The geometrical construction of Maschke's quartic surfaces

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Introduction.

There is, in the second (Cambridge, 1911) edition of Burnside's *Theory of Groups of Finite Order*, an example on p. 371 which must have aroused the curiosity of many mathematicians; a quartic surface, invariant for a group of $2^4.5!$ collineations, appears without any indication of its provenance or any explanation of its remarkable property. The example teases, whether because Burnside, if he obtained the result from elsewhere, gives no reference, or because, if the result is original with him, it is difficult to conjecture the process by which he arrived at it. But the quartic form which, when equated to zero, gives the surface, appears, together with associated forms, in a paper by Maschke[1], and it is fitting therefore to call both form and surface by his name.

Maschke, who solved his problem to perfection, was concerned with the forms which are invariant for a group of $2^6.6!$ quaternary linear substitutions; any such invariant form gives, when equated to zero, a surface which is invariant for a group of $2^4.6!$ quaternary collineations. This latter group is the famous one discovered by Klein and associated with a set of six linear complexes any two of which are in involution; it so happens, and will be explained geometrically below, that there are six quartic surfaces inherent in Klein's figure which are transitively permuted by this group, and each of which is therefore invariant for a group of $2^4.5!$ collineations. Burnside's example is thus "rationalised," and the quartic surface which occurs there is identified. But the geometry of the figure, and the place of the quartic surface in it, require elucidation; it is the purpose of the following paragraphs to make some contribution to this end.

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[1] *Math. Annalen*, 30 (1887), 496-515.

Dyck's Plane Configuration.

1. A quartic curve D whose equation is of the form

$$\alpha x^4 + \beta y^4 + \gamma z^4 = 0$$

is invariant for a group of 96 ternary collineations. Such a curve was first encountered by Dyck[1], and is generally named after him. The triangle Δ to which the curve is referred has notable relations with it; the intersections of the sides of Δ with D are flecnodes, i.e., points at which the tangents of D have four-point contact. Dyck called these points the a-points of D, and the tangents at the four a-points on a side of Δ are concurrent at the opposite vertex of Δ.

The 12 flecnodal tangents count among the bitangents of D; there are 16 other bitangents, which will be called f-lines. Their equations, together with those of the flecnodal tangents, were found by Cayley[2], and the properties of the configuration can easily be deduced either from these equations or from the fact of invariance under the group of collineations.

The f-lines fall into four sets of four, forming four quadrilaterals, q_1, q_2, q_3, q_4, each of which has Δ for its diagonal triangle. On each side of Δ there are two pairs of points, each harmonic to the two vertices of Δ which lie on that side and each constituting a pair of opposite vertices of *two* of the four quadrilaterals; on one side of Δ q_2 and q_3 have a common pair of opposite vertices, as also have q_1 and q_4; on another side of Δ the quadrilaterals are coupled q_3 with q_1 and q_2 with q_4, while on the remaining side of Δ q_1 is coupled with q_2 and q_3 with q_4. The 12 points so arising on the sides of Δ are collinear in triads on the 16 f-lines, four f-lines passing through each of the 12 points. Further: not only are the two pairs of points on a side of Δ harmonic to the vertices of Δ, but they are harmonic to each other, so that we have, on each side of Δ, three pairs of points such that each pair is harmonic to both the others; this arrangement of three pairs is sometimes called a regular sextuple.

The whole figure can be constructed linearly when Δ and one of the f-lines are given; it can also be constructed linearly when one of the quadrilaterals is given.

It is important, for a future application, to point out that the tetrad of a-points on a side of Δ is uniquely determined by the

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[1] *Math. Annalen*, **17** (1880), 510-516.

[2] *Educational Times*, **36** (1881), 64; *Mathematical Papers*, **10**, 603.

regular sextuple on that side. The three pairs of any regular sextuple give, when squared, binary quartics belonging to the same pencil. The tetrad of a-points on any side of Δ belongs to the corresponding pencil, being that member which is, in the pencil, the harmonic conjugate, of the quartic which arises from the square of the pair of vertices of Δ, with respect to the quartics which arise from the squares of the other two pairs of the sextuple.

