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Galois Covers of Moduli of Curves

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Abstract. Moduli spaces of pointed curves with some level structure are studied. We prove that for so-called geometric level structures, the levels encountered in the boundary are smooth if the ambient variety is smooth, and in some cases we can describe them explicitly. The smoothness implies that the moduli space of pointed curves (over any field) admits a smooth finite Galois cover. Finally, we prove that some of these moduli spaces are simply connected.

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1. Level Structures over $\mathcal{M}_{g,n}$

In this first section we give a preliminary exposition of the theory of level structures over $M_{g,n}$, the coarse moduli scheme for smooth curves with n distinct marked points, from a stack (orbifold) theoretical point of view. So let $\mathcal{M}_{g,n}$ be the corresponding stack (orbifold), and $\overline{\mathcal{M}_{g,n}}$ its Knudsen's compactification. In the sequel we will actually switch very often between algebraic (stacks') and analytic (orbifolds') formalism, according to our necessities. We will abuse notation in that the symbols $\mathcal{M}_{g,n}$, $\overline{\mathcal{M}_{g,n}}$, etc., will mostly denote stacks over the field of complex numbers **C**, but sometimes (in Proposition 2.6 notably) they denote stacks over Spec(**Z**).

Given a closed compact oriented surface $S_{g,n}$ with n distinct points removed, the Teichmüller group $\Gamma_{g,n}$ is usually defined as the group of isotopy classes of orientation preserving homeomorphisms which fix the punctures pointwise. We will use the following algebraic characterization. Let $\Pi_{g,n}$ be a group abstractly isomorphic to the fundamental group of $S_{g,n}$, so its standard presentation is

$$\Pi_{g,n} \cong \langle \alpha_{\pm 1}, \ldots, \alpha_{\pm g}, \delta_1, \ldots, \delta_n | \delta_n \cdots \delta_1 [\alpha_1, \alpha_{-1}] \cdots [\alpha_g, \alpha_{-g}] \rangle.$$

Let $A(g,n)$ be the group of automorphisms of $\Pi_{g,n}$ which fix the conjugacy class of every δ_i and induce the identity on $H_2(\Pi_{g,0}, \mathbf{Z})$ (actually this last condition is needed only for the $n = 0$ case). We will refer to its elements as *geometric automorphisms* of $\Pi_{g,n}$. The inner automorphisms $I(g,n)$ of $\Pi_{g,n}$ clearly form a normal subgroup of $A(g,n)$. It is well known (in case $n = 0$ it is a classical result due to Nielsen) that there is a canonical isomorphism

$$\Gamma_{g,n} \cong A(g,n)/I(g,n).$$

In Teichmüller theory a *level structure* λ is a normal subgroup $\Gamma^{\lambda}_{g,n}$ of $\Gamma_{g,n}$ of finite index. The quotient $M^{\lambda}_{g,n} := T_{g,n}/\Gamma^{\lambda}_{g,n}$ is a (finite) Galois cover of $M_{g,n}$, which is algebraic by the generalized Riemann existence theorem. If $\Gamma^{\lambda}_{g,n}$ acts freely on $T_{g,n}$, the level λ is said to be *fine*, since in that case $M^{\lambda}_{g,n}$ represents a moduli functor for pointed curves with some extra structure. The level λ is said to be *geometric* if there exists an invariant subgroup (which means stable by geometric automorphism of the fundamental group) K^{λ} of $\Pi_{g,n}$, such that $\Gamma^{\lambda}_{g,n}$ is the kernel of the natural representation $\Gamma_{g,n} \to \mathrm{Out}(\Pi_{g,n}/K^{\lambda})$.

It can be enlightening to rephrase the above definitions in terms of orbifolds. The fundamental group and orbifold universal cover in the category of orbifolds share properties similar to those in the case of manifolds. In our case we can rephrase part of Teichmüller theory by saying that the universal cover of $\mathcal{M}_{g,n}$ is represented by a smooth analytic space $T_{g,n}$ and that the group of deck transformations for the cover $T_{g,n} \to \mathcal{M}_{g,n}$ equals the group $\Gamma_{g,n} = \pi_1(\mathcal{M}_{g,n}, a)$. Furthermore, $\mathcal{M}_{g,n}$ is the orbifold quotient of $T_{g,n}$ by the action of $\Gamma_{g,n}$ and so it is an analytic orbifold.

In this setting a level structure is just a (finite) étale Galois cover of the smooth analytic orbifold $\mathcal{M}_{g,n}$

$$\mathcal{M}^{\lambda}_{g,n} \xrightarrow{\pi} \mathcal{M}_{g,n}$$

and it is 'fine' when it is represented by an analytic variety $M^{\lambda}_{g,n}$.

We say that the level structure λ_2 dominates the level structure λ_1 (the notation is $\lambda_2 \geqslant \lambda_1$) when $\mathcal{M}^{\lambda_2}_{g,n}$ is an étale cover of $\mathcal{M}^{\lambda_1}_{g,n}$. This is clearly equivalent to $\Gamma^{\lambda_2}_{g,n} \leqslant \Gamma^{\lambda_1}_{g,n}$, therefore the set of level structure with the relation of domination is a lattice equivalent to the lattice of finite index normal subgroups of $\Gamma_{g,n}$.

As we mentioned above, the stack $\mathcal{M}_{g,n}$ has been given a canonical compactification $\overline{\mathcal{M}_{g,n}}$ by Deligne, Mumford and Knudsen. A nice way to restate a classical result of Teichmüller theory is then the following.

