cong RF$ *,* F *is fully faithful and strongly generating, and each object* FJk *,* $k \in \mathcal{K}$ *, is P-presentable.*

Proof. Necessity of the conditions is clear. For sufficiency first note that R reflects isomorphisms because Rf an isomorphism implies $\mathcal{C}(NA,\ Rf) \cong \mathcal{B}(FA,\ f)$ is isomorphism for all $A \in \mathcal{A}$, implies f an isomorphism since F strongly generates $\mathcal{B}$. By Theorem 3.1 it remains to prove that $R\big(K(k,\ C)\circ FJk\big) \cong K(k,\ C)\circ NJk \cong C$ for all $C \in \mathcal{C}$. Because NJ strongly generates $\mathcal{C}$ this follows from

$$\begin{aligned} \mathcal{C}\Big(NJh,\ R\big(K(k,\ C)\circ FJk\big)\Big) &\cong \mathcal{B}\big(FJh,\ K(k,\ C)\circ FJk\big) \\ &\cong K(k,\ C)\circ \mathcal{B}(FJh,\ FJk) \\ &\cong K(k,\ C)\circ \mathcal{A}(Jh,\ Jk) \\ &\cong \mathcal{C}(NJh,\ C) \quad \text{for all } h \in \mathcal{K}\ . \qquad // \end{aligned}$$

4. Examples

EXAMPLE 4.1. If $N : \mathcal{A} \to \mathcal{C}$ is an arbitrary dense functor then there exists a standard coend presentation of N with $K = A$, $J = 1 : A \to A$, $K(A, C) = C(NA, C)$, and

$$\xi : C(NA, C) \circ NA \cong C .$$

It is a simple matter to recover the following from Theorem 3.3.

THEOREM 4.1.1 (Bunge [6]). *A category* B *is equivalent to* $[D^{op}, V]$ *for some small* D *iff* B *is cocomplete and has a small strongly generating subcategory* D *such that, for each* $D \in D$, $B(D, -) : B \to V$ *preserves all colimits.* //

EXAMPLE 4.2. If $N : A \to C$ is dense then there exists a comma-category density presentation of N . If P and Q denote the projections

$$\begin{tikzcd} & N/C \arrow[dl, "P"'] \arrow[dr, "Q"] & \\ A \arrow[dr, "N"'] & & C \arrow[dl, "1"] \\ & C & \end{tikzcd}$$

then we obtain a natural isomorphism

$$\int_A B(FNA, GA) \cong \int_k B(FQk, GPk)$$

which holds for all $F : C^{op} \to B$ and $G : A^{op} \to B$. Thus we obtain

$$C(Qk, C) \circ NPk \cong C .$$

Hence we recall that the V-comma category N/M of functors $N : A \to C \leftarrow B : M$ is constructed by taking as objects the morphisms $f \in C_0(NA, MB)$ and defining the objects of morphisms by means of pull-back diagrams of the form

$$\begin{tikzcd} A(AA') \arrow[d, "T(-B')"'] & & B(BB') \arrow[d, "T(A-)"] \\ {[T(A'B'),T(AB')]} \arrow[dr, "{[f',1]}"'] & & {[T(AB),T(AB')]} \arrow[dl, "{[f,1]}"] \\ & {[I,T(AB')]} & \end{tikzcd}$$

where $T(AB) = C(NA, MB)$.

https://doi.org/10.1017/S0004972700023509 Published online by Cambridge University Press

