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BEST DIFFERENCE EQUATION APPROXIMATION TO DUFFING'S EQUATION

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Abstract

Duffing's differential equation in its simplest form can be approximated by a variety of difference equations. It is shown how to choose a 'best' difference equation in the sense that the solutions of this difference equation are successive discrete exact values of the solution of the differential equation.

1. Introduction

A differential equation, such as the simple Duffing equation [2]

$$\ddot{x}(t) + ax(t) + bx^3(t) = 0, \tag{1.1}$$

can be approximated by difference equations in a variety of ways. In numerical analysis, the aim is to choose a difference equation (ΔE) which gives a high-order approximation to the differential equation (DE) and which can be solved computationally in a stable way without trouble from round-off errors.

The simplest ΔE approximation to (1.1) is

$$h^{-2}(x_{n+1} - 2x_n + x_{n-1}) + ax_n + bx_n^3 = 0, \tag{1.2}$$

where h is a chosen constant time interval, but for theoretical purposes this is not suitable because it cannot be solved in closed form, nor can the general behaviour of its solutions be analysed.

In a recent paper [3] it has been shown that an alternative ΔE approximation, namely

$$h^{-2}(x_{n+1} - 2x_n + x_{n-1}) + ax_n + \tfrac{1}{2}bx_n^2(x_{n+1} + x_{n-1}) = 0, \tag{1.3}$$

can be solved in closed form and for appropriate choice of the value of h periodic solutions are obtained with the same qualitative features as the solutions of the *DE* (1.1).

The problem solved in this paper is: find a best ΔE approximation to the *DE* (1.1).

By "best" we shall mean the following: if $x(t)$ is the solution of a *DE*, then a best ΔE approximation to the *DE* using a time interval h is one for which the solution x_n of the ΔE is given by

$$x_n = x(nh). \tag{1.4}$$

Geometrically, the x_n satisfying the ΔE are required to be points on the solution curve $x(t)$ of the *DE* at successive values $t = nh$, where n is an integer.

In previous papers [4], [5] this question has been answered for linear *ODE*'s with constant coefficients and for some non-linear *ODE*'s describing ecological problems. To answer the question for the Duffing equation (1.1) we consider the three cases which arise giving the Jacobian elliptic functions cn, dn and sn as periodic solutions and a further case of a non-periodic solution.

In general, there will be more than one ΔE with the property (1.4), but the particular difference equations we derive are simple in form and directly recognizable as approximations to the *DE*.

2. Case I

Consider first the equation (1.1) with initial conditions

$$x(0) = A \quad \text{and} \quad \dot{x}(0) = 0, \tag{2.1}$$

and for the range of parameter values

$$a > -\tfrac{1}{2}bA^2 \quad \text{where } b > 0. \tag{2.2}$$

Then the well-known [2] solution of (1.1) is

$$x(t) = A\,\mathrm{cn}\left[(a + bA^2)^{1/2}t \,|\, \tfrac{1}{2}bA^2/(a + bA^2)\right], \tag{2.3}$$

using the notation of [1].

To find a ΔE satisfied by this function we use the following.

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LEMMA. *If*

$$x(t_2) = f[t_2 - t_1, x(t_1)] \quad \textit{for all } t_1, t_2 \tag{2.4}$$

and

$$x_n = x(nh) \quad \textit{where } n \textit{ is an integer} \tag{2.5}$$

then

$$x_{n+1} = f[h, x_n], \tag{2.6}$$

and

$$x_{n-1} = f[-h, x_n]. \tag{2.7}$$

The proof of the lemma is immediate.

For the function

$$x(t) = A\,\mathrm{cn}[l(t_0 + t_1)\,|\,m] = A\,\mathrm{cn}[l(t_0 + t)] \tag{2.8}$$

the addition formula for cn can be used to derive from

$$x(t_2) = A\,\mathrm{cn}[l(t_2 - t_1) + l(t_0 + t_1)] \tag{2.9}$$

the relation

$$\begin{aligned} x(t_2)\{1 - m\,\mathrm{sn}^2[l(t_2 - t_1)]\mathrm{sn}^2[l(t_0 + t_1)]\} \\ = A\,\mathrm{cn}[l(t_2 - t_1)]\mathrm{cn}[l(t_0 + t_1)] \\ -A\,\mathrm{sn}[l(t_2 - t_1)]\mathrm{sn}[l(t_0 + t_1)]\mathrm{dn}[l(t_2 - t_1)]\mathrm{dn}[l(t_0 + t_1)]. \end{aligned} \tag{2.10}$$

Use of the lemma gives the ΔE

$$(x_{n+1} + x_{n-1})\{1 - m\,\mathrm{sn}^2(lh)[1 - A^{-2}x_n^2]\} = 2x_n\,\mathrm{cn}(lh). \tag{2.11}$$

Specific application to (2.3) accordingly gives

$$(x_{n+1} + x_{n-1})\{\mathrm{dn}^2 + x_n^2[\tfrac{1}{2}b/(a + bA^2)]\mathrm{sn}^2\} = 2x_n\,\mathrm{cn}, \tag{2.12}$$

where, in this ΔE, we use the abbreviated notation

$$\mathrm{sn} = \mathrm{sn}\left[(a + bA^2)^{1/2}h\,|\,\tfrac{1}{2}bA^2/(a + bA^2)\right] \tag{2.13}$$

and similarly for cn and dn.