PROPOSITION 1.1. *The stack* (*orbifold*) $\overline{\mathcal{M}_{g,n}}$ *is simply connected*.

Proof. We will prove that $\overline{\mathcal{M}_{g,n}}$ has only trivial étale covers. Let $X \xrightarrow{\pi} \overline{\mathcal{M}_{g,n}}$ be an étale cover, then the suborbifold $U = \pi^{-1}(\mathcal{M}_{g,n})$ has $T_{g,n}$ as universal cover. By general theory we know that a local chart for X is completely determined by the monodromy representation

$$\pi_1(B^{3g-3+n} \backslash \Delta) \to \Gamma_{g,n}/\pi_1(U),$$

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where $\mathcal{C} \to B^{3g-3+n}$ is a modular family of n-pointed stable curves, giving a local chart $B^{3g-3+n} \to \overline{\mathcal{M}_{g,n}}$ for the orbifold, and Δ is the locus in B corresponding to singular curves. For every $[C] \in \overline{\mathcal{M}_{g,n}}$ we can take $B \supset [C]$ so small that $[C]$ is the most degenerate curve in the family $C \to B$. The monodromy representation sends a standard generator of $\pi_1(B \backslash \Delta)$, given by a simple loop around the irreducible component of Δ corresponding to a certain singularity on C, to the Dehn twist along the corresponding vanishing cycle on C.

The map π is étale if and only if this representation is trivial for every possible B and in that case every Dehn twist is in $\pi_1(U)$. We have that $\pi_1(U) = \Gamma_{g,n}$ since the Teichmüller group is generated by Dehn twists. So the group $\Gamma_{g,n}/\pi_1(U)$ is trivial and therefore every étale cover of $\overline{\mathcal{M}_{g,n}}$ is trivial.

COROLLARY 1.2. *The scheme $\overline{M_{g,n}}$ is simply connected.*

Proof. The natural map of orbifolds $\overline{\mathcal{M}_{g,n}} \to \overline{M_{g,n}}$, induces a surjection on fundamental groups (indeed it is easy to see that every connected étale cover of $\overline{M_{g,n}}$ can be pulled back to an étale connected cover of $\overline{\mathcal{M}_{g,n}}$). □

Now we come to the definition of the compactified level structures. There is a canonical way to compactify a level structure $\mathcal{M}^{\lambda}_{g,n}$ over $\overline{\mathcal{M}_{g,n}}$; namely we just take the normalization of the proper stack $\overline{\mathcal{M}_{g,n}}$ in the function field of the stack $\mathcal{M}^{\lambda}_{g,n}$. We thus obtain a proper stack $\overline{\mathcal{M}^{\lambda}_{g,n}}$.

We say that the level λ is an Abelian level if it is geometric with $\Pi_{g,n}/K^{\lambda}$ equal to $H_1(S_{g,0}, \mathbf{Z}/l\mathbf{Z})$. We will usually denote an Abelian level by (l) (hence the corresponding normal subgroup by $\Gamma^{(l)}_{g,n}$ and the variety by $M^{(l)}_{g,n}$). If λ dominates an abelian level (l), with $l \geqslant 3$, then both $\mathcal{M}^{\lambda}_{g,n}$ and $\overline{\mathcal{M}^{\lambda}_{g,n}}$ are known to be represented by a scheme (for the latter, see [2]). We will assume that this is the case throughout the whole paper, and we will use the same notation for both stack and variety.

We know that the universal curve $\mathcal{C}_{g,n} \to \overline{\mathcal{M}_{g,n}}$ is isomorphic to $\overline{\mathcal{M}_{g,n+1}}$, and from the universal property of the fiber product it follows easily that the universal curve for the level λ over $\mathcal{M}^{\lambda}_{g,n}$, which is just the pull-back of $\mathcal{C}_{g,n}$, is isomorphic to $\mathcal{M}^{\lambda}_{g,n+1}$. We can actually prove more.

PROPOSITION 1.3. *The pull back on $\overline{\mathcal{M}^{\lambda}_{g,n}}$ of the universal curve over $\overline{\mathcal{M}_{g,n}}$ is canonically isomorphic to $\overline{\mathcal{M}^{\lambda}_{g,n+1}}$.*

Proof. We first prove that the pull back $\mathcal{C}^{\lambda}_{g,n}$ of the stable curve $\overline{\mathcal{M}_{g,n+1}} \to \overline{\mathcal{M}_{g,n}}$ on $\overline{\mathcal{M}^{\lambda}_{g,n}}$ is normal. Over $\mathcal{M}^{\lambda}_{g,n}$, the pull-back of $\overline{\mathcal{M}_{g,n+1}}$ is clearly smooth. The completion of a local ring at a closed point of the boundary is either isomorphic to

$$\hat{\mathcal{O}}_{\overline{\mathcal{M}^{\lambda}_{g,n}},p}[[x]] \quad \text{or to} \quad \hat{\mathcal{O}}_{\overline{\mathcal{M}^{\lambda}_{g,n}},p}[[x,y]]/xy - t^n,$$

where t is a regular element in $\hat{\mathcal{O}}_{\overline{\mathcal{M}^{\lambda}_{g,n}},p}$. In both cases it is normal because $\hat{\mathcal{O}}_{\overline{\mathcal{M}^{\lambda}_{g,n}},p}$ is normal.