EXAMPLE 4.3. Let $\mathsf{T} = (T, \mu, \eta)$ be a monad on C with Kleisli category $A = C_{\mathsf{T}}$ and Eilenberg-Moore category C^{T}. Let $N : A \to C^{\mathsf{T}}$ be the comparison functor and let $\mathcal{D}$ be the free V-category on the category

$$\left\{2 \underset{\varepsilon}{\overset{m}{\rightrightarrows}} 1\right\} .$$

Let $K = \mathcal{D} \otimes C^{\mathsf{T}}$ with index functor $J : K \to A$ defined by

$$J\big(m, (C, \zeta)\big) = \mu_C : T^2C \to TC ,$$

$$J\big(\varepsilon, (C, \zeta)\big) = T\zeta : T^2C \to TC ,$$

for each T-algebra (C, ζ). The coefficient functor $K : K^{\mathrm{op}} \otimes C^{\mathsf{T}} \to V$ is then defined by

$$\begin{aligned} K\big(d, (C, \zeta), (D, \zeta)\big) &= I \otimes C^{\mathsf{T}}(C, D) \\ &\cong C^{\mathsf{T}}(C, D) . \end{aligned}$$

Then we have

$$\begin{aligned} K(d, D, C) \circ NJ(d, D) &= C^{\mathsf{T}}(C, D) \circ NJ(d, D) \\ &\cong \int^{d} NJ(d, C) \end{aligned}$$

by the representation theorem applied to $D \in C^{\mathsf{T}}$

$$\begin{aligned} &\cong \operatorname{colim} NJ(d, C) \\ &\cong C \end{aligned}$$

because $T^2C \underset{T\zeta}{\overset{\mu}{\rightrightarrows}} TC \xrightarrow{\zeta} C$ is a coequaliser diagram in C^{T}. This isomorphism is natrual in $C \in C^{\mathsf{T}}$ and the coequaliser diagram is preserved by each functor $C^{\mathsf{T}}(NA, -) : C^{\mathsf{T}} \to V$; hence we obtain a density presentation of N. From Theorem 3.1 we obtain:

THEOREM 4.4.1 (Beck). *An adjunction* $F \dashv U : B \to C$ *is monadic iff* U *creates coequalisers of* U*-split pairs.* //

On applying the underlying-set functor $V : V \to S$ to Proposition 1.2, we obtain:

https://doi.org/10.1017/S0004972700023509 Published online by Cambridge University Press

PROPOSITION 4.3.2. *The comparison functor* $N : C_T \to C^T$ *is dense and, for each algebra* $(C, \zeta) \in C^T$ *, the natural transformations from* $C^T(N-, C)$ *to a prealgebra* $G : A^{op} \to V$ *correspond to the elements in the equaliser of*

$$VGC \underset{VGT\zeta}{\overset{VG\mu}{\rightrightarrows}} VGTC$$

where μ *and* $T\zeta$ *are regarded as morphisms in* C_T . //

More generally we have:

PROPOSITION 4.3.3. *If* $F \dashv U : B \to C$ *and* $\varepsilon : FU \to 1$ *is a regular epimorphism then the full image of* F *is dense in* B *if* B *has kernel pairs.*

Proof. This is essentially the same as for the monad case except that one presents each $B \in B$ as a coequaliser of the form

$$D(B) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} FUB \xrightarrow{\varepsilon} B$$

where D is the functorial kernel pair of ε . //

This example admits an evident generalisation. One may suppose that there is a given a functor $N : A \to C$, a "diagram" $\mathcal{D}$, and functors $J : \mathcal{D} \otimes C \to A$ and $H : \mathcal{D}^{op} \to V$ together with a natural transformation $\alpha_{dC} : Hd \to C(NJ(d, C), C)$ such that

$$\int^{\mathcal{D}} H(d) \cdot NJ(d, C) \cong C$$

holds and this coend is preserved by each of the functors $C(NA, -)$, $A \in A$. Then one obtains a strict presentation of N on taking the coefficient functor to be

$$K : \mathcal{D}^{op} \otimes C^{op} \otimes C \xrightarrow{H \otimes \mathrm{hom}C} V \otimes V \xrightarrow{\otimes} V .$$