For $t_0 = 0$ and $x_0 = A$, $x_1 = x_{-1}$ and the solution of (2.12) is

$$x_n = A\,\mathrm{cn}\left[(a + bA^2)^{1/2}nh\,|\,\tfrac{1}{2}bA^2/(a + bA^2)\right]. \tag{2.14}$$

The ΔE (2.12) is thus a best approximation to the DE (1.1) in the sense that, comparing (2.3) with (2.14),

$$x_n = x(nh). \tag{2.15}$$

It should be noted that that the constant time interval h does not need to be 'small' in any sense, nor is it restricted to being positive. As h is made smaller, the points x_n on the solution curve $x(t)$ simply come closer together.

The fact that in the limit $h \to 0$ the ΔE tends to the DE can be verified as follows. The ΔE (2.12) can be written

$$(x_{n+1} - 2x_n + x_{n-1})(a + bA^2)\mathrm{ns}^2\mathrm{dn}^2 + 2x_n(a + bA^2)\mathrm{ns}^2(\mathrm{dn}^2 - \mathrm{cn}) + \tfrac{1}{2}bx_n^2(x_{n+1} + x_{n-1}) = 0. \tag{2.16}$$

And since

$$(a + bA^2)\mathrm{ns}^2\mathrm{dn}^2 = h^{-2} + O(h^{-1}) \tag{2.17}$$

and

$$2(a + bA^2)\mathrm{ns}^2(\mathrm{dn}^2 - \mathrm{cn}) = a + O(h) \tag{2.18}$$

the ΔE (2.16) tends to the DE (1.1) as $h \to 0$.

It is interesting to note from (2.16) that this ΔE is obtained from the DE by replacing $\ddot{x}$ by $(x_{n+1} - 2x_n + x_{n-1})(a + \frac{1}{2}bA^2)\,\mathrm{ns}^2\mathrm{dn}^2$ instead of the usual $h^{-2}(x_{n+1} - 2x_n + x_{n-1})$. And as has been observed previously [3], x^3 has been replaced not by x_n^3 but by $\frac{1}{2}x_n^2(x_{n+1} + x_{n-1})$.

3. Case II

For the range of parameters

$$-bA^2 < a < -\tfrac{1}{2}bA^2 \quad \text{where } b > 0, \tag{3.1}$$

we transform the results for case I using the reciprocal parameter [1]

$$\mu = m^{-1} = 2(a + bA^2)/(bA^2) \quad \text{where } 0 < \mu < 1, \tag{3.2}$$

and

$$v = (\tfrac{1}{2}bA^2)^{1/2}h, \tag{3.3}$$

so that

$$\mathrm{sn}\left[(a + bA^2)^{1/2}h \,|\, \tfrac{1}{2}bA^2/(a + bA^2)\right] = \left[2(a + bA^2)/(bA^2)\right]^{1/2}\mathrm{sn}(v \,|\, \mu), \tag{3.4}$$

and similarly

$$\mathrm{cn} = \mathrm{dn}(v \mid \mu), \tag{3.5}$$

$$\mathrm{dn} = \mathrm{cn}(v \mid \mu). \tag{3.6}$$

Hence (2.12) becomes

$$(x_{n+1} + x_{n-1})\{\mathrm{cn}^2 + x_n^2 A^{-2} \mathrm{sn}^2\} = 2x_n \mathrm{dn}, \tag{3.7}$$

where

$$\mathrm{sn} = \mathrm{sn}\left[\left(\tfrac{1}{2}bA^2\right)^{1/2} h \mid 2(a + bA^2)/(bA^2)\right], \tag{3.8}$$

and similarly for cn and dn.

For the range of parameters (3.1) the solution of the *DE* (1.1) is [2]

$$x(t) = A\,\mathrm{dn}\left[\left(\tfrac{1}{2}bA^2\right)^{1/2} t \mid 2(a + bA^2)/(bA^2)\right], \tag{3.9}$$

while the solution of (3.7) is

$$x_n = A\,\mathrm{dn}\left[\left(\tfrac{1}{2}bA^2\right)^{1/2} nh \mid 2(a + bA^2)/(bA^2)\right]. \tag{3.10}$$

Again we have $x_n = x(nh)$ so that in the sense defined, (3.7) is a best ΔE approximation to the *DE* (1.1) for the range of parameters (3.1).