It may, lastly, be remarked that the points of contact of D with any f-line constitute the Hessian duad, on that line, of the triad of points in which the line is met by the sides of Δ.

Klein's Space Configuration.

2. This, one of the best-known of all configurations, arises from a set of six linear complexes any two of which are in involution[1]; it contains 15 pairs of *directrices*, each pair being a pair of opposite edges of three of 15 *fundamental tetrahedra*. It would be superfluous to give a detailed account of so well-known a figure; but it must here be emphasised that *the section of Klein's configuration by a face of a fundamental tetrahedron is Dyck's configuration.*

Each directrix contains pairs of vertices of three fundamental tetrahedra, and the three pairs form a regular sextuple (Klein, § 6). Thus a face π of a fundamental tetrahedron T_0 gives a triangle Δ formed by the edges of T_0 in π, while on each side of Δ is a regular sextuple with two vertices of Δ forming one of its three pairs. Moreover there are (Klein, § 8) 320 lines each of which contains vertices of three tetrahedra while at the same time lying in faces of three other tetrahedra; the number of such lines lying in a face, such as π, of a fundamental tetrahedron is therefore $(320 \times 3)/(4 \times 15)$, or 16; these are the f-lines of the Dyck configuration in π, and the triad of vertices of tetrahedra on one of these f-lines is constituted by its intersections with the sides of Δ.

An examination of much of the literature concerned with the Klein configuration has not disclosed any explicit acknowledgment of the Dyck configurations in the faces of the fundamental tetrahedra. But two remarks in Klein's § 8 are concerned with those planes and lines of the figure which go through a vertex of a fundamental tetrahedron: these statements may, as Klein himself remarks, be dualised,

[1] Klein : Zur Theorie der Linienkomplexe des ersten und zweiten Grades : *Math. Annalen*, 2 (1870), 198-226 ; *Gesammelte Math. Abhandlungen*, 1, 53-80.

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and, when they are, the Dyck configuration instantly appears. The section of the figure by a face of a fundamental tetrahedron is alluded to by E. Hess in the course of a long paper[1]; these allusions, however, which occur on pp. 125-6, 128, 164, are but slight. The nomenclature for the f-lines has been adopted from Hess.

The 15 fundamental tetrahedra can be grouped, if we may borrow a phrase from Sylvester[2], into six *synthematic totals* of five; the five tetrahedra of any total account, by their 30 edges, for all the directrices. Each tetrahedron belongs to two totals. If

$$T_0, T_1, T_2, T_3, T_4$$

be tetrahedra of a total, a face π of T_0 is met, by the faces of T_1, T_2, T_3, T_4, respectively, in the quadrilaterals q_1, q_2, q_3, q_4 of the Dyck configuration.

Any two tetrahedra of a total belong to a desmic triad. There are ten different pairs of tetrahedra in a total, and the ten tetrahedra which complete the desmic triads are the remaining tetrahedra, other than the five in the total, of the configuration. The tetrahedron which makes up a desmic triad with T_i and T_j may be denoted by U_{ij}.

Maschke's Quartic Surfaces.

3. Since there is a Dyck configuration in each face of a fundamental tetrahedron, there is a curve D in each face too. Now, as was pointed out in § 1, the four a-points in which D meets a side of Δ are uniquely determined by the regular sextuple on that side; it therefore follows that any two of the four curves in the faces of T_0 meet the edge which is common to these faces in the same four points. Hence the four curves D in the faces of T_0, or of any other fundamental tetrahedron, lie on a pencil of quartic surfaces: a detailed proof of this statement is given in *Proc. London Math. Soc.* (2), 47 (1941), 132-133. It can next be shown that the two pencils of quartic surfaces that are associated with two fundamental tetrahedra belonging to the same total have a surface in common.

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[1] *Nova Acta Leop.*, **55** (1891), 97-167.

[2] Elementary Researches in the Analysis of Combinatorial Aggregation: *Phil. Mag.*, **24** (1844), 285-296; *Mathematical Papers*, **1**, 91-102. The 15 synthemes formed by Sylvester from a set of six objects are, when these objects are linear complexes mutually in involution, equivalent to Klein's 15 fundamental tetrahedra.