EXAMPLE 4.4. Let $V = S$ and let C be a category with canonical $E - M$ factorisations for a proper $E - M$ factorisation system on C (Freyd and Kelly [12]). Let M also denote the category whose objects are those of C and whose morphisms are those in M . Let A be a full subcategory of $\mathcal{C}$ which is closed under E-images and let $N : \mathcal{A} \to \mathcal{C}$ denote the inclusion (usually omitted). On taking $K = \mathcal{A} \cap M$ and $K(A, C) = M(A, C)$ we have

$$\int^{A} M(A, C) \times \mathcal{C}(B, A) \cong \mathcal{C}(B, C)$$

naturally in $C \in M$ and $B \in \mathcal{A}$. Thus, if each object $C \in \mathcal{C}$ is the union $\int^{A} M(A, C) \cdot A$ of its subobjects which lie in K then $N : \mathcal{A} \to \mathcal{C}$ is dense and

$$\int_{\mathcal{A}} [\mathcal{C}(A, C), GA] \cong \int_{M} [M(A, C), GA]$$

for each functor $G : \mathcal{A}^{\mathrm{op}} \to S$. It follows from Proposition 1.3 that $\int^{A} M(A, C) \cdot A \cong C$ if $\mathcal{A}$ strongly generates $\mathcal{C}$ and each $A \in \mathcal{A}$ is P-presentable for this P . In the case $\mathcal{C} = S$ any category $\mathcal{A} \subset S$ which is closed under surjective images in S satisfies these conditions.

This presentation occurs in the study of pro-objects and model-induced monads (see Appelgate and Tierney [1], Hofmann [15], Kennison and Gildenhuys [17], and Lim [19]). Given $\mathcal{A} \subset \mathcal{C}$ we form

$$\begin{aligned} P\mathcal{C} &= \left\{C \in \mathcal{C};\ \int^{A} M(A, C)\cdot A \cong C\right\} \\ &= \left\{C \in \mathcal{C};\ \operatorname*{colim}_{M(A,C)} A \cong C\right\} . \end{aligned}$$

PROPOSITION 4.4.1. *If* $P\mathcal{C}$ *is closed in* $\mathcal{C}$ *under* E*-quotients and* $\operatorname*{colim}_{M(A,C)} A \in P\mathcal{C}$ *for all* $C \in \mathcal{C}$ *then* $P\mathcal{C}$ *is coreflective in* $\mathcal{C}$.

Proof. The coreflection RC of $C \in \mathcal{C}$ is defined by the $E - M$ factorisation of the canonical transformation $\operatorname{colim} A \to C$. //

We generalise a result of Lim [19], Theorem 5.1.

THEOREM 4.4.2. *If there exists* $U \dashv G : S \to P\mathcal{C}$ *then* UG *generates the density comonad on* $U_0 = UN$.

Proof.

$$\begin{tikzcd} \mathrm{A} \arrow[r, "N"] \arrow[dr, "U_0"'] & PC \subset C \arrow[d, shift right, "U"'] \\ & S \arrow[u, shift right, "G"'] \end{tikzcd} ,$$

$$GX \cong \int^A PC(A, GX) \cdot A \quad \text{by the representation theorem}$$

$$\cong \int^A S(UA, X) \cdot A$$

whence $UGX \cong \int^A S(U_0A, X) \cdot U_0A$. //

COROLLARY 4.4.3. *If* PC *is comonadic with respect to* $\mathrm{G} = UG$ *then* PC *is category equivalent to the category of coalgebras model-induced by* $U_0 = UN$. //

COROLLARY 4.4.4. *If each* $A \in \mathrm{A}$ *is* GX *for some* $X \in S$ *then* S_{G} *is dense and codense in* PC . //

Most of these results are valid for $V = Ab$; however, for general V , one must postulate that $E - M$ is suitably related to C in the first place.

EXAMPLE 4.5. If a composite functor of the form

$$K \xrightarrow{J} \mathrm{A} \xrightarrow{N} C$$

is dense then we can ask when the resultant expression $C(NJk, C) \circ NJk \cong C$ gives a density presentation of N . Clearly this is so iff

$$C(NJk, C) \circ \mathrm{A}(A, Jk) \cong C(NA, C)$$

for all $A \in \mathrm{A}$ and $C \in C$. If N is a full embedding this is so iff J is Cauchy dense in the sense that

$$\int^k \mathrm{A}(Jk, A') \otimes \mathrm{A}(A, Jk) \cong \mathrm{A}(A, A')$$

for all $A, A' \in \mathrm{A}$.

