STRUCTURE THEORY OF COMPLEMENTED BANACH ALGEBRAS

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1. Introduction. If A is an H^*-algebra, then the orthogonal complement of a closed right (left) ideal I is a closed right (left) ideal I^p. Saworotnow **(7)** considered Banach algebras which are Hilbert spaces and in which the closed right ideals satisfy the complementation property of an H^*-algebra. In our right complemented Banach algebras we drop the requirement of the existence of an inner product and only assume that for every closed right ideal I there is a closed right ideal I^p which behaves like the orthogonal complement in a Hilbert space (Definition 1). Thus our algebras may be considered as a generalization of Saworotnow's right complemented algebras.

We show that a semi-simple right complemented Banach algebra A contains a primitive idempotent (Theorem 1) and that this leads to a determination of the structure of A in terms of its minimal closed two-sided ideals (Theorem 4). If A is a simple right complemented Banach algebra and I is a minimal left ideal in A, then an inner product can be introduced in I for which I is a Hilbert space (Theorem 5). If, in addition, A is a left annihilator algebra (Definition 3), then for every closed left ideal J there is a closed left ideal J' such that $J \cap J' = (0)$ and $\overline{J + J'} = A$ (Theorem 6). Moreover, A is continuously isomorphic to an algebra of completely continuous operators on a Hilbert space (Theorem 7), and therefore is an annihilator algebra. From this it follows that a semi-simple left annihilator right complemented Banach algebra is an annihilator algebra (Theorem 8). If A is a simple bicomplemented Banach algebra with the annihilator properties, then A is dual (Theorem 9). Finally, if a simple annihilator right complemented Banach algebra A has the minimal norm property, then A is equivalent to the algebra of all completely continuous operators on a Hilbert space (Theorem 10).

In notation and terminology we follow **(6)** and **(7)**. An idempotent will always be understood to be a non-zero element, and the radical is taken in the sense of Jacobson **(3)**. If S is any set in a Banach algebra A, $\bar{S}$ will denote the closure of S in A.

2. Right complemented algebras. Let A be a Banach algebra. We shall denote the lattice of all closed right (left) ideals in A by L_r (L_l).

DEFINITION 1. *Let A be a complex Banach algebra. We shall call A a right*

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complemented Banach algebra if there exists a mapping $I \to I^p$ of L_r into L_r such that:

(1) *If $I \in L_r$, then $I \cap I^p = (0)$*
(2) *If $I \in L_r$, then $(I^p)^p = I$*
(3) *If $I \in L_r$, then $I \oplus I^p = A$*
(4) *If I_1 and I_2 are in L_r, $I_1 \subseteq I_2$, then $I_2^p \subseteq I_1^p$.*

Analogously we define a left complemented Banach algebra. If A is both a left and a right complemented Banach algebra, we shall call A a bicomplemented Banach algebra. I^p will be called the complement of I.

It is clear that whatever statement we make about a right complemented Banach algebra also holds for a left complemented Banach algebra if we replace everywhere the word "right" by the word "left" and the word "left" by the word "right." From now on we shall call a right complemented (bicomplemented) Banach algebra simply a right complemented (bicomplemented) algebra.

For any set S in a Banach algebra A, let $l(S) = \{x \in A : xS = (0)\}$ and $r(S) = \{x \in A : Sx = (0)\}$. It is clear that $l(S)$ and $r(S)$ are closed left and right ideals respectively. A Banach algebra A is called an annihilator algebra if $l(A) = r(A) = (0)$, and if for every closed right ideal I and every closed left ideal J, $l(I) \neq (0)$ and $r(J) \neq (0)$. If, in addition, $r[l(I)] = I$ and $l[r(J)] = J$, then A is called a dual algebra. We shall also use I_l and J_r for $l(I)$ and $r(J)$ respectively.

Examples. An annihilator B^*-algebra is a right complemented algebra. In fact, let A be an annihilator B^*-algebra. Let I be a proper closed right ideal in A and I_l be the left annihilator of I. Then $(I_l)^*$ is the complement of I, that is, $I^p = (I_l)^*$. From (**1**, p. 157) we know that $(I_l)^* \oplus I = A$, $(I_l)^* \cap I = (0)$ and that if I and J are proper closed right ideals in A, $I \subseteq J$, then $(J_l)^* \subseteq (I_l)^*$. We shall show now that $(I^p)^p = ([(I_l)^*]_l)^* = I$. Clearly $(I_l)^* = (I^*)_r$ for $xI = (0)$ if and only if $I^*x^* = (0)$. Therefore $(I^p)^p = ((I^p)_l)^* = ((I^p)^*)_r = ([(I_l)^*]^*)_r = I$. In particular, the algebra of all completely continuous operators on a Hilbert space is a right complemented algebra (**6**). In fact, all annihilator B^*-algebras are bicomplemented algebras.