4. Case III

For the range of parameters

$$a > -bA^2 \quad \text{where } b < 0, \tag{4.1}$$

we use the 'negative parameter' transformation [1]

$$\mu = \frac{-m}{1 - m} = \frac{-bA^2}{2a + bA^2} \quad \text{where } 0 < \mu < 1, \tag{4.2}$$

and

$$v = \left(a + \tfrac{1}{2}bA^2\right)^{1/2} h, \tag{4.3}$$

so that

$$\mathrm{sn}\left[(a + bA^2)^{1/2} h \mid \tfrac{1}{2}bA^2/(a + bA^2)\right] = (a + bA^2)^{1/2}\left(a + \tfrac{1}{2}bA^2\right)^{-1/2} \mathrm{sd}(v \mid \mu) \tag{4.4}$$

and similarly

$$\mathrm{cn} = \mathrm{cd}(v \mid \mu), \tag{4.5}$$

$$\mathrm{dn} = \mathrm{nd}(v \mid \mu). \tag{4.6}$$

Hence (2.12) becomes

$$(x_{n+1} + x_{n-1})\{1 + x_n^2[b/(2a + bA^2)]\text{sn}^2\} = 2x_n \text{cn dn}, \tag{4.7}$$

where

$$\text{sn} = \text{sn}\left[(a + \tfrac{1}{2}bA^2)^{1/2}h \mid -\tfrac{1}{2}bA^2/(a + \tfrac{1}{2}bA^2)\right], \tag{4.8}$$

and similarly for cn and dn.

For the range of parameters (4.1) the solution of the *DE* (1.1) satisfying the conditions

$$x(0) = 0 \text{ and } |x| = A \quad \text{when } \dot{x} = 0, \tag{4.9}$$

is

$$x(t) = A\,\text{sn}\left[(a + \tfrac{1}{2}bA^2)^{1/2}t \mid -\tfrac{1}{2}bA^2/(a + \tfrac{1}{2}bA^2)\right]. \tag{4.10}$$

A solution of the ΔE (4.7) is

$$x_n = A\,\text{cd}\left[(a + \tfrac{1}{2}bA^2)^{1/2}(t_0 + nh) \mid -\tfrac{1}{2}bA^2/(a + \tfrac{1}{2}bA^2)\right]. \tag{4.11}$$

If we choose t_0 so that

$$(a + \tfrac{1}{2}bA^2)^{1/2}t_0 = -K, \tag{4.12}$$

where K is the quarter-period of sn, then

$$x_n = A\,\text{sn}\left[(a + \tfrac{1}{2}bA^2)^{1/2}nh \mid -\tfrac{1}{2}bA^2/(a + \tfrac{1}{2}bA^2)\right] \tag{4.13}$$

satisfying $x_0 = 0$ and max $x = A$. Comparing (4.13) with (4.10) we again see that $x_n = x(nh)$.

5. A case of a non-periodic solution

The Duffing equation

$$\ddot{x} + \omega^2 x - 2\omega^2 A^{-2}x^3 = 0, \tag{5.1}$$

with

$$x(0) = A \quad \text{and} \quad \dot{x}(0) = 0,$$

has values for the parameters outside the ranges considered in the above three cases since $a = -\tfrac{1}{2}bA^2 < -bA^2$ and $b < 0$. The *DE* (5.1) has the non-periodic solution

$$x(t) = A \sec \omega t. \tag{5.2}$$

A best approximating ΔE to the DE (5.1) in this case is obtained using the addition formula for sec, namely

$$A\sec(\omega t_2) = A\sec[\omega(t_2 - t_1) + \omega t_1]$$
$$= A/\{\cos\omega(t_2 - t_1)\cos\omega t_1 - \sin\omega(t_2 - t_1)\sin\omega t_1\},$$

so that

$$(x_{n+1} + x_{n-1})\{1 - A^{-2}x_n^2\sin^2(\omega h)\} = 2\,x_n\cos(\omega h) \tag{5.3}$$

or

$$(x_{n+1} - 2x_n + x_{n-1})/\left(4\omega^{-2}\sin^2\tfrac{1}{2}\omega h\right) + \omega^2 x_n - \omega^2 A^{-2}x_n^2(x_{n+1} + x_{n-1})\cos^2\tfrac{1}{2}\omega h = 0. \tag{5.4}$$

Again the ΔE (5.3) or (5.4) is a best approximation to the DE (5.1) in the sense that the solution of the ΔE is

$$x_n = A\sec(\omega nh) = x(nh). \tag{5.5}$$

The form (5.4) is revealing in showing that $\ddot{x}$ in (5.1) is replaced by $(x_{n+1} - 2x_n + x_{n-1})/(4\omega^{-2}\sin^2\frac{1}{2}\omega h)$ and x^3 by $\frac{1}{2}x_n^2(x_{n+1} + x_{n-1})\cos^2\frac{1}{2}\omega h$.

6. Discussion

This paper has shown that it is possible to find from the infinite choices available a difference equation which in an obvious sense is a best approximation to Duffing's differential equation in its simplest form. The result extends to an important classical non-linear differential equation a general theory applicable to linear differential equations with constant coefficients. Because the method is based on the knowledge of the solution of the differential equation it cannot be expected to apply to equations for which closed form solutions are not known. For these, recourse is often made to perturbation methods, as for example in Duffing's equation (1.1) when there is an additional small friction term $\varepsilon\dot{x}$. Then it is an advantage to use for the unperturbed equation the difference equation derived above.

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