For example, let $V = \mathrm{R}^+$ with the usual monoidal closed structure generated by addition on R^+ (Lawvere [18]). Then V-categories are

https://doi.org/10.1017/S0004972700023509 Published online by Cambridge University Press

simply metric spaces in a suitably general sense. The unit circle is V-dense in the unit disc in $\mathbb{C}$, a presentation being given by the rational points on the circle. The Kan extension (Day [8]) of addition of the rational points is the multiplication of the points in the disc.

EXAMPLE 4.6. Suppose that $N : A \to C$ is a functor and, for each $C \in |C|$, there exists a free V-category K_C and a functor $P_C : K_C \to A$ together with a colimit cone $\xi_C : \mathrm{colim}_k\, NPk \cong C$ such that $\mathrm{colim}_k\, A(A, Pk) \cong C(NA, C)$ for all $A \in A$. Then (P, K, ξ) determines a density presentation of C on setting

$$K = \sum_{|C|} K_C \quad \text{and} \quad J = (P_C) : K \to A ,$$

and defining $K : K^{\mathrm{op}} \otimes |C| \to V$ by the Kronecker condition

$$\begin{cases} K(k, C) = I \text{ if } k \in K_C , \\ K(k, C) = 0 \text{ if } k \notin K_C . \end{cases}$$

For example, if $V = S$ then the "discrete" comma-category presentation of a dense functor N is given by $K_C = N/C$ for each $C \in C$. Furthermore, given *any* functor $N : A \to C$ the condition $\mathrm{colim}_k\, A(A, Pk) \cong C(NA, C)$ is satisfied for all $A \in A$ and $C \in C$ by the representation theorem. Thus, by Proposition 1.3, we have:

PROPOSITION 4.6.1. *$N : A \to C$ is dense iff N is strongly generating and $C(NA, \mathrm{colim}_k\, NPk) \cong \mathrm{colim}_k\, A(A, Pk)$ for all $A \in A$ and $C \in C$.* //

In particular, if C is an S-based category and α is a regular cardinal number then $A \in C$ is called *α-presentable* if $C(A, -) : C \to S$ preserves α-filtered colimits (see Gabriel, Ulmer [13]). Let C_α denote the full subcategory of C determined by the α-presentable objects.

COROLLARY 4.6.2 (Gabriel, Ulmer [13], Theorem 7.4). *A composite inclusion of the form $A \subset C_\alpha \subset C$ is dense if A strongly generates C and A/C is α-filtered for each $C \in C$; in particular, if A is closed* *under α-filtered colimits in* $\mathcal{C}$. //

A cocomplete category $\mathcal{C}$ is then said to be locally α-presentable if it contains a small strongly generating set of α-presentable objects. For results involving the characterisation of locally α-presentable categories we refer directly to Diers [*10*], Example 5.3.

Again suppose that $V = S$. An object $C \in \mathcal{C}$ is a *Z-object* (is "connected") if the functor $\mathcal{C}(C, -) : \mathcal{C} \to S$ preserves coproducts in $\mathcal{C}$. An object $C \in \mathcal{C}$ is *Z-generated* if $C = \sum C_i$ with each C_i a Z-object.

