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SUBNORMAL EMBEDDING OF RELATIVELY COMPLETE GROUPS

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Abstract

In this paper we show that a group A is embedded in any finite group G as a subnormal subgroup with low degree of complication, provided that the automorphism group of A satisfies a condition depending on some Fitting class (which coincides with completeness for the Fitting class of all groups). A criterion is given for these groups as to whether they can be embedded subnormally in the commutator subgroup of some finite group or not.

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Introduction

A group is called complete, if its centre is trivial and all automorphisms are inner. It has been shown earlier [3], that the way these groups are embedded in a group as subnormal subgroup is rather restricted: if A is directly indecomposable, complete and subnormal in G, if F is the Fitting subgroup of the normal closure A^G of A, then A^G/F is the direct product of all conjugates of AF/F in G/F, provided A is not isomorphic to the holomorph of some cyclic 3-group (see [3, Theorem C]).

When considering subnormal embedding, we find that the most common operations are forming normal products and reducing to normal subgroups; the

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Fitting classes are by definition the classes of groups which are closed with respect to just these operations. In this note we want to generalize the statement mentioned before for Fitting classes. We call a group G complete with respect to a Fitting class $\mathfrak{F}$ if $Z(G) = 1$ and certain extensions do not belong to $\mathfrak{F}$ (see below for the exact definition). Then for Fitting classes which are closed with respect to epimorphic images (called briefly Q-closed) we are able to establish

THEOREM. *If A is a directly indecomposable subnormal subgroup of a finite group G such that*

(i) *A is complete with respect to a Q-closed Fitting class $\mathfrak{F}$,*

(ii) *A is not isomorphic to the $\{2,p\}$-Hall subgroup of the holomorph of some cyclic p-group, where $p \equiv 3 \bmod 4$, a prime,*

then

(a) *the nilpotent residual A^* of A is normal in A^G and $(A^*)^G$ is the direct product of all conjugates of A^* in G,*

(b) *if $F = F(A^G)$, AF is normal in A^G and A^G/F is the direct product of all conjugates of AF/F in G/F.*

Closure with respect to epimorphic images is needed here, as a counterexample will show.

NOTATION. The *nilpotent residual* G^* of a group G is the intersection of all normal subgroups K of G with nilpotent quotient group G/K. If $\mathfrak{F}$ is a Fitting class, the *$\mathfrak{F}$-radical* of G is the product of all normal subgroups of G belonging to $\mathfrak{F}$. A normal subgroup B of a group G is called *big*, if every nontrivial normal subgroup of G intersects B nontrivially.

DEFINITION. The group A is complete with respect to a Fitting class $\mathfrak{F}$, if the following two conditions are satisfied:

(1) $Z(A) = 1$;
(2) for every group U with $\mathrm{Inn}(A) \subseteq U \subseteq \mathrm{Aut}(A)$, the subgroup $\mathrm{Inn}(A)$ is the $\mathfrak{F}$-radical of U.

All notation yet unexplained should be standard (compare with Huppert [4] for instance).

All groups considered in this note are finite.

Four lemmas have been crucial for the proof of [3, Theorem C]. We will first state and prove modifications of these lemmas, which in part are also more general when applied to the case of the class of all finite groups. The remainder (Lemmas 5 and 6) is needed to replace Gagen's Theorem [2].

LEMMA 1. *Suppose that $G = AN$, where N is a nilpotent normal subgroup of G and A is a subnormal subgroup of G which is complete with respect to some given Fitting class $\mathfrak{F}$. Then $G = AK$ where K is the hypercentre of G and $A \cap K = 1$.*

PROOF. If L is a normal subgroup of G which is contained in the hypercentre of G, then clearly $A \cap L$ is contained in the hypercentre of A which is trivial. It suffices therefore to show that G is generated by A and the hypercentre of G.