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From this description we also see that $\mathcal{C}^{\lambda}_{g,n}$ is smooth in codimension one. Now we want to apply Serre's criterion for normality (see [4], Chapter IV, Section 5.8). We only have to verify that $\mathcal{C}^{\lambda}_{g,n}$ satisfies property S_2 of Serre. By Remark 6.4.3 in Chapter IV of [4], it is enough to verify this property on the completed local rings, in which case it is satisfied because they are normal.

By the universal property of the normalization we can conclude that we have a finite map

$$\overline{\mathcal{M}_{g,n+1}} \times_{\overline{\mathcal{M}_{g,n}}} \overline{\mathcal{M}^{\lambda}_{g,n}} \to \overline{\mathcal{M}^{\lambda}_{g,n+1}}.$$

By a previous remark we know that this map is also birational, hence it is an isomorphism. □

E. Looijenga has proved the existence of level structures over $\overline{\mathcal{M}_g}$ such that their compactification is a smooth variety (see [13]), and other levels with a similar property have been defined by M. Pikaart and A. J. De Jong (see [18]). The following proposition shows that the stable curve one has over such a smooth covering does not yield a smooth covering of $\overline{\mathcal{M}^{\lambda}_{g,1}}$.

PROPOSITION. 1.4. *The stable curve* $\overline{\mathcal{M}^{\lambda}_{g,n+1}}$ *over* $\overline{\mathcal{M}^{\lambda}_{g,n}}$ *is singular for any level* λ *defined on* $\overline{\mathcal{M}_{g,n}}$.

Proof. For every proper subgroup $\Gamma^{\lambda}_{g,n} < \Gamma_{g,n}$ we can find a Dehn twist τ_{γ} which does not belong to $\Gamma^{\lambda}_{g,n}$. We consider a neighbourhood in $\overline{\mathcal{M}^{\lambda}_{g,n}}$ of $[C]$ where C has one singularity such that γ is a vanishing cycle. A local chart U around $[C]$ will be given by the ramified cover

$$\begin{aligned} U &\to B^{3g-3+n}, \\ t &\mapsto t^m. \end{aligned}$$

Here B is the base of the universal deformation of C, and m is the smallest natural number such that $\tau^m_{\gamma} \in \Gamma^{\lambda}_{g,n}$.

The pull-back over U of the local equation $xy = t$ for the singularity in the special fiber over B, is then $xy = t^m$, yielding a singularity for $\overline{\mathcal{M}^{\lambda}_{g,n+1}}$. □

2. The Deligne–Mumford Boundary of Geometric Level Structures

A description of the boundary for the stack of n-pointed stable curves $\overline{\mathcal{M}_{g,n}}$ was given by Knudsen in [11, II, Sec. 3]. In this section we will give an analogous one for the boundary of geometric level structures (i.e. corresponding to geometric subgroups of the Teichmüller group).

Let us recall Knudsen's results. Let $H = \{h_1, h_2, \ldots, h_{n_1}\}$ and $K = \{k_1, k_2, \ldots, k_{n_2}\}$ be complementary subsets of $\{1, 2, \ldots, n\}$ of cardinality n_1 and n_2, respectively. Let g_1 and g_2 be nonnegative integers with $g = g_1 + g_2$ and satisfying the condition that $n_i \geqslant 2$ if $g_i = 0$. There are finite morphisms

$$\beta_0 \colon \overline{\mathcal{M}_{g-1,n+2}} \to \overline{\mathcal{M}_{g,n}}$$

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and

$$\beta_{g_1,g_2,H,K}\colon \overline{\mathcal{M}_{g_1,n_1+1}} \times \overline{\mathcal{M}_{g_2,n_2+1}} \to \overline{\mathcal{M}_{g,n}}.$$

These two maps can be described as follows. If $[C]$ is a point of $\overline{\mathcal{M}_{g-1,n_1+1}}$, then $\beta_0([C])$ is the class of the curve obtained by identifying the labeled points P_{n+1} and P_{n+2} of C to a node. Similarly, if $([C_1],[C_2])$ is a point of $\overline{\mathcal{M}_{g_1,n_1+1}} \times \overline{\mathcal{M}_{g_2,n_2+1}}$, then $\beta_{g_1,g_2,H,K}([C_1],[C_2])$ is the class of the curve obtained from $C_1 \amalg C_2$ by identifying the points $P_{n_1+1} \in C_1$ and $P_{n_2+1} \in C_2$ to a node.

These maps define closed substacks $B^0_{g,n}$ and $B_{g_1,g_2,H,K}$ of $\overline{\mathcal{M}_{g,n}}$, which are irreducible components of the boundary, and all the irreducible components of the boundary can be obtained in this way. In general β_0 and $\beta_{g_1,g_2,H,K}$ are not embeddings. This can be seen as follows. Each irreducible component of the boundary corresponds in a unique way to a certain kind of singularity on a curve of genus g. The fact that on the same curve we can have several singularities of the same type translates into the statement that the corresponding irreducible component of the boundary has self-intersection. Moreover, the map β_0 and, for $n = 0$ and $g_1 = g_2 = g/2$, the map $\beta_{g/2,g/2}$ factorize over the projection to the quotient $\overline{\mathcal{M}_{g-1,n+2}}/S_2$, respectively, $S^2(\overline{\mathcal{M}_{g/2,1}})$. In the first case the symmetric group acts by permuting the two last labeled points on the curve and in the second by permuting the two components of genus $g/2$.