PROPOSITION 4.6.3 (R.-E. Hoffmann). *The inclusion of the category of Z-objects in the category of Z-generated objects is dense.* //

EXAMPLE 4.7. Suppose $\xi_C : K_C(k) \circ NJ_C k \cong C$ is given for each $C \in \mathcal{C}$ where $K_C : \mathsf{K}_C^{\mathrm{op}} \to V$ and $J_C : \mathsf{K}_C \to \mathsf{A}$ and $K_C(k) \circ \mathsf{A}\left(A, J_C k\right) \cong \mathcal{C}(NA, C)$ for all $A \in \mathsf{A}$ and $C \in \mathcal{C}$. Then a presentation of $N : \mathsf{A} \to \mathcal{C}$ is given by

$$\mathsf{K} = \sum_{C \in |\mathcal{C}|} \mathsf{K}_C \quad \text{and} \quad J = \left(J_C\right) : \mathsf{K} \to \mathsf{A}$$

with

$$\begin{cases} K(k, C) = K_C(k) & \text{if } k \in \mathsf{K}_C , \\ \\ K(k, C) = 0 & \text{if } k \notin \mathsf{K}_C . \end{cases}$$

For example, if $V = S$ and $N : \mathsf{A} \to \mathcal{C}$ is dense then $N_B : N/B \to \mathcal{C}/B$ is dense for all $B \in \mathcal{C}$ because $N/C \cong N_B/g$ for all $g \in \mathcal{C}(C, B)$. A presentation of N_B is given on setting $\mathsf{K}_g = N/C$ and $J_g = N/g : N/C \to N/B$ for each $g \in \mathcal{C}(C, B)$ and $C \in \mathcal{C}$. The coefficients are then given by $K_g : N/C^{\mathrm{op}} \to S$, $K_g(x) = \mathcal{C}(NA, C)$ where $x : NA \to C$. The result is the "reduced" coend presentation of N_B :

https://doi.org/10.1017/S0004972700023509 Published online by Cambridge University Press

$$\begin{tikzcd} \mathcal{C}(NA, C) \circ NA \arrow[r, phantom, "\cong"] \arrow[d, dashed] & C \arrow[dl, "g"] \\ B & \end{tikzcd}$$

This example extends to any *cartesian* closed category V .

5. Linear monads

Many of the density presentations arising in monad theory are not strict. Thus a more complex form of Theorem 3.1 is often needed. Here, however, we are mainly concerned with the presentations themselves and results along the lines of Theorem 3.1 will appear elsewhere.

Suppose that a monad $\mathsf{T} = (T, \mu, \eta)$ is given on $\mathcal{C}$ and denote the usual resolution of this monad into a Kleisli category and an Eilenberg-Moore category by

$$\begin{tikzcd} \mathcal{C}_{\mathsf{T}} \arrow[rr, "M"] \arrow[dr, shift left, "U"] & & \mathcal{C}^{\mathsf{T}} \arrow[dl, shift left, "U"] \\ & \mathcal{C} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F"] & \end{tikzcd}$$

where M is the dense comparison functor.

PROPOSITION 5.1. *Given a dense functor* $N : \mathcal{A} \to \mathcal{C}_{\mathsf{T}}$ *the composite* $MN : \mathcal{A} \to \mathcal{C}^{\mathsf{T}}$ *is dense.*

Proof. Because N is dense we have $C \cong \mathcal{C}_{\mathsf{T}}(NA, C) \circ NA$ in $\mathcal{C}_{\mathsf{T}}$. Thus $MC \cong \mathcal{C}_{\mathsf{T}}(NA, C) \circ MNA$ for each $(C, \zeta_C) \in \mathcal{C}^{\mathsf{T}}$. Thus there exists a density presentation of MN with structure $\zeta_C : MC \to C$. //

The concept of a density presentation is related to the idea of a monad with rank (see Barr [2] and [3], Freyd and Kelly [12], Gabriel, Ulmer [13], and Linton [20]). Suppose $N : \mathcal{A} \to \mathcal{C}$ is a full embedding with density presentation $P = (J, K, \xi)$.

DEFINITION 5.2. (a) T is a *P-monad* if $\mathcal{C}(NA, T\xi)$ and $K(k, C) \circ \mathcal{C}(NA, TNJk) \to \mathcal{C}\big(NA, T\big(K(k, C) \circ NJk\big)\big)$ are both epimorphisms for all $A \in \mathcal{A}$ and $C \in \mathcal{C}$.