We proceed by induction on the defect of A in G. For the initial step suppose that A is normal in G. Denote by C the centralizer of A in G. Since A is a normal subgroup of G, so is C, and $A \cap C = Z(A) = 1$. Now $G/C = (AC/C)(NC/C)$, $AC/C \cong A$, and $NC/C \cong N/N \cap C$ is nilpotent, and we may assume without loss of generality $C = 1$. Assume $A \neq G$. We choose an element x of N such that xA is an element of $Z(G/A)$, but x is not in A. This is possible since $G/A = AN/A \cong N/N \cap A$ is nilpotent. The smallest normal subgroup K of G containing x is contained in N and therefore nilpotent. The intersection $K \cap A \neq 1$ is a normal subgroup of A and belongs therefore to $\mathfrak{F}$. Now also K and KA belong to $\mathfrak{F}$, and $C_{KA}(A) \neq 1$ since otherwise KA is isomorphic to a subgroup of $\mathrm{Aut}(A)$ containing $\mathrm{Inn}(A)$. But then $C_G(A) \neq 1$, the final contradiction. So $G = A \times C_G(A)$ and $C_G(A) = AC_G(A)/A = AN/A \cong N/A \cap N$ is nilpotent.

It is now clear that $C_G(A)$ is the hypercentre of G, and the initial step of the induction is carried out.

Assume now that the statement is proved for all pairs A, L where A is of defect $n \geq 1$ at most in L, and assume that A is of defect $n+1$ in $G = AN$. To derive a contradiction, assume further that the hypercentre of G is trivial and A is different from G.

Let $L = A^G$. By modular law, $L = A(N \cap L)$, and we may apply the induction hypothesis for the pair A, L. We obtain $L = AT$ where T is the hypercentre of A. If $T \neq 1$, also $Z(L) \neq 1$, and $Z(L)$ is a normal subgroup of G.

We consider two different cases:

If $Z(L) \cap N = 1$, we have $Z(L) \subseteq Z(LN) = Z(G)$,

if $Z(L) \cap N \neq 1$, we have $1 \neq Z(L) \cap Z(N) \subseteq Z(LN) = Z(G)$. Both cases lead to a contradiction, so $Z(L) = 1$ and $L = A$, that is A is normal in G. This case we have treated in the initial step, so Lemma 1 is shown.

LEMMA 2. *Suppose that A is a subnormal subgroup of the group H such that the intersection A_H of all conjugates of A in H is a big normal subgroup of A. If A is complete with respect to some Fitting class $\mathfrak{F}$, then $A^H = AK$ where K is the hypercentre of A^H. If furthermore H belongs to $\mathfrak{F}$, and $\mathfrak{F}$ is also Q-closed, then there is a normal subgroup N in H such that $H = AN$ and $A \cap N = 1$.*

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PROOF. We begin with the general case and proceed by induction on the defect d of A in H. Nothing is to be shown if A is normal in G; we consider the case that A is of defect 2 in H. As in the proof of Lemma 1 we may assume that the hypercentre of A^H is trivial. If A^H is different from A, there is a conjugate B of A which is different from A. So $AB \neq A$ and AB belongs to the Fitting class $\mathfrak{F}$. By completeness of A with respect to $\mathfrak{F}$ we have $C_{AB}(A) \neq 1$ and so $R = C_{A^H}(A) \neq 1$.

If D is a big normal subgroup of A, then $x^{-1}Dx$ is a big normal subgroup of $x^{-1}Ax$. Applying this to $D = A_H$ yields

$$1 = R \cap A = R \cap A_H = R \cap x^{-1}Ax$$

for all x in H, and so $R = Z(A^H)$. So the hypercentre of A^H is nontrivial contrary to our assumption. This contradiction shows that $A^H = A$ if the centralizer of A^H is trivial, and so $A^H = AK$ where K is the hypercentre of AK.