Let us consider more closely the preimages of these boundary divisors in the coverings $\overline{M^\lambda_{g,n}}$. In the cases we are considering we have a Galois morphism $\overline{\mathcal{M}^\lambda_{g,n}} \xrightarrow{\pi} \overline{\mathcal{M}_{g,n}}$ which ramifies only over the Deligne–Mumford boundary of $\overline{\mathcal{M}_{g,n}}$. Purity of branch locus (see [20] Theorem 3.1) tells us that the ramification locus must be a union of irreducible boundary components. This means that the restriction of π to each open stratum of $\overline{M^\lambda_{g,n}}$ is étale. Here we are referring to the natural stratification of the Deligne–Mumford compactification. Thus in the square diagrams

$$\begin{array}{ccc} X_0 & \xrightarrow{\pi'} & \mathcal{M}_{g-1,n+2} \\ \big\downarrow{\scriptstyle \beta_0'} & & \big\downarrow{\scriptstyle \beta_0} \\ \overline{\mathcal{M}^\lambda_{g,n}} & \xrightarrow{\pi} & \overline{\mathcal{M}_{g,n}} \end{array} \qquad \begin{array}{ccc} X_1 & \xrightarrow{\pi'} & \mathcal{M}_{g_1,n_1+1} \times \mathcal{M}_{g_2,n_2+1} \\ \big\downarrow{\scriptstyle \beta'_{g_1,g_2}} & & \big\downarrow{\scriptstyle \beta_{g_1,g_2}} \\ \overline{\mathcal{M}^\lambda_{g,n}} & \xrightarrow{\pi} & \overline{\mathcal{M}_{g,n}} \end{array},$$

where X_0 and X_1 are connected components of the fiber products, the maps π' are Galois étale morphisms.

Differently from β_0 and β_{g_1,g_2}, the morphisms β_0' and β'_{g_1,g_2} are isomorphisms onto their images as soon as the level λ dominates an Abelian level. To show this we have to prove that the action of S_2 on $\beta_0'(X_0)$ and $\beta'_{g/2,g/2}(X_1)$ is not trivial.

This is clear for a point $[C] \in \beta'_{g/2,g/2}(X_1)$, because the action induced by permuting the two genus $g/2$ components of C always moves some nontrivial cycle in the homology of C, thus it cannot leave $[C]$ fixed.

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For $[C] \in \beta_0'(X_0)$, the action, which is induced by permutation of the two distinguished points on the normalization $\tilde{C}$ of C, topologically is given by half a Dehn twist along a simple closed curve β bounding the two distinguished points. Its action on the homology of C is that of reversing the orientation of the cycle supported on the pinched genus 0 surface bounded by β, so it is never trivial if the homology is taken at least with coefficients in $\mathbf{Z}/m\mathbf{Z}$ with $m \geqslant 3$.

We can now prove

PROPOSITION 2.1. *Let* λ *be a level structure over* $M_{g,n}$ *such that its compactification* $\overline{M^{\lambda}_{g,n}}$ *is smooth, then an irreducible component of the boundary does not have self-intersection, hence it is smooth.*

Proof. From the local monodromy description of the smooth Galois cover $\overline{M^{\lambda}_{g,n}} \to \overline{\mathcal{M}_{g,n}}$, we know that the Deligne–Mumford boundary of $\overline{M^{\lambda}_{g,n}}$ is a divisor with normal crossings, hence it suffices to prove that there are no irreducible boundary components with self-intersection.

Suppose there is an irreducible boundary divisor D with self intersection. Take a small neighbourhood U of D and a point x in $U\backslash D$. Then the fibre over x admits (at least) two models $(S, x_1, \ldots, x_n, \gamma_1)$ and $(S, x_1, \ldots, x_n, \gamma_2)$, where $(S, x_1, \ldots, x_n)$ represents the smooth n-pointed curve x, and the γ_i are simple closed nonisotopic curves on $S\backslash x_1, \ldots, x_n$ which we can and will choose disjoint. The hypotheses imply the existence of an element f in $\Gamma^{\lambda}_{g,n}$ such that $f(\gamma_1) = \gamma_2$.

Let us consider first the case in which the γ_i are nonseparating simple closed curves; $\gamma_1 = f(\gamma_2)$ implies $\tau_{\gamma_1} = \tau_{f(\gamma_2)} = f \cdot \tau_{\gamma_2} \cdot f^{-1}$, and hence $\tau_{\gamma_1} \cdot \tau_{\gamma_2}^{-1} \in \Gamma^{\lambda}_{g,n}$. But for nonseparating curves we know that $\tau^{k}_{\gamma_i} \in \Gamma^{\lambda}_{g,n}$ only for $|k|$ at least 2 (see Proposition 1.1). This relation yields a singularity in each point of $\overline{M^{\lambda}_{g,n}}$ corresponding to a singular curve for which γ_1 and γ_2 are vanishing loops.

It remains to rule out the case in which the γ_i are separating loops. Let S_i and S_i' be the connected components of $S\backslash\gamma_i$, such that S_i has least genus. We have that $f(S_1) = S_2$, maybe upon interchanging S_2 and S_2' in case they have equal genus. Now f maps a nonseparating curve in S_1 to a nonseparating curve in S_2, and by the above argument this yields a singularity. □

A consequence of the previous proposition is the following.