(b) T is *P-linear* if $K(k, C) \circ TNJk$ exists in $\mathcal{C}$ for all $C \in \mathcal{C}$ and the canonical maps $K(k, C) \circ TNJk \to TC$ are epimorphisms for all $C \in \mathcal{C}$.

PROPOSITION 5.3. *A P-monad is P-linear.*

Proof. Consider the diagram

$$\begin{tikzcd} \int^k K(k,C) \otimes \mathcal{C}(NA, TNJk) \arrow[r] \arrow[d] & \mathcal{C}\big(NA, T(K(k,C) \circ NJk)\big) \arrow[d, "\mathcal{C}(1, T\xi)"] \\ \mathcal{C}\big(NA, K(k,C) \circ TNJk\big) \arrow[r] & \mathcal{C}(NA, TC) \end{tikzcd}$$

The diagonal is an epimorphism because T is a P-monad. Thus, by density of N, the maps $K(k, C) \circ TNJk \to TC$ are epimorphisms. //

For the given density presentation P of N define categories K' and A' and functors J' and N' as follows: A' is the full image of $FN : A \to \mathcal{C}_T$, K' is the full image of $FJ : K \to A'$, and $N' : A' \to \mathcal{C}_T$, $J' : K' \to A'$ are the induced functors such that $FN = N'F$ and $FJ = J'F$:

$$\begin{tikzcd} K' \arrow[r, "J'"] & A' \arrow[r, "N'"] & \mathcal{C}_T \\ K \arrow[u, "F"] \arrow[r, "J"'] & A \arrow[u, "F"] \arrow[r, "N"'] & \mathcal{C} \arrow[u, "F"] \end{tikzcd}$$

A coefficient functor $K' : (K')^{op} \times |\mathcal{C}_T| \to V$ is defined as $K'(Fh, FC) = K'(Fh, Fk) \circ K(k, C)$, and $\xi' : K'(Fh, Fc) \circ N'J'(Fh) \to FC$ is defined by:

$$\begin{aligned} K'(Fh, FC) \circ N'J'(Fh) &= \big(K'(Fh, Fk) \circ K(k, C)\big) \circ N'J'(Fh) \\ &\cong K(k, C) \circ N'J'(Fk) \quad \text{by the representation theorem} \\ &\cong K(k, C) \circ FNJk \\ &\cong F\big(K(k, C) \circ NJk\big) \\ &\to FC , \end{aligned}$$

thus giving a presentation P' of N'.

THEOREM 5.4. *T is a P-monad iff P' is a density presentation of N'.*

Proof. Because $F \dashv U$, $\mathcal{C}_T(N'FA, \xi'_{FC})$ is an epimorphism for all

$FA \in A'$ iff $C(NA, T\xi_C)$ is an epimorphism for all $A \in A$;

$$C(N'FA, \xi'_{FC}) \cong C(FNA, \xi'_{FC}) \cong C(NA, T\xi_C) .$$

Moreover,

$$\begin{aligned} K'(Fk, FC) \circ A'(FA, J'Fk) &= (K'(Fk, Fh) \circ K(h, C)) \circ A'(FA, J'Fk) \\ &\cong K(h, C) \circ A'(FA, J'Fh) \\ &\qquad \text{by the representation theorem} \\ &\cong K(h, C) \circ C_T(N'FA, N'J'Fh) \\ &\cong K(h, C) \circ C^T(FNA, FNJh) \\ &\cong K(h, C) \circ C(NA, TNJh) \\ &\to C(NA, T(K(k, C) \circ NJk)) \\ &\cong C(NA, U(K(k, C) \circ FNJk)) \quad \text{since } F \dashv U \\ &\cong C_T(FNA, K'(Fk, FC) \circ N'J'Fk) \end{aligned}$$

which is an epimorphism iff

$$K(h, C) \circ (NA, TNJh) \to C(NA, T(K(h, C) \circ NJh))$$

is an epimorphism. //

In order to demonstrate the relationship of P-monads to monads with rank consider Proposition 4.6.2 where $V = S$ and α is a regular cardinal. Suppose each A/C is α-filtered and $P_C : A/C \to A$ be projection. Suppose A is small and C is cocomplete.