Assume now that the lemma is proved for all pairs A, L where A is of defect $n \geq 2$ at most, and that A is of defect $n+1$ in H. Then A is of defect n in $L = A^H$, and $A^L = AM$, where M is the hypercentre of A^L. Since M is nilpotent and subnormal in H, M^H is a nilpotent normal subgroup of H.

If $A_H \cap M^H \neq 1$, there is a collection of conjugates M^{x_i} and one conjugate M^y such that $A_H \cap M^y Q \neq 1$ and $A_H \cap Q = 1$, where $Q = \langle \dots, M^{x_i}, \dots \rangle$ and there is, furthermore, an index s such that

$$A_H \cap Z_s(A^y M^y)Q = 1 \text{ while } A_H \cap Z_{s+1}(A^y M^y)Q \neq 1.$$

But now

$$[A^y, A_H \cap Z_{s+1}(A^y M^y)Q] \subseteq A_H \cap Z_s(A^y M^y)Q = 1$$

and $Z(A^y) \neq 1$, a contradiction. So $A_H \cap M^H = 1$, which yields $A \cap M^H = 1$, and application of Lemma 1 yields that M^H is contained in the hypercentre of A^H. Now AM^H/M^H is of defect 2 in H/M^H, and our initial step of the induction shows that

$$A^H/M^H = (AM^H/M^H)^{H/M^H} = (AM^H/M^H)(T/M^H)$$

where T/M^H is the hypercentre of A^H/M^H.

Now $A^H = AT$ and T is the hypercentre of A^H. If $\mathfrak{F}$ is Q-closed and H belongs to $\mathfrak{F}$, denote by V a normal subgroup of H containing the hypercentre T of A^H such that $A \cap V = 1$ but $A \cap W \neq 1$ for all normal subgroups of H which have V as a proper subgroup. Since $\mathfrak{F}$ is Q-closed, H/V belongs to $\mathfrak{F}$ and has the normal subgroup $AV/V \cong A/(A \cap V) = A$. By maximality of V, $C_{H/V}(AV/V) = 1$.

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Now H/V can be mapped faithfully into $\mathrm{Aut}(A)$ such that $\mathrm{Inn}(A)$ is the image of AV/V, and by relative completeness of A, we have

$$H/V = AV/V \quad \text{and} \quad AV = H.$$

LEMMA 3. *Suppose that there are two normal subgroups K and L of a group A which is complete with respect to a Fitting class $\mathfrak{F}$ and $K \cap L = 1$. Then there are two normal subgroups H and J of A such that $H \cap J = 1$ and $Z_2(A/HJ) = Z(A/HJ)$. If $\mathfrak{F}$ is also Q-closed, H and J can be found with $Z(A/HJ) = 1$.*

PROOF. Choose a normal subgroup H of A which is maximal with respect to containing K and having trivial intersection with L. Once H is chosen, take a normal subgroup J which is maximal with respect to containing L and having trivial intersection with H. Assume now $Z(A/J) = M/J \neq 1$. Then

$$[A, H \cap M] = [A, H] \cap [A, M] \subseteq H \cap J = 1$$

and $H \cap M \neq 1$ is contained in $Z(A)$, a contradiction. So $Z(A/J) = Z(A/H) = 1$.

The subgroup $\{(aH, aJ) | a \in A\} = R$ of the direct product $D = (A/H) \times (A/J)$ is known to be isomorphic to A. Let $W/HJ = Z_2(A/HJ)$ and consider

$$S = \{(xH, yJ) | xW = yW, x, y \in A\}.$$

If $T = \{(uH, vJ) | u, v \in W\}$ and $U = \{(rH, sJ) | r, s \in HJ\}$, then

$$S = RT \supseteq U \quad \text{and} \quad T/U = Z_2(D/U).$$

We deduce that R is subnormal in S. If R is not normal in S, we have a conjugate V of R in S such that V and R are normal subgroups of VR. Since $\mathfrak{F}$ is a Fitting class, VR belongs to $\mathfrak{F}$. Every conjugation by an element of VR induces an automorphism of R, by this we define a homomorphism of VR into $\mathrm{Aut}(R)$ mapping R onto $\mathrm{Inn}(R)$. Since R is complete relative to $\mathfrak{F}$, $C_{VR}(R) \neq 1$ and $C_D(R) \neq 1$. If (bH, cJ) belongs to $C_D(R)$ we have