LEMMA 1. *Let A be a right complemented algebra such that $r(A) = (0)$. If I is a closed two-sided ideal, then $l(I) = r(I) = I^p$. Moreover, every closed two-sided ideal in A is a right complemented algebra.*

Proof. For any closed two-sided ideal I, we have $I^pI \subset I^p \cap I = (0)$. Let $J = I \cap l(I)$. Then $AJ = (J \oplus J^p)J = (0)$, and so $J = (0)$, $I \cap l(I) = (0)$. Similarly $I \cap r(I) = (0)$ and therefore $l(I) = r(I)$. Also, since $A = I \oplus I^p$, $I^p \subset l(I)$ and $I \cap l(I) = (0)$, it follows that $I^p = l(I)$. Now let R be any closed right ideal in I. Then R is a closed right ideal in A and $R' = R^p \cap I$ is the complement of R in I.

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COROLLARY. *If A is a right complemented algebra such that $r(A) = (0)$, then $l(A) = (0)$.*

LEMMA 2. *Let I be a proper closed modular right ideal in a right complemented algebra A. Then I^p contains a left identity modulo I. If e is a left identity modulo I that belongs to I^p, then e is an idempotent, $I = \{x - ex : x \in A\}$ and $I^p = eA$.*

Proof. If e is a left identity modulo I and $e \in I^p$, then $e^2 = e$ (**7**, p. 50). Let u be a left identity modulo I. Then $u = d + e$ with $u \in I$ and $e \in I^p$, and we have $ex = x$ for all $x \in I^p$, in particular, $e^2 = e$. Also $eI = (0)$ and therefore $I = \{x - ex : x \in A\}$.

LEMMA 3. *Let J be a left ideal in a Banach algebra A such that every element of J is right quasi-regular. Then J is contained in the radical of A.*

Proof. Let R be the right ideal generated by the elements of J. Since left (right) quasi-regularity for $(\alpha + x)y$ is equivalent to left (right) quasi-regularity for $y(\alpha + x)$, where α is a complex number (**6**, p. 17), R is a quasi-regular right ideal. Thus J belongs to the radical of A.

THEOREM 1. *If A is a non-radical right complemented algebra, then A contains a primitive idempotent.*

Proof. Let $z \in A$ be such that z is not right quasi-regular. Then $Q = \{x - zx : x \in A\}$ is a proper modular right ideal. Let M be a maximal modular right ideal containing Q. Then $z = d + e$ with $d \in M$ and $e \in M^p$, and by Lemma 2, $e^2 = e$. Since M^p is a minimal closed right ideal, e is primitive. Also we have $ez = e$ and $(ze)^2 = z(ez)e = ze$; $ze \neq 0$, otherwise $e \in M \cap M^p = (0)$.

COROLLARY. *If A is a semi-simple right complemented algebra, then every proper left (right) ideal in A contains a primitive idempotent.*

Proof. Let I be a proper left ideal. By Lemma 3, I contains an element z which is not right quasi-regular. As in the proof of Theorem 1, $ez = e \in I$ and e is a primitive idempotent. If I is a proper right ideal, then $ze \in I$ and since Ae is a minimal left ideal, ze is a primitive idempotent.

From Lemma 2 it follows that every maximal modular right ideal M in a right complemented algebra A is of the form $M = \{x - ex : x \in A\}$ where e is a primitive idempotent. If a right complemented algebra A is such that $r(A) = (0)$ we shall say that A is proper. We end this section with the following theorem.

THEOREM 2. *Let A be a proper right complemented algebra, and R be the radical of A. Then R^p is a semi-simple right complemented algebra.*

Proof. Let $A' = R^p$. Since by Lemma 1 $l(A') = R$, every idempotent $e \in A$ belongs to A' and thus every maximal modular right ideal M' in A' is given by $M' = M \cap A'$, where M is a maximal modular right ideal in A.

Hence if R' is the radical of A', $R' \subseteq R$ and since $R \cap A' = (0)$, $R' = (0)$. By Lemma 1, A' is a right complemented algebra.