PROPOSITION 2.2. *Let* λ *be as in Proposition* 2.1, *then every irreducible boundary component of* $\overline{M^{\lambda}_{g,n}}$, *lying over* $\overline{\mathcal{M}_{g-1,n+2}} \to \overline{\mathcal{M}_{g,n}}$, *is isomorphic to* $\overline{M^{\lambda_0}_{g-1,n+2}}$, *for some level* λ_0 *with smooth compactification.*

Proof. Let X be an irreducible component of the boundary of $\overline{M^{\lambda}_{g,n}}$, lying over $\overline{\mathcal{M}_{g-1,n+2}} \to \overline{\mathcal{M}_{g,n}}$. We know that the dense open stratum X^0 of X is an étale Galois cover of $\mathcal{M}_{g-1,n+2}$ and, hence, by definition X^0 is isomorphic to $M^{\lambda_0}_{g-1,n+2}$, for some level λ_0. From the previous proposition we know that X is a smooth

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compactification of $M^{\lambda_0}_{g-1,n+2}$. Furthermore, it is finite over $\overline{\mathcal{M}_{g-1,n+2}}$. Hence X is isomorphic to the normalization of $\overline{\mathcal{M}_{g-1,n+2}}$ in the function field of $M^{\lambda_0}_{g-1,n+2}$. □

In case the level λ is geometric, we are able to give a simple complete description of the irreducible components of the boundary lying over the divisors in $\overline{\mathcal{M}_{g,n}}$ parametrizing reducible curves.

THEOREM 2.3. *Let λ be a geometric level structure over $\mathcal{M}_{g,n}$ with smooth compactification. Let X, respectively X^0, denote the closed, respectively open, stratum in $\overline{\mathcal{M}_{g,n}}$ lying above $\mathcal{M}_{g_1,n_1+1} \times \mathcal{M}_{g_2,n_2+1}$. Then X^0 is canonically isomorphic to $\mathcal{M}^{\lambda_1}_{g_1,n_1+1} \times \mathcal{M}^{\lambda_2}_{g_2,n_2+1}$, where λ_1 and λ_2 are geometric levels naturally induced by λ. Furthermore, its closure X is smooth and canonically isomorphic to $\overline{\mathcal{M}^{\lambda_1}_{g_1,n_1+1}} \times \overline{\mathcal{M}^{\lambda_2}_{g_2,n_2+1}}$.*

Proof. As in the previous proof, we have that X^0 is an étale Galois cover of $\mathcal{M}_{g_1,n_1+1} \times \mathcal{M}_{g_2,n_2+1}$, hence $\pi_1(X^0, P)$ is a normal subgroup of $\Gamma_{g_1,n_1+1} \times \Gamma_{g_2,n_2+1}$, so we have to prove that if $(a_1, a_2) \in \pi_1(X^0, P)$, then $(a_1, 1)$ and $(1, a_1)$ are already in $\pi_1(X^0, P)$.

By assumption, $\overline{M^{\lambda}_{g,n}}$ is smooth in an analytic neighbourhood of X^0, thus, if α is a loop in X^0 whose class is (a_1, a_2) in $\pi_1(X^0, P)$, we can lift it along the normal line bundle $\mathcal{N}_{X^0/\overline{M^{\lambda}_{g,n}}}$ to a loop $\tilde{\alpha}$ in $M^{\lambda}_{g,n}$.

We choose a λ-Teichmüller structure for the universal curve $\mathcal{C}^{\lambda}_{g,n} \to M^{\lambda}_{g,n}$, i.e. a $\Gamma^{\lambda}_{g,n}$ orbit of Teichmüller markings, and we denote the corresponding marked Riemann surface by $S_{g,n}$.

If γ is a vanishing loop on $S_{g,n}$ corresponding to the specialization to $[C] \in X^0$, we denote the two connected component of $S_{g,n} \backslash \gamma$ by S_1 and S_2 (cf. the proof of Proposition 2.1). Let C_1 and C_2 be the two corresponding irreducible components of C, and choose markings on T_{g_1,n_1+1} and T_{g_2,n_2+1} compatible with this correspondence.

Deformation theory tells us that the complex analytic deformation of the curve C along the normal direction at X^0 is given by the smoothing of the singular point of C, and that it is trivial outside a small neighbourhood of the singularity. On the other hand we can assume the deformation of C along α trivial inside the same small neighbourhood.

Thus the lifting of $\tilde{\alpha}$ to $T_{g,n}$ can be represented in $\Gamma_{g,n}$ as a product $\tilde{a}_1 \cdot \tilde{a}_2$ of two homeomorphisms supported respectively on S_1 and S_2 and trivial inside a small neighbourhood of γ, such that they project to a_1 and a_2 in $\pi_1(X^0)$. We are reduced then to prove that either $\tilde{a}_1$ or $\tilde{a}_2$ (and hence both) project to the identity on $\pi_1(X^0)$, which in turn is equivalent to show that $\tilde{a}_1$ is in the same class as some power of τ_γ in $\Gamma_{g,n}/\Gamma^{\lambda}_{g,n}$.

Here we come to the part of the proof in which we need the fact that our level is geometric. We have chosen $\tilde{a}_1 \cdot \tilde{a}_2$ such that it fixes a neighbourhood of γ in $S_{g,n}$, so let us take p in S_2 and inside such neighbourhood. By hypothesis $\tilde{a}_1 \cdot \tilde{a}_2$ acts

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on $\pi_1(S_{g,n}, p)/I$ like an inner automorphism. Let us take a set of generators for $\pi_1(S_{g,n}, p)$ compatible with the decomposition of $S_{g,n}$ in S_1 and S_2. The support of $\tilde{a}_1$ is on S_1, hence it acts trivially on the generators supported on S_2. On the other hand, modulo inner automorphisms, $\tilde{a}_1$ acts on the generators supported on S_1 like $\tilde{a}_2^{-1}$, which is supported on S_2, and hence it can act on S_1 only like conjugation by γ^k, for some k. This proves that $\tilde{a}_1 \equiv \tau_\gamma^k \bmod \Gamma_{g,n}^{\lambda}$ for some k, which yields $(a_1, 1) \in \pi_1(X^0)$, as it was to prove.