$$(H, J) = [(bH, cJ), (aH, aJ)] = ([b, a]H, [c, a]J) \quad \text{for all } a \text{ in } A$$

and $bH \in Z(A/H)$, $cJ \in Z(A/J)$ so $(bH, cJ) = (H, J)$ and $C_D(R) = 1$. This is a contradiction showing that R is normal in S, and $Z_2(D/U) = Z(D/U)$ is now deduced easily. If $\mathfrak{F}$ is Q-closed, we proceed analogously to obtain that S and R have to be identical and so $Z(D/U) = 1$. The corresponding statements for A/HJ are now immediate.

COROLLARY. *If A is complete with respect to a Q-closed Fitting class and directly irreducible, then A^* is no (proper) direct product of A-invariant subgroups.*

LEMMA 4. *If* (i) *A and B are two subnormal subgroups of a group G,*

(ii) *A^* and B^* are normal subgroups of G,*

(iii) *A and B are directly indecomposable and complete with respect to a Q-closed Fitting class $\mathfrak{F}$,*

then $A^* = B^*$ or $A^* \cap B^* = 1$ and $A \cap B$ is contained in the Fitting subgroup of G.

The proof is exactly that of [3, Lemma 4] except that our Lemma 2 is substituted for [3, Lemma 2]. Remember that A^* is a big normal subgroup of A if $Z(A) = 1$.

LEMMA 5. *If $\mathfrak{F}$ is a Q-closed Fitting class and the extension of a cyclic group of order p^n by a cyclic group of order r dividing $p-1$ that has trivial centre belongs to $\mathfrak{F}$, then also the group T with the following properties belongs to $\mathfrak{F}$:*

(i) *T is the extension of p-group P by a cyclic group $Y = \langle y \rangle$ of order dividing r.*

(ii) *P can be generated by a generating set $\{x_1, \ldots, x_k\}$ such that y is in the normalizer of $\langle x_i \rangle$ for all i and the orders of the elements x_i divide p^n.*

PROOF. We proceed by induction on the nilpotency class c of P. If $c = 1$ and p is abelian, without loss of generality we may choose x_i such that

$$P = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_k \rangle.$$

Then T is isomorphic to a normal subgroup of the direct product of the quotient groups $T/\langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k \rangle$ all of which belong to $\mathfrak{F}$. Assume now that P is of class $c = m+1$ and that the lemma is proved whenever P is of smaller nilpotency class. Consider $W = L/K_{m+2}(L)$ where L is the free product of k cyclic groups $\langle a_i \rangle$ of order p^n and $K_{m+2}(L)$ is the $(m+2)$th term of its lower central series; and denote $a_i K_{m+2}(L)$ by b_i.

There is an automorphism of W fixing all of these b_i except one, b_j, say and mapping this b_j onto some power b_j^s. We take such an automorphism of order r for every j and call it σ_j. We want to show that $\langle d_j, W | d_j^r = 1, d_j^{-1} w d_j = w^{\sigma_j} \rangle = V_j$ belongs to $\mathfrak{F}$: V_j is the normal product of W and $U_j = \langle d_j, h^{-1} b_j h | h \in \langle b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_k \rangle \rangle$ and the normal closure of an element in a nilpotent group is (abelian or) of lower nilpotency class than the original group. So we may apply our induction hypothesis for U_j and with U_j also V_j belongs to $\mathfrak{F}$. The different automorphisms σ_j commute with each other, so there is a normal product of groups V_j, namely

$$\langle d_1, \ldots, d_j, \ldots, d_k, W | [d_i, d_j] = 1, d_j^{-1} w d_j = w^{\sigma_j} \text{ for all } w \in W \text{ and all } i, j \rangle.$$

Assume now that $y^{-1} x_i y = x_i^{u_i}$ where $(u_i)^r \equiv 1 \bmod p^n$.