3. Semi-simple right complemented algebras.

DEFINITION 2. *An idempotent e in a right complemented algebra A is called a left projection if the complement R^p of $R = \{x - ex : x \in A\}$ is eA. If e is primitive we shall say that e is a primitive left projection.*

If A is a non-radical right complemented algebra, then A contains a proper closed modular right ideal and by Lemma 2, A therefore contains a left projection. Moreover, if A is a semi-simple annihilator right complemented algebra, then every proper closed right ideal contains a primitive left projection. This follows from the fact that every proper closed right ideal I is contained in a maximal modular right ideal. If e and f are primitive left projections in I and I^p respectively, then $ef = fe = 0$. By Zorn's Lemma there exists a maximal family $\{e_\alpha : \alpha \in \mathfrak{J}\}$ of orthogonal primitive left projections in a semi-simple annihilator right complemented algebra.

THEOREM 3. *If $\{e_\alpha : \alpha \in \mathfrak{J}\}$ is a maximal family of orthogonal primitive left projections in a semi-simple annihilator right complemented algebra A, then the ideals*

$$\sum_{\alpha \in \mathfrak{J}} e_\alpha A$$

and

$$\sum_{\alpha \in \mathfrak{J}} A e_\alpha$$

are dense in A.

Proof. Let

$$I = \overline{\sum_{\alpha \in \mathfrak{J}} e_\alpha A}$$

and

$$J = \bigcap_{\alpha \in \mathfrak{J}} M_\alpha ,$$

where $M_\alpha = \{x - e_\alpha x : x \in A\}$. Then $e_\alpha A \subset I$ for all $\alpha \in \mathfrak{J}$ implies that $I^p \subset J$. But $J \subset M_\alpha$, hence $J^p \supset e_\alpha A$ for all $\alpha \in \mathfrak{J}$ and therefore $J^p = I$, $J = I^p$. If I is proper, then I^p contains a primitive left projection e, and $e_\alpha e = ee_\alpha = 0$ for all $\alpha \in \mathfrak{J}$, a contradiction. Hence $I = A$. Let

$$L = \overline{\sum_{\alpha \in \mathfrak{J}} A e_\alpha}$$

Then

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$$l(L) \subset \bigcap_{\alpha \in \mathfrak{J}} M_\alpha$$

and since $I = A$, $l(L) = (0)$ and so $L = A$.

Let A be a semi-simple right complemented algebra and $\Omega = \{e_\alpha : \alpha \in \Lambda\}$ be the set of all primitive idempotents in A. It is easy to see that the sum

$$\sum_{\alpha \in \Lambda} e_\alpha A$$

is a two-sided ideal in A. This follows from the fact that if e is a primitive idempotent in A, then the ideal aeA (or Aea) is either (0) or minimal for every $a \in A$ (**1**, p. 158 or **6**, p. 45). The ideal

$$\sum_{\alpha \in \Lambda} e_\alpha A$$

(which is equal to the ideal

$$\sum_{\alpha \in \Lambda} Ae_\alpha)$$

is called the socle of A and we denote it by $\mathfrak{S}$.

LEMMA 5. *If A is a semi-simple right complemented algebra, then the socle $\mathfrak{S}$ is dense in A.*

Proof. Let

$$I = \overline{\sum_{\alpha \in \Lambda} e_\alpha A}.$$

If $x \in I^p$, then by Lemma 1, $xe_\alpha = 0$ for all $\alpha \in \Lambda$. Thus I^p is contained in the intersection of all the maximal modular right ideals. Since A is semi-simple, $I^p = (0)$ and we have $I = A$.

THEOREM 4 (*structure theorem*). *A semi-simple right complemented algebra A is the direct topological sum of its minimal closed two-sided ideals, each of which is a simple right complemented algebra.*

Proof. Let K be the topological sum of all the minimal closed two-sided ideals in A. Since a closed two-sided ideal $[I]$ generated by a minimal right ideal I is a minimal closed two-sided ideal (**1**, p. 158) and since by Lemma 5 the socle is dense in A, it follows that $K = A$. By the proof of Theorem 6 in (**1**), K is also a direct topological sum.

For further analysis of semi-simple right complemented algebras we turn now to simple right complemented algebras.

4. Simple right complemented algebras.

Unless otherwise stated, A will stand for a simple right complemented algebra in the following discussion.

