ON PARTIAL SUMS OF LAGRANGE'S SERIES WITH APPLICATION TO THE THEORY OF QUEUES

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1. Introduction

Lagrange's theorem on the reversion of power series may be stated in the following form (e.g. Whittaker and Watson [3]).

THEOREM L. *Let $k(z)$ be a function of z analytic on and inside a contour C surrounding a point x and let y be such that for all points z on C*

(1.1) $$|yk(z)| < |z-x|.$$

Then the equation

(1.2) $$z = x+yk(z),$$

regarded as an equation in z, has one root, ζ, in the interior of C and if $\phi(z)$ is any function of z analytic on and inside C

(1.3) $$\phi(\zeta) = \phi(x) + \sum_{m=1}^{\infty} \frac{y^m}{m!} \frac{d^{m-1}}{dx^{m-1}} [\{k(x)\}^m \phi'(x)].$$

By a partial sum of Lagrange's series we mean an expression of the form

(1.4) $$\phi^n(x) = \phi(x) + \sum_{m=1}^{n} \frac{y^m}{m!} D^{m-1}[\{k(x)\}^m D\phi(x)]$$

where $D \equiv d/dx$. In this paper we consider expressions $\phi_j^n(x)$ of the form (1.4) where $\phi(x) = \phi_j(x) = x^j$, $x \geqq 0$, $j \geqq 0$. We show that the $\phi_j^n(x)$ satisfy a certain set of difference equations which occur in the theory of queues. This result gives a simple proof of a conjecture of Finch [2] which has since been proved by Brockwell [1] using quite different methods.

2. The difference equation of partial sums

We prove the following

THEOREM. *Let*

(2.1) $$k(z) = \sum_{j=0}^{\infty} k_j z^j$$

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be analytic in some region surrounding the point x, and let

$$\phi_j^n(x) = \phi_j(x) + \sum_{m=1}^{n} (y^m/m!)D^{m-1}[\{k(x)\}^m D\phi_j(x)] \tag{2.2}$$

where $\phi_j(x) = x^j$ *and* $D \equiv d/dx$.

Then, $\phi_0^n(x) \equiv 1$ *and*

$$\phi_j^{n+1}(x) - x\phi_{j-1}^{n+1}(x) = y\sum_{i=0}^{\infty} k_i \phi_{j+i-1}^n(x), \qquad j \geqq 1,\ n \geqq 1. \tag{2.3}$$

PROOF. By definition $\phi_0^n(x) = \phi_0(x) \equiv 1$. For $j \geqq 1$ we obtain from (2.2)

$$y\sum_{i=0}^{\infty} k_i \phi_{j+i-1}^n(x) = x\phi_{j-1}(x) - x\phi_{j-1}^{n+1} + \sum_{m=1}^{\infty} \frac{y^m}{m!}\psi_j^m(x), \tag{2.4}$$

where

$$\psi_j^m(x) = xD^{m-1}[\{k(x)\}^m D\phi_{j-1}(x)] + mD^{m-2}[\{k(x)\}^{m-1} D\{\phi_{j-1}(x)k(x)\}]. \tag{2.5}$$

Expanding the terms on the right of (2.5) by Leibnitz' theorem and rearranging we obtain

$$\psi_j^m(x) = \sum_{s=0}^{m-1} \binom{m-1}{s} [D^s\{k(x)\}^m][xD^{m-s}\phi_{j-1}(x) + (m-s)D^{m-1-s}\phi_{j-1}(x)]. \tag{2.6}$$

It can be verified readily or proved easily by induction that

$$xD^{m-s}\phi_{j-1}(x) + (m-s)D^{m-1-s}\phi_{j-1}(x) = D^{m-1}\phi_j(x).$$

Thus from (2.6)

$$\psi_j^m(x) = D^{m-1}[\{k(x)\}^m D\phi_j(x)].$$

Substituting in (2.4) we obtain (2.3). This proves the theorem.

COROLLARY. *If* $k(z)$ *is analytic at* $z = 0$ *then the quantities*

$$\phi_j^n = (j/m)\sum_{m=j}^{n} \{(m-j)!\}^{-1}[D^{m-j}\{k(x)\}^m]_{x=0}, \qquad n \geqq j \geqq 1, \tag{2.7}$$

satisfy the equations

$$\phi_j^{n+1} = \sum_{i=0}^{\infty} k_i \phi_{j+i-1}^n \tag{2.8}$$

with $\phi_0^n \equiv 1$ *and* $\phi_j^n = 0$, $1 \leqq n < j$ *and initial conditions* $\phi_0^1 = 1$, $\phi_1^1 = k_0$, $\phi_j^1 = 0, j > 1$.

The corollary is proved easily by putting $x = 0$, $y = 1$ in the theorem.

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3. Application to the queueing system *GI/M/1*.

Consider the queueing system $GI/M/1$ in which the times at which customers arrive form a renewal process with inter-arrival distribution $A(x)$ and in which the service-time distribution is exponential with parameter μ. Let P_j^n be the probability that the nth arrival finds j customers in the system, then

$$P_j^{n+1} = \sum_{i=0}^{\infty} k_i P_{j+i-1}^n, \qquad n \geqq 1, j \geqq 1 \tag{3.1}$$

where
$$k_m = (m!)^{-1} \int_0^{\infty} (\mu x)^m e^{-\mu x} dA(x)$$

and
$$k(z) = \sum_{m=0}^{\infty} k_m z^m \text{ is analytic at } z = 0.$$

Write $Q_j^n = \sum_{m=j}^{\infty} P_m^{n+1}$, then Q_j^n is the probability that the $(n+1)$ the arrival finds j or more customers in the system, and from (3.1)

$$Q_j^{n+1} = \sum_{i=0}^{\infty} k_i Q_{j+i-1}^n, \qquad n \geqq 1, j \geqq 1, \tag{3.2}$$

with $Q_0^n \equiv 1$.

If the queueing system starts from emptiness, so that $P_0^1 = 1$, $P_j^1 = 0$, $j > 0$, we have $Q_0^1 = 1$, $Q_1^1 = k_0$ and $Q_j^1 = 0$ $j > 1$. Since the Q_j^{n+1} are uniquely determined by the Q_j^n we can appeal to the corollary of the theorem of the last section and deduce

$$Q_j^n = (j/m) \sum_{m=j}^{n} \{(m-j)!\}^{-1} [D^{m-j}\{k(x)\}^m]_{x=0}, \qquad n \geqq j \geqq 1. \tag{3.3}$$

When $k'(1) > 1$ a limiting distribution of queue size exists and $Q_j^n \to q^j$, $j \geqq 1$ where $0 < q < 1$ and q is the only root within the unit circle of the equation $z = k(z)$. When $k'(1) \geqq 1$ there is no root of this eauation within the unit circle and a limiting distribution of queue size does not exist, in this case $Q_j^n \to 1$, $j \geqq 1$.

Equation (3.3) establishes the conjecture of Finch [2] that the probabilities Q_j^n are given by the partial sums of the Lagrange series for q^j. As noted earlier this result has been proved by Brockwell, [1].

References

[1] Brockwell, P. J., The transient behaviour of the queueing system *GI/M/1*, Journ. Aust. Math. Soc.

[2] Finch, P. D., The single Server queueing system with non-recurrent input process and Erlang Service time, Journ. Aust. Math. Soc.

[3] Whittaker, E. T. and Watson, G. N., *Modern Analysis* (Cambridge Univ. Press, 1950).

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