There is an automorphism τ of W such that $b_i^\tau = b_i^{u_i}$, and a product g of the elements d_i such that $g^{-1} b_i g = b_i^{u_i}$.

There is a normal subgroup R of W such that W/R is isomorphic to P, such that the elements b_i are mapped onto the elements x_i. Since y induces in P by conjugation the automorphism corresponding to that induced in W by g, this isomorphism can be extended and R is invariant under conjugation by g. Now

$$\langle g, W\rangle/R \cong \langle y, P\rangle = T$$

and T belongs to $\mathfrak{F}$.

LEMMA 6. *Suppose that A satisfies the following conditions*

(i) *A is a directly indecomposable group which is closed with respect to some Q-closed Fitting class $\mathfrak{F}$,*

(ii) *A^* is an abelian p group,*

(iii) *$A/F(A)$ is abelian of exponent r dividing $p-1$,*

then A is isomorphic to a Hall subgroup of the holomorph of some cyclic p-group.

PROOF. We deduce from Lemma 3 that A^* is not a (proper) direct product of two A-invariant subgroups since A is no (proper) direct product. So if x is an element of A which does not belong to $F(A)$, conjugation by x induces in A^* a fixedpointfree automorphism.

We obtain

(1) Every p'-Hall subgroup of A is cyclic, its elements induce a power automorphism in A^* by conjugation.

Now we may derive from the relative completeness of A that

(2) A/A^* is the direct product of a cyclic group $\langle y, A\rangle/A$ of order r and a group P/A which is isomorphic to a p-Sylow subgroup of $\mathrm{Aut}(A^*)$.

Assume that A^* is noncyclic. We may describe A^* as a direct product of cyclic groups $\langle f_1\rangle, \dots, \langle f_m\rangle$ in such a way that there is an element in A which conjugates f_i onto $f_i f_{i+1}$. There is an automorphism of A^* which is of order r and which fixed all f_i except one, f_j say, which is mapped onto some power f_j^k. This automorphism can be extended to an automorphism of A, we call it σ.

Now $B = \langle z, P | z^{-1} g z = g^\sigma \text{ for all } g \in P,\ z^r = 1\rangle$ belongs to the class $\mathfrak{F}$ since P is a nilpotent p-group generated by $f_1, \dots, f_m$ and elements t_{ij} such that

$$t_{ij}^{-1} f_u t_{ij} = f_u \quad \text{for } u \neq i,$$
$$t_{ij}^{-1} f_i t_i = f_i f_j^{m_{ij}},$$

and σ maps all of these elements onto powers of themselves, they are furthermore of order dividing p^n. So B belongs to $\mathfrak{F}$ by Lemma 5. Now also

$$AB = \langle z, A | yz = zy, gz = zg \text{ for } g \in P\rangle$$

is contained in $\mathfrak{F}$ and has trivial centre. Now the completeness of A with respect to the Fitting class $\mathfrak{F}$ leads to a contradiction. We have found

(3) A^* is cyclic.

Assume now that r and $(p-1)/r$ are not relatively prime, that is, there is a prime power q^m dividing r such that q^{m+1} divides $p-1$ but not r. If X is an extension of a cyclic group of order p^n by a cyclic group of order q^m such that $Z(X)=1$, and if S is a group of order q, we consider the wreath product $X \operatorname{wr} S$.