To conclude we have to describe explicitly the levels λ_1 and λ_2 as geometric level structures. Any geometric automorphism of $\pi_1(S_i)$ can be lifted to a geometric automorphism of $\pi_1(S_{g,n})$, thus $I_i := \Pi_{g_i,n_i+1} \cap I$, $i = 1, 2$, is an invariant subgroup of Π_{g_i,n_i+1}, and it is clear that $a \in \Gamma_{g,n}^{\lambda_i}$, $i = 1, 2$, if and only if a acts on $\Pi_{g_i,n_i+1}/I_i$, $i = 1, 2$, like an inner automorphism. Therefore, the levels λ_i, $i = 1, 2$, are the geometric levels defined by I_i, $i = 1, 2$. □

Now we will describe some explicit geometric level structures which yield regular coverings of $\overline{M_{g,n}}$. Let $(S, x_1, \ldots, x_n)$ be a punctured surface of genus g and let $\Pi_{g,n}$ be a group abstractly isomorphic to the fundamental group of $S \backslash \{x_1, \ldots, x_n\}$. So we have a presentation

$$\Pi_{g,n} \cong \langle \alpha_{\pm 1}, \ldots, \alpha_{\pm g}, \delta_1, \ldots, \delta_n | \delta_n \cdots \delta_1 [\alpha_1, \alpha_{-1}] \cdots [\alpha_g, \alpha_{-g}] \rangle.$$

We write Π_g if $n = 0$. Let $\Pi_{g,n}^{[k]}$ be the kth term in the lower central series, i.e.

$$\Pi_{g,n}^{[1]} := \Pi_{g,n}, \qquad \Pi_{g,n}^{[k+1]} := [\Pi_{g,n}, \Pi_{g,n}^{[k]}].$$

We have that $[\Pi_{g,n}^{[k]}, \Pi_{g,n}^{[l]}] \subset \Pi_{g,n}^{[k+l]}$ and this implies that the associated graded Abelian group $\mathrm{Gr}(\Pi_{g,n}) := \oplus_{k \geqslant 1} \Pi_{g,n}^{[k]}/\Pi_{g,n}^{[k+1]}$ carries the natural structure of a Lie algebra. The Lie bracket is induced by the commutator bracket.

Suppose for a moment that $n = 0$ and write L^k for the quotient $\Pi_g^{[k]}/\Pi_g^{[k+1]}$. Notice that L^1 is nothing else than $H_1(S, \mathbf{Z})$. It is proven in [12, Thm.] that the Lie algebra $L^\bullet$ is freely generated over $\mathbf{Z}$ by L^1 with the unique relation $\sum_{i=1}^g [a_i, a_{-i}] = 0$, where $a_{\pm i}$ is the image of $\alpha_{\pm i}$ in L^1. Write ω for the intersection form $\sum_{i=1}^g a_i \wedge a_{-i}$, then there are exact sequences describing L^2 and L^3

$$0 \to \mathbf{Z} \to L^1 \wedge L^1 \xrightarrow{[\,,\,]} L^2 \to 0,$$

where 1 is mapped to the intersection form $\omega \in L^1 \wedge L^1$, and

$$0 \to \wedge^3 L^1 \xrightarrow{i} L^1 \otimes L^2 \xrightarrow{[\,,\,]} L^3 \to 0,$$

where $i(a \wedge b \wedge c) = a \otimes [b, c] + b \otimes [c, a] + c \otimes [a, b]$.

In case $n \geqslant 1$ we need a finer filtration if we insist on the property that the associated graded be determined by the first homology of the curve. First we introduce

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the weight filtration, defined as follows. Consider the surjective homomorphism $\Pi_{g,n} \to \Pi_g$ obtained by filling in the punctures; write N for its kernel. Define

$$\begin{aligned} W^1\Pi_{g,n} &:= \Pi_{g,n}, \\ W^2\Pi_{g,n} &:= N \cdot \Pi_{g,n}^{[2]}, \\ W^{k+1}\Pi_{g,n} &:= [\Pi_{g,n}, W^k\Pi_{g,n}] \cdot [N, W^{k-1}\Pi_{g,n}]. \end{aligned}$$

In terms of a standard presentation as above

$$\Pi_{g,n} \cong \langle \alpha_{\pm 1}, \ldots \alpha_{\pm g}, \delta_1, \ldots, \delta_n | \delta_n \cdots \delta_1 [\alpha_1, \alpha_{-1}] \cdots [\alpha_g, \alpha_{-g}] \rangle,$$

we have assigned weight 1 to the $\alpha_{\pm i}$ and weight 2 to the δ_j. We clearly have $W^k\Pi_{g,n} \supset \Pi_{g,n}^{[k]} \supset W^{2k-1}\Pi_{g,n}$. Put $V^{n(k)}\Pi_{g,n} := \Pi_{g,n}^{[k]}$, where $n(k) := 1 + 2 + \cdots + k = k(k+1)/2$ and put $V^{n(k)+l}\Pi_{g,n} := (\Pi_{g,n}^{[k]} \cap W^{k+l}\Pi_{g,n}) \cdot \Pi_{g,n}^{[k+1]}$ for $l \in \{0, \ldots, k\}$. Thus $V^k\Pi_{g,n}$ is the lift of the induced weight filtration on the quotients $\Pi_{g,n}^{[k]}/\Pi_{g,n}^{[k+1]}$; notice that there are $k+1$ graded sub-quotients.

