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CRITICAL GRAPHS FOR ACYCLIC COLORINGS

BY

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Introduction. The concept of acyclic colorings of graphs, introduced by Grünbaum [2], is a generalization of point-arboricity. An *acyclic coloring* of a graph is a proper coloring of its points such that there is no two-colored cycle. We denote by $a(G)$, the *acyclic chromatic number* of a graph G, the minimum number of colors for an acyclic coloring of G. We call G *k-critical* if $a(G)=k$ but $a(G')<k$ for any proper subgraph G'. For all notation and terminology not defined here, see Harary [3].

Kronk and Mitchem [4] and Bollobás and Harary [1] showed the existence of graphs of every possible order critical for point-arboricity. In this paper we prove the analogous result for acyclic colorings:

THEOREM. *For each $k\geq 3$ and $n\geq k$ there exists a k-critical graph of order n.*

We note that the only 2-critical graph is K_2.

Proof of the Theorem. We first note without proof the following simple lemma.

LEMMA. *If $G=A+B$ then in any acyclic coloring of G, either all the points of A or else all the points of B must receive distinct colors.*

The theorem is proved by presenting constructions for five classes of critical graphs.

PROPOSITION 1. *The theorem is true for $n=2k-l$, where $5\leq l\leq k$ and k and l are of the same parity.*

Proof. Let $G=(\bar{K}_{k-l}\cup K_{l-3})+C_{k-l+3}$. G can be colored either with $k-3$ colors for $\bar{K}_{k-l}\cup K_{l-3}$ and three more for the cycle, or else with $l-3$ colors for $\bar{K}_{k-l}\cup K_{l-3}$ and $k-l+3$ more for the cycle. Thus $a(G)=k$.

To show that G is critical, let $V(\bar{K}_{k-l})=\{p_1\cdots p_{k-l}\}$, $V(K_{l-3})=\{q_1\cdots q_{l-3}\}$ and $V(C_{k-l+3})=\{r_1\cdots r_{k-l+3}\}$.

(i) Delete line q_i, q_j. Then make q_i; q_j and all of $\bar{K}_{k-l}$ color 1. Use $k-2$ more colors for the remaining $k-2$ points.

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(ii) Delete line r_i, r_{i+1}. Then two-color the cycle and use $k-3$ more colors for $\bar{K}_{k-l} \cup K_{l-3}$.

(iii) Delete line q_i, r_j. Then color the cycle with $k-l+3$ colors. Use the same color for q_i as for r_j and use $l-4$ more colors for the remaining points of $\bar{K}_{k-l} \cup K_{l-3}$.

(iv) Delete line p_i, r_j. Then make p_i and r_j both color 1. Alternate colors 2 and 3 for the rest of the cycle. (Note that since $k-l+3$ is odd the two neighbours of r_j are colored differently.) Then use $k-4$ more colors for the remaining $k-4$ points of $\bar{K}_{k-l} \cup K_{l-3}$.

The proofs of Propositions 2 through 5 are similar to that of Proposition 1, and the details are left to the reader.

PROPOSITION 2. *The theorem is true for $n = 2k - l$ where $5 \le l \le k$, and k and l are of opposite parity.*

Proof. Let G be as above, but for each i delete the line p_i, r_1.

PROPOSITION 3. *The theorem is true for $n \ge 2k-4$ where k and n are of the same parity.*

Proof. Let $G = \bar{K}_{k-3} + C_{n-k+3}$.

PROPOSITION 4. *The theorem is true for $n > 2k-4$ where k and n are of opposite parity.*

Proof. Let $G = \bar{K}_{k-3} + C_{n-k+3}$, but for each $i = 1$ to $k-3$ delete the line p_i, r_1 for $p_i \in \bar{K}_{k-3}$ and $r_1 \in C_{n-k+3}$.

PROPOSITION 5. *The theorem is true for $n = 2k-4$ where k and n are of opposite parity.*

Proof. Let $G = \bar{K}_{k-3} + C_{n-k+3}$, but for each $i = 2$ to $k-3$ delete the line p_i, r_1 for $p_i \in \bar{K}_{k-3}$ and $r_1 \in C_{n-k+3}$.

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