LEMMA 6. *Let I be a minimal left ideal in A and let L be the lattice of all closed linear subspaces of I. Then there is a mapping $S \to S^p$ of L into L such that:*

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(a) *If* $S \in L$, *then* $S \cap S^p = (0)$

(b) *If* $S \in L$, *then* $(S^p)^p = S$

(c) *If* $S \in L$, *then* $S \oplus S^p = I$

(d) *If* S_1, S_2 *are in* L, $S_1 \subseteq S_2$, *then* $S_2^p \subseteq S_1^p$.

Proof. Let R be a closed right ideal in A, that is, $R \in L_r$. Then $x \in I = Ae$ can be written in the form $x = x_1 + x_2$ with $x_1 \in R$ and $x_2 \in R^p$, and since $x = xe$ we have $x_1e = x_1$ and $x_2e = x_2$. Thus $x_1 \in R \cap I$ and $x_2 \in R^p \cap I$. If we let $S = R \cap I$ and $S^p = R^p \cap I$, then $S \cap S^p = (0)$, $(S^p)^p = S$ and $S \oplus S^p = I$. Moreover, if R_1, R_2 are in L_r, $R_1 \subseteq R_2$ and if $S_i = R_i \cap I$, $S_i^p = R_i^p \cap I$, $i = 1, 2$, then $S_1 \subseteq S_2$ and $S_2^p \subseteq S_1^p$. Let L_R be the set of all closed linear subspaces S in I such that $S = R \cap I$ for some $R \in L_r$. Then the mapping $S \to S^p$ of L_R into L_R enjoys all the properties (a), (b), (c), and (d). Hence to complete the proof we need only show that $L_R = L$. Let S be a proper closed linear subspace of I, and let R_S be the right ideal SA. Let $R = \bar{R}_S$. Then $S \subseteq R \cap I$. If $z \in R \cap I$ then z is the limit of a sequence $\{z_n\}$, $z_n \in R_S$. Since

$$z = ze = \lim_{n \to \infty} z_n e$$

and $R_S e \subset S$, it follows that $z \in S$. Thus $S = R \cap I$ and R is a proper closed right ideal. We have $L_R = L$.

THEOREM 5. *Let I be a minimal left ideal in a simple right complemented algebra A. Then an inner product (x, y) can be introduced in I for which I is a Hilbert space and the norm $|x| = (x, x)^{\frac{1}{2}}$ is equivalent to the given norm $||x||$ in I.*

Proof. By Lemma 6, $S \to S^p$ is an involutory anti-automorphism of L onto L and therefore by Theorem 2 in **(5)** an inner product (x, y) can be introduced in I for which I is a Hilbert space and such that the corresponding norm $|x| = (x, x)^{\frac{1}{2}}$ is equivalent to the given norm $||x||$. With respect to this inner product S and S^p become orthogonal complements of each other.

LEMMA 7. *Let e be a left projection in a simple right complemented algebra A. Let I be a minimal left ideal with an inner product (x, y). Then $(ex, y) = (x, ey)$ for all x, y in I.*

Proof. Let $R = eA$. Then $R^p = \{x - ex : x \in A\}$ and if x, y are in I then $x = x_1 + x_2$ and $y = y_1 + y_2$ with x_1, y_1 in R and x_2, y_2 in R^p. Hence $(ex, y) = (x_1, y_1 + y_2) = (x_1, y_1)$ and $(x, ey) = (x_1 + x_2, y_1) = (x_1, y_1)$ and therefore $(ex, y) = (x, ey)$ for all x, y in I.

If e is a primitive idempotent in A then eAe is isomorphic to the complex number field, and therefore every $0 \neq a \in eA$ gives rise to a unique non-zero bounded linear functional ϕ_a on Ae. In fact $||\phi_a|| \leqslant (||a||)/(||e||)$ **(1**, p. 161**)**. Thus $a \to \phi_a$ is a continuous isomorphism of eA into $(Ae)^*$, the dual of Ae. Let Φ be the set of all linear functionals in $(Ae)^*$ corresponding to the elements of eA. Considering Ae as a Hilbert space Φ can be identified by way of Riesz' representation theorem with a subset of Ae, which we shall denote by S_Φ.