PROPOSITION 5.5. *The following conditions are equivanlent:*

(a) $\displaystyle\int^B C(NB, C) \times C(NA, TNB) \cong C(NA, TC)$;

(b) $\displaystyle\int^A C(NA, C) \cdot TNA \cong TC$;

(c) T *preserves* $\operatorname{colim} NP_C k$ *for all* $C \in C$;

(d) T *preserves* α*-filtered colimits.*

Proof. (b) $\Rightarrow$ (d) since $NA \in C_\alpha$ for all $A \in A$. (d) $\Rightarrow$ (c) because A/C is α-filtered for all $C \in C$. (c) $\Rightarrow$ (b) because

$$\int^{A} C(NA,\, C) \cdot TNA \cong \int^{A} C\big(NA,\, \mathrm{colim}\; NP_{C}k\big) \cdot TNA$$

$$\cong \mathrm{colim} \int^{A} C\big(NA,\, NP_{C}k\big) \cdot TNA \quad \text{because} \quad NA \in C_{\alpha}$$

$$\cong \mathrm{colim}\; TNP_{C}k \quad \text{by the representation theorem}$$

$$\cong T\big(\mathrm{colim}\; NP_{C}k\big) \quad \text{by } (c)$$

$$\cong TC\ .$$

Finally *(a)* $\Leftrightarrow$ *(c)* because

$$\int^{B} C(NB,\, C) \times C(NA,\, TNB) \cong \int^{B} C\big(NB,\, \mathrm{colim}\; NP_{C}k\big) \times C(NA,\, TNB)$$

$$\cong \mathrm{colim} \int^{B} C\big(NB,\, NP_{C}k\big) \times C(NA,\, TNB)$$

$$\text{because} \quad NB \in C_{\alpha}$$

$$\cong \mathrm{colim}\; C\big(NA,\, TNP_{C}k\big)$$

$$\text{by the representation theorem}$$

$$\cong C\big(NA,\, \mathrm{colim}\; TNP_{C}k\big) \quad \text{because} \quad NA \in C_{\alpha}$$

$$\cong C\big(NA,\, T\big(\mathrm{colim}\; NP_{C}k\big)\big) \cong C(NA,\, TC)\ . \qquad //$$

Suppose that $A_0 \in A$ is a base object such that $NA \cong C\big(NA_0,\, NA\big) \cdot NA_0$ and let P be the standard coend presentation of $N : A \to C$.

PROPOSITION 5.6. *T is a P-monad iff*

$$\int^{B} C(NB,\, C) \times C\big(NA_0,\, TNB\big) \to C\big(NA_0,\, TC\big)$$

is an epimorphism.

Proof. Necessity is clear. For sufficiency, we have

$$\begin{aligned}\int^B C(NB, C) \times C(NA, TNB) &\cong \int^B C(NB, \operatorname{colim} NP_C k) \times C(NA, TNB) \\ &\cong \operatorname{colim} C(NA, TNP_C k) \\ &\cong C(NA, \operatorname{colim} TNP_C k) \\ &\cong [C(NA_0, NA), C(NA_0, \operatorname{colim} TNP_C k)] \\ &\cong \left[C(NA_0, NA), \int^B C(NB, C) \times C(NA_0, TNB)\right] \\ &\twoheadrightarrow [C(NA_0, NA), C(NA_0, TC)] \\ &\qquad\qquad \text{because epimorphisms split in } S \\ &\cong C(NA, TC) . \qquad //\end{aligned}$$

Thus, if we regard $C(NA_0, TC)$ as the set of C-ary operations of T then T is a P-monad iff each C-ary operation factors appropriately through some NA-ary operation. With suitable modifications NA_0 can clearly be replaced by a dense subcategory of C which lies in A .

Instances of P-monads occur widely in finitary universal algebra over a suitable closed category V (see Borceux and Day [4]).