This wreath product contains a subnormal subgroup isomorphic to $X' \operatorname{wr} S$ which is easily seen to be contained in $\mathfrak{F}$, also the base group X^S belongs to $\mathfrak{F}$. Since both together generate the wreath product, it is also contained in $\mathfrak{F}$. But there is also a subnormal subgroup Y of $X \operatorname{wr} S$ which contains X'^S such that Y/X'^S is cyclic of order q^{m+1}. Since q^{m+1} divides $p-1$, the abelian normal subgroup X'^S of Y splits onto the direct product of some Y-invariant cyclic normal subgroups, and since $\mathfrak{F}$ is Q-closed, there are extensions of cyclic groups of order p^n by cyclic groups of order q^m which have trivial centre and belong to $\mathfrak{F}$. It is now easy to form an extension of A which is still contained in $\mathfrak{F}$ such that the centralizer of A is trivial, and to derive a contradiction in this case. So

(4) A is isomorphic to a Hall subgroup of $\operatorname{Aut}(A^*)$.

Now Lemma 6 is proved by (3) and (4).

PROOF OF THE THEOREM. We will proceed by induction on the defect of A in G. In the initial step of the induction A is normal in G and the theorem is trivially true.

We assume now that the theorem has been proved for all pairs (A^+, G^+) satisfying the hypothesis with A^+ of smaller defect in G^+ than that of A in G. We apply the theorem to the pair (A, A^G). Then A^* is of defect 2 in A^G. We assume further the existence of a conjugate $B = g^{-1}Ag$ of A such that A^* is not normal in $W = \langle A, B\rangle$. We plan to show that A is one of the excluded groups in this case.

We can now follow the proof of Theorem C in [3] in several steps, substituting our Lemma 3 for [3, Lemma 3], and we obtain

(1) $(A^*)^W = (B^*)^W$,

(2) A^* is abelian,

(3) $(A^*)^W = \langle A^*, t^{-1}A^*t\rangle$ for some t in B,

(4) $A/F(A)$ is abelian,

(see statements (7), (8), (9) and (10), pages 440–441 of [3]). We deduce from (2) and our corollary to Lemma 3 that

(5) A^* is a p-group for some prime p

and now $Z(A) = 1$ yields

(6) $F(A)$ is a p-group.

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It follows that $A/F(A)$ is an abelian group of order prime to p and $A/A^* = Q/A^* \times F(A)/A^*$, where Q/A^* is the p'-Hall subgroup of A/A^*. Since A^* does not split into a direct product of A-invariant subgroups and $A/F(A) \cong Q/A^*$ is abelian, furthermore A^* is selfcentralizing in Q, we have

(7) $A/F(A)$ is cyclic.

By symmetry, $B/F(B)$ is cyclic as well. By induction hypothesis, $(A^*)^W = \langle A^*, t^{-1}A^*t\rangle$ is the direct product of all W-conjugate subgroups of A^*, so, in particular, if x is an element of B, x^2 leaves invariant A^* and $t^{-1}A^*t$. Again by symmetry

(8) $(B^*)^W = \langle B^*, u^{-1}B^*u\rangle$ for some u in A,

and x^2 leaves invariant B^* and $u^{-1}B^*u$. By the well known theorem of Wielandt [5], $\langle A, B\rangle^* = A^*B^*$ and so

$$A^*B^* = (A^*B^*)^W = (A^*)^W(B^*)^W = (A^*)^W = (B^*)^W.$$

Considering the order of $A^*B^* = \langle A^*, t^{-1}A^*t\rangle = \langle B^*, u^{-1}B^*u\rangle$ we obtain

$$A^* \cap B^* = A^* \cap u^{-1}B^*u = 1.$$

Assume now that x^2 is not contained in $F(B)$ and of order prime to p. Then conjugation of B^* by x^2 is a fixedpointfree automorphism of B^*. On the other hand $[[\cdots[\langle B^*, u^{-1}B^*u\rangle, x^2]\cdots], x^2]$ is contained in B^* for suitably high commutators. Since x^2 is of order prime to p and $u^{-1}B^*u$ is a p-group, this means that x^2 centralizes $u^{-1}B^*u$.