LEMMA 8. *Let Ae be a minimal left ideal in A, then Φ is dense in $(Ae)^*$.*

Proof. Let (x, y) denote the inner product in Ae and let S be the orthogonal complement of the closed linear subspace $\bar{S}_\Phi$. Since for each $a \in eA$ there is a unique $y_a \in S_\Phi$ such that $ax = (x, y_a)e$, $x \in Ae$, it follows that if $x \in S$ then $ax = 0$ for all $a \in eA$; that is, $eAx = (0)$ for $x \in S$. Hence $S = (0)$, S_Φ is dense in Ae and so Φ is dense in $(Ae)^*$.

DEFINITION 3. *Let A be a Banach algebra. If for every proper closed right ideal I in A, $l(I) \neq (0)$, we shall say that A is a left annihilator algebra.*

LEMMA 9. *Let Ae be a minimal left ideal in a simple left annihilator right complemented algebra A. Then $\Phi = (Ae)^*$.*

Proof. It suffices to show that every $\phi \in (Ae)^*$ belongs to Φ. Let $\phi \neq 0$ be an element of $(Ae)^*$ and let S be the null space of ϕ. Let $u \in Ae$ be such that $\phi(u) = 1$. Then $S \oplus (u) = Ae$ and $l(S) \neq (0)$ (**1**, p. 160). If $0 \neq a \in l(S) \cap eA$, then $aS = (0)$ and $au \neq 0$. Hence if we choose $a \in l(S) \cap eA$ such that $au = e$, then $\phi(x)e = ax$ and so $\phi \in \Phi$. Thus, $\Phi = (Ae)^*$ and it follows now that $a \to \phi_a$ is a continuous isomorphism of eA onto $(Ae)^*$.

COROLLARY 1. *The isomorphism $a \to \phi_a$ is a homeomorphism of eA onto $(Ae)^*$.*

COROLLARY 2. *If I is a minimal right ideal in A, then an inner product (x, y) can be introduced in I for which I is a Hilbert space and the norm $|x| = (x, x)^{\frac{1}{2}}$ is equivalent to the given norm $||x||$ in I.*

THEOREM 6. *Let J be a proper closed left ideal in a simple left annihilator right complemented algebra A. Then there is a left ideal J' in A such that $J \cap J' = (0)$ and $\overline{J + J'} = A$. Moreover, if J_1 and J_2 are closed left ideals, $J_1 \subseteq J_2$, then $J_2' \subseteq J_1'$.*

Proof. Let e be a primitive idempotent in J. Let $S = J \cap eA$ and S^p be the orthogonal complement of S in the Hilbert space eA. Then $\overline{AS^p}$ is a proper closed left ideal in A. Let $J' = \overline{AS^p}$. Then $J \cap J' = (0)$ and by the proof of Lemma 6, $S^p = J' \cap eA$. Thus $J + J' \supset eA$ and therefore $\overline{J + J'} = A$.

THEOREM 7 (*representation theorem*). *Let A be a simple left annihilator right complemented algebra. Then there is a continuous isomorphism of A onto an algebra B of completely continuous operators on a Hilbert space. B contains all the operators of finite rank.*

Proof. Let $I = Ae$ be a minimal left ideal in A. We represent A as an algebra of operators on Ae as follows: for each $a \in A$ let the operator T_a be defined by T_a: $a \to ax$, $x \in Ae$. Let B be the algebra of all operators T_a, $a \in A$. It is clear that if $a \neq 0$, then $T_a \neq 0$. Since

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$$|T_a| = \sup_{\|x\|=1} ||ax|| \leqslant ||a||, x \in Ae,$$

$a \to T_a$ is a continuous isomorphism of A onto B. Let F be the algebra of all operators of finite rank on Ae. Then by Lemma 9, $F \subset B$ and by Lemma 5, F is dense in B. Let (x, y) denote the inner product in Ae and $||x||_1 = (x, x)^{\frac{1}{2}}$. Since the norms $|| \quad ||$ and $|| \quad ||_1$ are equivalent, $a \to T_a$ is thus a continuous isomorphism of A onto B, an algebra of completely continuous operators on the Hilbert space Ae. By **(6**, Theorem 2.8.23**)**, B is also an annihilator algebra.

COROLLARY. *A simple left annihilator right complemented algebra is an annihilator algebra.*

THEOREM 8. *A semi-simple left annihilator right complemented algebra A is an annihilator algebra.*

Proof. Since every minimal closed two-sided ideal I in A is a simple left annihilator right complemented algebra **(1**, Theorem 8**)**, by Corollary to Theorem 7 and **(6**, Theorem 2.8.29**)** it follows that A is an annihilator algebra.