Notice that $V^{n(k)+l}\Pi_{g,n} = V^{n(k+1)}\Pi_{g,n}$ if $n = 1$ and $1 \leqslant l \leqslant k$. We write $M^\bullet$ for the associated graded of the filtration $V^*\Pi_{g,n}$. For the convenience of the reader, we list the first few terms of the filtration V^*.

$$\begin{array}{ccccccccccccc} \Pi_{g,n} & \supset & W^2\Pi_{g,n} & \supset & \Pi_{g,n}^{[2]} & \supset & W^3\Pi_{g,n} & \supset & W^4\Pi_{g,n} \cdot \Pi_{g,n}^{[3]} & \supset & \Pi_{g,n}^{[3]} & \supset & W^4\Pi_{g,n} \\ \| & & \| & & \| & & \| & & \| & & \| & & \| \\ V^1 & \supset & V^2\Pi_{g,n} & \supset & V^3\Pi_{g,n} & \supset & V^4\Pi_{g,n} & \supset & V^5\Pi_{g,n} & \supset & V^6\Pi_{g,n} & \supset & V^7\Pi_{g,n}. \end{array}$$

As in the case without punctures, we have that $[V^k\Pi_{g,n}, V^l\Pi_{g,n}] \subset V^{k+l}\Pi_{g,n}$, so that $M^\bullet$ is again a Lie algebra. It is generated over $\mathbf{Z}$ by $a_{\pm i}$ for $i = 1$ to g in degree 1 and d_j for $j = 1$ to n in degree 2 (where the d_j are the images of the δ_j in M^2) with the unique relation

$$\sum_{i=1}^{g} [a_i, a_{-i}] + \sum_{j=1}^{n} d_j,$$

see [12, Thm.]. Filling in the punctures provides us with canonical isomorphisms $M^1 \cong L^1$ and $M^2 \cong \mathrm{Ker}(H_1(C \backslash \{x_1, \ldots, x_n\}, \mathbf{Z}) \to H_1(C, \mathbf{Z}))$. Notice that M^1 and M^2 induce on $H_1(C \backslash \{x_1, \ldots, x_n\}, \mathbf{Z})$ its weight filtration, which implies that also in this case the associated graded is determined by the homology provided it is equipped with its weight filtration. We have the following identifications, which are easily proved using the rank formulas from [10, Prop. 1] and the obvious surjections from the right-hand side onto the left-hand side

$$\begin{aligned} M^1 &\cong L^1 \cong H_1(C, \mathbf{Z}), \\ M^2 &\cong \mathrm{Ker}(H_1(C \backslash \{x_1, \ldots, x_n\}, \mathbf{Z}) \to L^1), \end{aligned}$$

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$$M^3 \cong L^1 \wedge L^1,$$
$$M^4 \cong L^1 \otimes M^2,$$
$$M^5 \cong M^2 \wedge M^2,$$
$$M^6 \cong (L^1 \otimes M^3)/i(\wedge^3 L^1),$$

Generators for M^3, M^4, M^5 and M^6 are, respectively, $[a_{\pm i}, a_{\pm j}]$, $[a_{\pm i}, d_j]$, $[d_i, d_j]$ and $[a_{\pm i}, [a_{\pm j}, a_{\pm k}]]$.

In order to obtain a filtration with finite quotients, we set $V^{k,l}\Pi_{g,n} := V^k\Pi_{g,n} \cdot \Pi^l_{g,n}$, where $\Pi^l_{g,n}$ means the subgroup of $\Pi_{g,n}$ generated by all lth powers. We write $M^{k,l}$ for the quotient $V^{k,l}\Pi_{g,n}/V^{k+1,l}\Pi_{g,n}$ and l_d for $l/\gcd(l, d)$. [18, Lem. 6.3] implies immediately the following.

LEMMA 2.4. *Notations as above. Then we have*

$$M^{1,l} \cong L^1/lL^1 \cong H_1(C, \mathbf{Z}/l\mathbf{Z}),$$
$$M^{2,l} \cong M^2/lM^2,$$
$$M^{i,l} \cong M^i/l_2M^i \quad \text{for } i \in \{3, 4, 5\},$$
$$M^{6,l} \cong M^6/A,$$

where the sublattice A is given as follows. The quotient M^6 is generated by elements $[x, [y, z]]$, where x, y and z can be taken from the set $\{a_{\pm i} | i = 1, \dots, g\}$. In fact, we obtain a basis if for every pair of distinct elements (x, y) we only take, say, $[x, [x, y]]$ and $[y, [x, y]]$ and for every triple (x, y, z) of distinct elements, we only take, say, $[x, [y, z]]$ and $[y, [z, x]]$. (This is the Jacobi relation.) The submodule A is generated by elements of the form $l_6[x, [x, y]]$, $l_6[y, [x, y]]$ and $n[x, [y, z]] + m[y, [z, x]]$ such that $l_6|n, m$ and $l_2|n + m$.

Finally, we note that whether an element of $\Pi_{g,n}$ is in $V^{k,l}\Pi_{g,n}$ for $k \leqslant 7$ can be read off from the associated graded.