Now x^2 induces by conjugation the identity automorphism on $A^*u^{-1}B^*u/A^*$ and a fixedposintfree automorphism on A^*B^*/A^*, a contradiction, since $A^*B^* = A^*u^{-1}B^*u$.

This contradiction shows that x^2 is the identity element, therefore we have $|B/F(B)| = 2$ and

(9) $|A/F(A)| = 2$.

Application of Lemma 6 shows that A is one of the excluded groups. So if A does not belong to this class of groups the induction step is successful and A^* is normal in A^G. Since no two different conjugates of A^* are operator-isomorphic in A^G, the product $(A^*)^G$ is the direct product of all conjugates of A^*. The uniqueness of this direct product in A^G yields also that A^G/F is the direct product of all conjugates of AF/F in G/F, and this proves the theorem.

COUNTEREXAMPLE Let G be generated by a, b, c, d, e, f subject only to

$$\begin{aligned}
&a^7 = b^7 = c^7 = d^3 = e^3 = f^4 = 1,\\
&[a,b] = [b,c] = [c,a] = [[d,e],e] = [[d,e],d] = 1,\\
&d^{-1}ad = a^2, d^{-1}bd = b^4, [d,c] = 1,\\
&e^{-1}ae = b, e^{-1}be = c, e^{-1}ce = a,\\
&f^{-1}df = e, f^{-1}ef = d^{-1},\\
&f^{-1}af = a^2bc^4, f^{-1}bf = ab^2c^4, f^{-1}cf = a^4b^4c^4.
\end{aligned}$$

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This is an extension of an elementary abelian 7-group $\langle a,b,c\rangle$ of rank three by the extension of a nonabelian group $\langle d,e\rangle$ of order 27 and exponent 3 by a cyclic group $\langle f\rangle$ of order 4.

The subgroups $\langle a,b,d\rangle$ and $\langle a^2bc^4, ab^2c^4, e\rangle$ are conjugate subnormal subgroups of G. We describe a Fitting class $\mathfrak{F}$ with respect to which they are complete. Every element of a solvable group induces by conjugation a linear mapping in every chief factor; its determinant is an element of the corresponding prime field. The class of all extensions of 7-groups by 3-groups in which for every element of the group, the product of all these determinants on 7-chief factors is 1, is a (normal) Fitting class (see Blessenohl and Gaschütz [1, Satz 3.3]). Now $\langle a,b,d\rangle$ is complete with respect to this Fitting class. The normal closure of $\langle a,b,d\rangle$ in G is $\langle a,b,c,d,e\rangle$, and statements (a) and (b) of the theorem are not true. So we see that the Q-closedness of $\mathfrak{F}$ is indispensable in the theorem.

Considering the subnormal embedding in the commutator subgroup of a group, we obtain the following consequence of our theorem:

CONCLUSION. *If A is a directly indecomposable subnormal subgroup of a finite group G such that*

(i) *A is complete with respect to a Q-closed Fitting class $\mathfrak{F}$*

(ii) $\mathrm{Inn}(A)$ *is not contained in* $(\mathrm{Aut}(A))'F(\mathrm{Inn}(A))$,

then A is not contained in G'.

PROOF. If A is isomorphic to the $\{2,p\}$-Hall subgroup of the holomorph of some cyclic p-group, we use an argument by determinants analogous to that in [3, p. 436] and show that A is not contained in G'. Assume now that A satisfies, in addition, condition (ii) of the theorem. Then we may use the results of the theorem: $A^G/F(A^G)$ is the direct product of the conjugates of $AF(A^G)/F(A^G)$ in $G/F(A^G)$.