For consider the pencil of quartic curves in which the face π of T_0 is met by the pencil Π_1 of quartic surfaces associated with T_1. The curve D which lies in a face of T_1 has the side of q_1, in which this face meets π, as a bitangent, and *its points of contact are the same as those of the curve D in π*; for, owing to the desmic relationship of T_0 and T_1, the line meets the sides of Δ in points which are also on edges of T_1, and the Hessian duad of this triad is the pair of points of contact, with the line, of both plane curves. Thus every surface of Π_1 touches the curve D in π at each of eight points; wherefore one surface of Π_1 contains the whole curve. Since this may be identified as that surface of Π_1 which contains any one of the four a-points on any one of the three sides of Δ (for none of these 12 points belongs to the base curve of Π_1), it follows that the same surface also contains those curves D which lie in the other three faces of T_0, and therefore that it contains the whole of the base curve of the pencil Π_0 associated with T_0. The surface thus belongs both to Π_0 and to Π_1.

Let this surface be called M. *It contains all the* 20 *curves D in the faces of the tetrahedra of the total*; for it has been obtained as containing eight of them, and it touches any one of the others at each of 16 points. The Klein configuration therefore involves six quartic surfaces M, one associated with each total. Two surfaces M associated with different totals intersect in the four curves D which lie in the faces of the tetrahedron common to the totals; thus the pencil determined by any two of the six surfaces M includes a member which consists of the four faces of a tetrahedron, and all 15 fundamental tetrahedra are accounted for in this way.

The six totals were taken by Klein[1] as representative of a sextic resolvent of a sextic equation; the resolvent may, we now perceive, equally well be represented by the six surfaces M.

There are, according to Schubert[2], 600 points on a non-singular quartic surface at which the tangent plane meets the surface in a curve with a biflecnode at the point of contact. For M, 120 of the 600 points can be identified immediately; they are the intersections of M

[1] Über eine geometrische Repräsentation der Resolventen algebraischer Gleichungen: *Math. Annalen*, 4 (1871), 346-358; *Gesammelte Math. Abhandlungen*, 2, 262-274. The fact that a sextic equation has a sextic resolvent distinct from the equation itself is due to the existence, first pointed out by Sylvester in 1844 in the paper to which reference has already been made above, of a function of six variables which, when the variables are permuted, takes six different values.

[2] *Kalkül der abzählenden Geometrie* (Leipzig, 1879), 246.

with the edges of the tetrahedra of the associated total. For any such intersection is an a-point on both the curves D which lie in the faces of the tetrahedron which pass through the edge, so that the lines which join the point to the two vertices on the opposite edge are both flecnodal tangents of M.

Maschke's Quartic Forms.

4. Let us take T_0 as tetrahedron of reference. Then, since T_1 is desmic to T_0, we may suppose the equations of its faces to be

$$-x+y+z+t=0,\ x-y+z+t=0,\ x+y-z+t=0,\ x+y+z-t=0.$$

In the face $t=0$ of T_0 there is a Dyck curve which has the four sides

$$-x+y+z=0,\ x-y+z=0,\ x+y-z=0,\ x+y+z=0,$$

of the quadrilateral q_1 as bitangents; the equation of this curve is of the form $\alpha x^4+\beta y^4+\gamma z^4=0$. Now it is easily shown that this curve has the line $lx+my+nz=0$ for a bitangent if, and only if,

$$\alpha:\beta:\gamma=l^4:m^4:n^4.$$

Thus $\alpha=\beta=\gamma$. This discussion shows that the four curves D in the faces of T_0 have equations

$$\left.\begin{matrix}y^4+z^4+t^4=0\\ x=0\end{matrix}\right\},\quad \left.\begin{matrix}z^4+t^4+x^4=0\\ y=0\end{matrix}\right\},\quad \left.\begin{matrix}t^4+x^4+y^4=0\\ z=0\end{matrix}\right\},\quad \left.\begin{matrix}x^4+y^4+z^4=0\\ t=0\end{matrix}\right\}.$$

They therefore constitute the base of the pencil of quartic surfaces $x^4+y^4+z^4+t^4+\lambda xyzt=0$; as λ varies this surface generates the pencil Π_0.