References

[1] H. Appelgate and M. Tierney, "Categories with models", *Seminar on triples and categorical homology theory*, 156-244 (Lecture Notes in Mathematics, **80**. Springer-Verlag, Berlin, Heidelberg, New York, 1969).

[2] Michael Barr, "Coequalisers and free triples", *Math. Z.* **116** (1970), 307-322.

[3] Michael Barr, "Exact categories", *Exact categories and categories of sheaves*, 1-120 (Lecture Notes in Mathematics, **236**. Springer-Verlag, Berlin, Heidelberg, New York, 1971).

[4] Francis Borceux and Brian Day, "Universal algebra in a closed category", *J. Pure Appl. Algebra* (to appear).

[5] Francis Borceux and G.M. Kelly, "A notion of limit for enriched categories", *Bull. Austral. Math. Soc.* **12** (1975), 49-72.

[6] Marta C. Bunge, "Relative functor categories and categories of algebras", *J. Algebra* 11 (1969), 64-101.

[7] Brian Day, "On closed categories of functors II", *Category seminar*, 20-54 (Proc. Sydney Category Seminar 1972/1973. Lecture Notes in Mathematics, 420. Springer-Verlag, Berlin, Heidelberg, New York, 1974).

[8] B.J. Day, "Varieties of a closed category", *Bull. Austral. Math. Soc.* 16 (1977), 131-145.

[9] Yves Diers, "J-adjonction, limite J-absolue et J-théorie algébrique", *C.R. Acad. Sci. Paris Sér. A* 278 (1974), 1009-1012.

[10] Yves Diers, "Type de densité d'une sous-catégorie pleine", *Ann. Soc. Sci. Bruxelles Sér. I* 90 (1976), 25-47.

[11] Samuel Eilenberg and G. Max Kelly, "Closed categories", *Proc. Conf. Categorical Algebra*, La Jolla, California, 1965, 421-562 (Springer-Verlag, Berlin, Heidelberg, New York, 1966).

[12] P.J. Freyd and G.M. Kelly, "Categories of continuous functors, I", *J. Pure Appl. Algebra* 2 (1972), 169-191.

[13] Peter Gabriel, Friedrich Ulmer, *Lokal präsentierbare Kategorien* (Lecture Notes in Mathematics, 221. Springer-Verlag, Berlin, Heidelberg, New York, 1971).

[14] Dion Gildenhuys and Chong-Keang Lim, "Free pro-C-groups", *Math. Z.* 125 (1972), 233-254.

[15] Karl Heinrich Hofmann, "Category theoretical methods in topological algebra", *Categorical topology*, 345-403 (Proc. Conf. Mannheim, 1975. Lecture Notes in Mathematics, 540. Springer-Verlag, Berlin, Heidelberg, New York, 1976).

[16] J.R. Isbell, "Adequate subcategories", *Illinois J. Math.* 4 (1960), 541-552.

[17] J.F. Kennison and Dion Gildenhuys, "Equational completions, model induced triples and pro-objects", *J. Pure Appl. Algebra* 1 (1971), 317-346.

[18] F. William Lawvere, "Metric spaces, generalized logic, and closed categories", Lecture Notes, Istituto di Matematica, Università di Perugia.

[19] Lim Chong-Keang, "Tripleableness of pro-C-groups", *J. Austral. Math. Soc. Ser. A* 21 (1976), 299-309.

[20] F.E.J. Linton, "Some aspects of equational categories", *Proc. Conf. Categorical Algebra*, La Jolla, California, 1965, 84-94 (Springer-Verlag, Berlin, Heidelberg, New York, 1966).

[21] S. Mac Lane, *Categories for the working mathematician* (Graduate Texts in Mathematics, 5. Springer-Verlag, New York, Heidelberg, Berlin, 1971).

[22] Friedrich Ulmer, "Properties of dense and relative adjoint functors", *J. Algebra* 8 (1968), 77-95.

Department of Pure Mathematics,
University of Sydney,
Sydney,
New South Wales.