LEMMA 10. *Let A be a simple right complemented algebra. Let R be a closed right ideal, I a minimal left ideal in A, and $S = R \cap I$. If an element a in A is such that $ax \in S$ for all $x \in I$, then $a \in R$.*

Proof. Since $a = a_1 + a_2$ with $a_1 \in R$ and $a_2 \in R^p$, $ax = a_1x + a_2x \in S$ for all $x \in I$ implies that $a_2I = (0)$. Thus $a_2 = 0$ and $a \in R$.

COROLLARY. *If R_1 and R_2 are closed right ideals in A with $S_1 = R_1 \cap I$ and $S_2 = R_2 \cap I$, then $S_1 = S_2$ if and only if $R_1 = R_2$.*

THEOREM 9. *A simple annihilator bicomplemented algebra A is dual.*

Proof. Let R be a proper closed right ideal and $I = Ae$ be a minimal left ideal in A. Let R_l be the left annihilator of R, $S = R \cap I$ and $S_l = R_l \cap eA$. It is clear that $S_l \neq (0)$. Let (x, y) denote the inner product in Ae. Then for each $y \in Ae$ there is a unique $b_y \in eA$ such that $(x, y)e = b_yx$, $x \in Ae$, and for each $b \in eA$ there is a unique y_b in Ae such that $(x, y_b)e = bx$, $x \in Ae$. Since $ax \in S$ for $a \in R$ and $x \in Ae$, $(ax, y)e = (b_ya)x$ implies that if $y \in S^p$ then $b_y \in S_l$. Conversely, if $b \in S_l$ then $y_b \in Ae$ belongs to S^p. Thus there is a one-one correspondence between S^p and S_l. Let $S_{lr} = R_{lr} \cap I$. Then $S \subset S_{lr}$. If $x \in S_{lr}$ then $(x, y) = 0$ for $y \in S^p$ and therefore $x \in S$. Thus $S \supset S_{lr}$ and hence $S = S_{lr}$. By the Corollary to Lemma 10, $R_{lr} = R$. Similarly we can show that for every proper closed left ideal J, $J_{rl} = J$. We use a minimal right ideal $I = eA$ and the fact that $(eA)^*$ is equivalent to Ae.

A Banach algebra A is said to have the minimal norm property if $| \quad |$ is a second norm with $|a| \leqslant ||a||$ for all $a \in A$, then $| \quad | = || \quad ||$. A Banach algebra A is a $B^\#$ algebra if for each $a \in A$ there exists an element $a^\# \in A$ such that $a^\# \neq 0$ and $\lim_{n\to\infty} ||(a^\# a)^n||^{1/n} = ||a^\#|| \, ||a||$. Bonsall **(2**, Theorem 4**)** has shown that if A is an annihilator $B^{\#}$ algebra, then A has the minimal norm property. We conclude this section with the following theorem.

THEOREM 10. *If a simple annihilator right complemented algebra A has the minimal norm property or is a $B^{\#}$ algebra, then A is bicontinuously isomorphic to the algebra of all completely continuous operators on a Hilbert space.*

Proof. Let $I = Ae$ be a minimal left ideal in A. We use the notation as given in the proof of Theorem 7. If A has the minimal norm property then $| \ | = || \ ||$ and since the norms $|| \ ||$ and $|| \ ||_1$ are equivalent, it follows that B is the algebra of all completely continuous operators on Ae.

5. Conclusion. We conclude with the following observations: Let A be a semi-simple algebra such that for every closed right ideal I the complement I^p is a closed left ideal, then A is commutative. Indeed, since $II^p \subset I \cap I^p = (0)$ we have $r(I) \supset I^p$ and $l(I^p) \supset I$. Since A is semi-simple $I \cap r(I) = (0)$ and $I^p \cap l(I^p) = (0)$; $r(I) = I^p$ and $l(I^p) = I$. Thus I and I^p are closed two-sided ideals. Hence every minimal right ideal I is a minimal closed two-sided ideal. Thus $I = eA = Ae$ and it follows that I is isomorphic to the complex number field. By Theorem 4, the sum $\sum I$ of all minimal closed two-sided ideals is dense in A and since $\sum I$ is commutative, A is commutative. It also follows from Theorem 4 that if each minimal closed two-sided ideal in a semi-simple right complemented algebra A is commutative, then A is commutative.

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