Now a corresponding argument can be applied to the four curves D in the faces of T_1; from this it appears that they constitute the base of the pencil of quartic surfaces

$$(y+z+t-x)^4+(z+t+x-y)^4+(t+x+y-z)^4+(x+y+z-t)^4$$
$$+\mu(y+z+t-x)(z+t+x-y)(t+x+y-z)(x+y+z-t)=0,$$

the surface generating the pencil Π_1 as μ varies.

If, now, Π_0 and Π_1 have a surface in common, it must be possible to choose λ and μ so that the left-hand sides of the two equations are proportional. In order that terms such as y^2z^2 should disappear from the equation of a surface of Π_1, the value of μ must be -12; the left-hand side of the equation is then $16(x^4+y^4+z^4+t^4-12xyzt)$, so that the equation of the Maschke surface, common to Π_0 and Π_1, is

$$x^4+y^4+z^4+t^4-12xyzt=0.$$

The quaternary quartic on the left of this equation is, in Maschke's notation (see his equations numbered 11 and 7), $-\frac{1}{2}\Phi_6$.

A harmonic inversion with respect to any face and opposite vertex of T_0 interchanges any two tetrahedra, such as T_i and U_{0i}, which form a desmic triad with T_0; the total

$$T_0, T_1, T_2, T_3, T_4,$$

therefore becomes the second total

$$T_0, U_{01}, U_{02}, U_{03}, U_{04},$$

to which T_0 belongs, and the Maschke surfaces associated with these two totals are interchanged by the harmonic inversion. Now such an inversion is accomplished analytically by changing the sign of one of the four coordinates; hence the equation of the Maschke surface associated with the second total is

$$x^4 + y^4 + z^4 + t^4 + 12xyzt = 0.$$

The form on the left of this equation is, in Maschke's notation, $-\frac{1}{2}\Phi_5$, and is the one given by Burnside.

Any one of the four remaining totals contains a pair of tetrahedra T_i and U_{0i}, so that it is not changed by harmonic inversion with respect to any face and opposite vertex of T_0. Wherefore the Maschke form associated with it can only contain even powers of the coordinates. It will be enough to obtain one of these four outstanding forms, say that associated with the total

$$U_{01}, T_1, U_{12}, U_{13}, U_{14}.$$

This form, when taken with $x^4 + y^4 + z^4 + t^4 + 12xyzt$, must determine a pencil of which one member is

$$x^4 + y^4 + z^4 + t^4 - 2\,(y^2z^2 + z^2x^2 + x^2y^2 + x^2t^2 + y^2t^2 + z^2t^2) + 8xyzt,$$

for this latter form, when equated to zero, gives the four faces[1] of U_{01}. Thus Maschke's form, since it is not to contain the term in $xyzt$, must be

$$\begin{aligned}\Phi_1 &\equiv 3\,\{x^4 + y^4 + z^4 + t^4 - 2\,(y^2z^2 + z^2x^2 + x^2y^2 + x^2t^2 + y^2t^2 + z^2t^2) + 8xyzt\} \\ &\qquad - 2\,\{x^4 + y^4 + z^4 + t^4 + 12xyzt\} \\ &\equiv x^4 + y^4 + z^4 + t^4 - 6\,(y^2z^2 + z^2x^2 + x^2y^2 + x^2t^2 + y^2t^2 + z^2t^2).\end{aligned}$$

[1] The equations of all the faces and the coordinates of all the vertices of the fundamental tetrahedra, referred to one of them as tetrahedron of reference, are given by Hess: *loc. cit.*, 107-8.

The remaining Maschke forms are found to be

$$\Phi_2 \equiv x^4 + y^4 + z^4 + t^4 + 6\,(-y^2z^2 + z^2x^2 + x^2y^2 - x^2t^2 + y^2t^2 + z^2t^2),$$
$$\Phi_3 \equiv x^4 + y^4 + z^4 + t^4 + 6\,(y^2z^2 - z^2x^2 + x^2y^2 + x^2t^2 - y^2t^2 + z^2t^2),$$
$$\Phi_4 \equiv x^4 + y^4 + z^4 + t^4 + 6\,(y^2z^2 + z^2x^2 - x^2y^2 + x^2t^2 + y^2t^2 - z^2t^2).$$

Maschke's Forms and Segre's Four-Dimensional Configuration.