By Lemma 1 we have $AF(A^G) = AK$ and $A\cap K = 1$, where K is the hypercentre of $AF(A^G)$. Choose a conjugate BL of AK such that $BL \neq AK$, B is conjugate to A and L is the hypercentre of BL. Conjugation of BL by elements of A induce in the quotient group $BL/L \cong B$ automorphisms of B, and there is consequently a mapping ρ of A into $\mathrm{Aut}(B)$ with kernel T. Since $\mathfrak{F}$ is Q-closed, $A^\rho = A/T$ belongs to $\mathfrak{F}$, and since A is subnormal in A^G, $A^\rho\,\mathrm{Inn}(B)$ belongs to $\mathfrak{F}$. Since B is complete with respect to $\mathfrak{F}$, we have that A^ρ belongs to $\mathrm{Inn}(B)$.

Choose any element y of A. Since $A^G/F(A^G)$ is the direct product of $AF(A^G)/F(A^G)$ and its conjugates, conjugation by y will fix all cosets $BF(A^G)$ in $BF(A^G) = B(F(B)L)$. We obtain

$$[BL, y] \subseteq F(B)L,$$

and there is an integer k such that

$$[[BL, y]\underbrace{, \dots ,}_{k \text{ times}} y] \text{ is contained in } (B^* \cap F(B))L.$$

On the other hand, B^* is a normal subgroup of A^G and so $A \cap B^*$ is a normal subgroup of A which is by Lemma 4 nilpotent and has trivial intersection with A^*. Since $Z(A) = 1$, we obtain that $A \cap B^* = 1$, and there is an integer m such that

$$[[B^*, y]\underbrace{, \dots ,}_{m \text{ times}} y] \subseteq A \cap B^* = 1.$$

We have found

$$[[BL, y]\underbrace{, \dots ,}_{k+m \text{ times}} y] \subseteq L$$

and since ρ maps y onto some inner automorphism of B, this inner automorphism is induced by some element of $F(B)$. We conclude

$$A^\rho \subseteq F(\mathrm{Inn}(B)).$$

Let $U_1 = AK, U_2, \dots$ be the set of all conjugates of AK, and denote the hypercentre of U_i by V_i. Now all U_i/V_i are isomorphic, we fix isomorphisms from $A \cong U_1/V_1$ to U_i/V_i and call them $\sigma(i)$ taking the identity for $\sigma(1)$. If g is any element of G, denote by $\tau(g)$ the mapping of U_i/V_i onto $g^{-1}U_i g/g^{-1}V_i g$ by g and $\pi(g,\ )$ the induced permutation of the indices of the U_i such that

$$g^{-1}U_i g = U_{\pi(g,i)}$$

The product

$$f(g) = \prod_i \sigma(i)\tau(g)\sigma(\pi(g,i))^{-1}$$

is an element of $\mathrm{Aut}(A)$, it is modulo $(\mathrm{Aut}(A))'$ independent of the order of the factors in the product. So the mapping of g onto $f(g)(\mathrm{Aut}(A))'F(\mathrm{Inn}(A))$ is a homomorphism of G into $\mathrm{Aut}(A)/(\mathrm{Aut}(A))'F(\mathrm{Inn}(A))$; its kernel contains the commutator subgroup of G. Now consider an element y of A such that the inner automorphism induced by y in A is not contained in $\mathrm{Aut}(A))'F(A))$. By hypothesis such an element y exists. Now $\tau(y)$, restricted to $U_1/V_1 = AK/K$ is not contained in $(\mathrm{Aut}(A))'F(\mathrm{Inn}(A))$, while all other $\sigma(i)\tau(y)\sigma(i)^{-1}$ belong to $F(\mathrm{Inn}(A))$ by the argument above. This shows that $f(y)(\mathrm{Aut}(A))'F(\mathrm{Inn}(A)) \neq (\mathrm{Aut}(A))'F(\mathrm{Inn}(A))$, y is not in the kernel of our homomorphism and accordingly not contained in G'. So A is not contained in G', which was to be shown.

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