5. The six Maschke forms Φ_i are such that their sum vanishes identically; they may then be regarded as homogeneous coordinates of a point in a [4]. But, since they are homogeneous polynomials in only four variables, they must satisfy one further identical relation; this appears, quite casually, in the course of Maschke's work, and is

$$4\,\Sigma\,\Phi_i\,\Phi_j\,\Phi_k\,\Phi_l = \{\Sigma\ \Phi_i\,\Phi_j\}^2.$$

But this, when the Φ_i are homogeneous coordinates, whose sum vanishes, in [4], is the equation of the quartic primal with 15 nodal lines which constitute the famous configuration studied by Segre[1] and Castelnuovo[2], and expounded in great detail by Baker[3]. The equations of one of the nodal lines are

$$\Phi_1 = \Phi_2, \quad \Phi_3 = \Phi_4, \quad \Phi_5 = \Phi_6;$$

any one of the lines is given by three such equations, and the lines answer to the synthemes of Sylvester.

The situation thus is that to every point (x, y, z, t), with the consequent values of Φ_i, there corresponds a unique point of this primal Γ; whereas, given a point of Γ, there correspond to it all those points in [3] for which the six forms Φ_i take the necessary values. There are, in general, 16 such points, forming a Kummer set. For the group of $2^6.6!$ quaternary substitutions is the direct product of a group of order $6!$, which subjects the six Maschke forms to all possible permutations, and a group of 2^6 quaternary substitutions for which each individual Φ_i is invariant. But each of x, y, z, t can be multiplied by the same fourth root of unity without altering the position of the point of which they are the homogeneous coordinates, and the 2^6 quaternary substitutions give, for this reason, only 2^4 collineations. This representation, of Kummer sets of 16 points in [3] by points of

[1] *Atti d. Reale Accad. di Scienze di Torino*, 22 (1887), 791, where the dual configuration of 15 planes appears.

[2] *Atti Ist. Veneto* (6), 6 (1888), 547-565. Several forms for the equation of the primal are given by Richmond : *Quart. J. of Math.*, 34 (1903), 142 *et seq.*

[3] *Principles of Geometry*, 4 (Cambridge, 1925 and 1940), Chapter 5.

Γ, is given by Baker[1], whose manner of obtaining it, however, makes no use of the Maschke forms; it seems doubtful whether the foundations of the representation can properly be apprehended, or its symmetry sufficiently clearly displayed, without them.

If a point of Γ is on one of the nodal lines, it is found that the corresponding Kummer set becomes a set of eight points, counted twice, of which four lie on each of a pair of directrices of the Klein configuration. Baker, while pointing out that each nodal line of Γ is associated with a pair of directrices, omits to explain that to different points of the nodal line correspond different sets of points on the directrices; perhaps, therefore, the following brief analytical substantiation should be given.

If $$\Phi_1 = \Phi_2, \quad \Phi_3 = \Phi_4, \quad \Phi_5 = \Phi_6,$$

simultaneously, then

$$z^2t^2 + x^2y^2 = z^2x^2 + y^2t^2 = xyzt = 0;$$

any point whose coordinates satisfy these latter equations lies either on the line $x = t = 0$ or on the line $y = z = 0$, and these two lines constitute a pair of directrices. Conversely: any point on $x = t = 0$ has coordinates $(0, \eta, \zeta, 0)$ and, for this point

$$\Phi_1 = \Phi_2 = \eta^4 + \zeta^4 - 6\,\eta^2\zeta^2,$$
$$\Phi_3 = \Phi_4 = \eta^4 + \zeta^4 + 6\,\eta^2\zeta^2,$$
$$\Phi_5 = \Phi_6 = -\,2\,(\eta^4 + \zeta^4).$$

The ratios of these values of Φ_i to one another vary with $\eta : \zeta$; but the eight points

$$(0, \eta, \zeta, 0), \quad (0, \zeta, \eta, 0), \quad (0, -\eta, \zeta, 0), \quad (0, \zeta, -\eta, 0),$$
$$(\eta, 0, 0, \zeta), \quad (\zeta, 0, 0, \eta), \quad (-\eta, 0, 0, \zeta), \quad (\zeta, 0, 0, -\eta),$$

of which four lie on $x = t = 0$ and four on $y = z = 0$, all give the same values for the Φ_i.

If $\eta = \zeta$ the above eight points coalesce, in pairs, at the points

$$(0, 1, 1, 0), \quad (0, -1, 1, 0), \quad (1, 0, 0, 1), \quad (1, 0, 0, -1);$$

also, for these four points, $\Phi_1 = \Phi_2 = \Phi_5 = \Phi_6$ and $\Phi_3 = \Phi_4$. The corresponding point of Γ is thus common to three of the nodal lines,

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[1] *loc. cit.*, 208-211.

and is what Richmond calls a *cross-point.* Thus it is seen that the Kummer set which corresponds to a cross-point on Γ consists of the vertices of a fundamental tetrahedron, taken four times; the 15 cross-points give one each of the 15 fundamental tetrahedra.

Maschke's Invariant.

6. The lowest degree for which an invariant of the group of $2^6.6!$ linear substitutions exists is eight; there is one invariant of this degree, which Maschke (p. 505) gives as

$$I \equiv x^8 + y^8 + z^8 + t^8 + 14\,(y^4z^4 + z^4x^4 + x^4y^4 + x^4t^4 + y^4t^4 + z^4t^4) + 168\,x^2y^2z^2t^2.$$

This, being an invariant of the whole group, is also an invariant for that sub-group of $2^6.5!$ linear substitutions for which

$$-\tfrac{1}{2}\Phi_5 \equiv x^4 + y^4 + z^4 + t^4 + 12\,xyzt$$

is invariant. Now this sub-group must also possess, among its invariants, the Hessian of Φ_5; this Hessian may be taken as

$$H \equiv \begin{vmatrix} x^2 & zt & yt & yz \\ zt & y^2 & xt & zx \\ yt & xt & z^2 & xy \\ yz & zx & xy & t^2 \end{vmatrix}$$

$$\equiv 2\,xyzt\,(x^4 + y^4 + z^4 + t^4 - xyzt) - y^4z^4 - z^4x^4 - x^4y^4 - x^4t^4 - y^4t^4 - z^4t^4.$$

The sub-group therefore has for invariants, all of degree eight, I, Φ_5^2 and H; these, however, are linearly connected, since

$$I \equiv \tfrac{1}{4}\Phi_5^2 - 12\,H.$$

This identity shows that the quartic surface $\Phi_5 = 0$ is met by its Hessian in the same curve in which it is met by the surface $I = 0$. Since any one of the six Maschke surfaces can be transformed into any other by operations of the whole group of collineations, and since I is invariant for this group, the surface $I = 0$ must be related to each of the Maschke surfaces in this way. Whence the following:—

The parabolic curves of the six Maschke surfaces inherent in a Klein configuration all lie on the same surface of order eight; it is this surface which furnishes the invariant of lowest degree for the group of $2^6.6!$ linear substitutions.

It will be observed that the Hessian of a Maschke surface *has triple points at the vertices of all the tetrahedra of the total with which the surface is associated.* The polar quadric of any such vertex with respect to the surface is the opposite face, taken twice, of the tetrahedron.

The quadric

$$\Phi_1^2 + \Phi_2^2 + \Phi_3^2 + \Phi_4^2 + \Phi_5^2 + \Phi_6^2 = 0$$

is invariantly related to the Segre configuration, for it meets each of the 15 lines in the Hessian duad of the triad of cross-points on the line. When the forms are substituted for the Φ_i, the left-hand side of the equation of the quadric becomes $12I$, and so the surface $I = 0$ is the locus of those Kummer sets which correspond to the points of the surface of intersection of the quadric with Γ. Of these Kummer sets 30, corresponding to the intersections of the quadric with the nodal lines of Γ, are repeated octads; each such octad includes four intersections of $I = 0$ with each of a pair of directrices, and the 30 octads together account for all the intersections of $I = 0$ with the 30 directrices.

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