Another reason for which we need the weight filtration is that the lower central series is not *strict*. This can be seen as follows. Let e be a simple closed loop dividing S into the connected components S_1 and S_2 such that S_1 has genus h and carries m of the n marked points. The inclusion $S_1 \hookrightarrow S$ induces an inclusion of fundamental groups

$$\begin{array}{l} \Pi_{h,m} \cong \langle \{\alpha_{\pm i}\}_{i=1}^h, \{\delta_j\}_{j=1}^m \varepsilon | \prod_{j=1}^m \delta_j \prod_{i=1}^h [\alpha_i, \alpha_{-i}]\varepsilon \rangle \\ \downarrow \\ \Pi_{g,n} \cong \langle \{\alpha_{\pm i}\}_{i=1}^g, \{\delta_j\}_{j=1}^n | \prod_{j=1}^n \delta_j \prod_{i=1}^g [\alpha_i, \alpha_{-i}] \rangle, \end{array}$$

which is not strict, i.e. it is not true in general that $\Pi_{h,m} \cap \Pi_{g,n}^{[k]} = \Pi_{h,m}^{[k]}$. Consider for example the case $n = m > 0$, then ε is contained in $\Pi_{g,n}^{[2]}$ but not in $\Pi_{h,m}^{[2]}$ (as one easily sees, this is the only 'obstruction').

We want to prove that the first steps of the filtration are strict in some cases.

LEMMA 2.5. *Notations as above. Let l be an integer at least* 3 *and let k be an element of* $\{2, \ldots, 7\}$. *Then we have*

$$\Pi_{h,m} \cap V^{2,l}\Pi_{g,n} = V^{2,l}\Pi_{h,m},$$

$$\Pi_{h,m} \cap V^{4,l}\Pi_{g,n} = V^{4,l}\Pi_{h,m} \text{ if and only if } l \text{ odd or } m = 0 \text{ or } 0 < m < n.$$

$$k \neq 2, 4: \quad \Pi_{h,m} \cap V^{k,l}\Pi_{g,n} = V^{k,l}\Pi_{h,m}, \text{ if and only if } m = 0 \text{ or } 0 < m < n,$$

Proof. The inclusions '$\supset$' are trivial. The first equality follows directly from the observation

$$\Pi_{h,m}/W^{2,l}\Pi_{h,m} \cong H_1(S_1, \mathbf{Z}/l\mathbf{Z}) \hookrightarrow H_1(S, \mathbf{Z}/l\mathbf{Z}) \cong \Pi_{g,n}/W^{2,l}\Pi_{g,n},$$

whereas equality in case $k = 3$ follows from the observation that the natural map

$$H_1(S_1 \backslash \{x_1, \ldots, x_m\}, \mathbf{Z}/l\mathbf{Z}) \to H_1(S \backslash \{x_1, \ldots, x_n\}, \mathbf{Z}/l\mathbf{Z})$$

is injective if and only if either $m = 0$ or $0 < m < n$.

For the other equalities, notice that Lemma 2.4 applies to both S and S_1, so that we only have to consider elements involving ε. These are the 'only' elements whose behaviour with respect to the filtration V^k depends on the integers m and n.

First suppose $m = 0$. Then $\varepsilon \in V^3\Pi_{g,n}$ and $\varepsilon^d \in V^{i,l}\Pi_{g,n}$ if and only if $l_2 \mid d$, for $i \in \{4, 5, 6, 7\}$. The same holds for $\Pi_{h,m}$ instead of $\Pi_{g,n}$.

Next suppose $0 < m < n$. Then ε is contained in $V^2\Pi_{g,n}$ but not in $V^3\Pi_{g,n}$ and $\varepsilon^d \in V^{i,l}\Pi_{g,n}$ if and only if $l|d$, for $i \in \{4, 5, 6, 7\}$. The same holds for $\Pi_{h,m}$ instead of $\Pi_{g,n}$.

Finally suppose $0 < m = n$. Then ε is contained in $V^3\Pi_{g,n}$ and $V^2\Pi_{h,m}$ but not in $V^3\Pi_{h,m}$ and we use the lines above.

This finishes the case $k = 4$; for the cases $k = 5, 6$ and $k = 7$ we still need to consider the elements $[\alpha_{\pm i}, \varepsilon]$ and $[\delta_j, \varepsilon]$.

If $m = 0$, then $[\alpha_i, \varepsilon] \in V^6\Pi_{g,n}$ and if $0 < m < n$, then $[\alpha_i, \varepsilon]$ is contained in $V^4\Pi_{g,n}$ but not in $V^5\Pi_{g,n}$. The same holds for $\Pi_{h,m}$ instead of $\Pi_{g,n}$. If $0 < m = n$ then $[\alpha_i, \varepsilon]$ is contained in $V^6\Pi_{g,n}$ and $V^4\Pi_{h,m}$ but not in $V^5\Pi_{h,m}$. It follows that in case $k = 5$ or 6 and $0 < m < n$, we have $V^{k,l}\Pi_{g,n} \cap \Pi_{h,m} \neq V^{k,l}\Pi_{h,m}$, since $[\alpha_i, \varepsilon]$ is contained in the left-hand side but not in the right-hand side.

Finally we consider the elements $[\delta_j, \varepsilon]$. If $m = 0$, then $[\delta_j, \varepsilon]$ is not contained in $\Pi_{h,m}$ at all. If $0 < m < n$, then $[\delta_j, \varepsilon]$ is contained in $V^5\Pi_{g,n}$ and in $V^5\Pi_{h,m}$. If $0 < m = n$ then $[\delta_j, \varepsilon]$ is contained in $V^7\Pi_{g,n}$ and $V^5\Pi_{h,m}$ but not in $V^6\Pi_{h,m}$ and we conclude as above. This finishes the cases $k = 5, 6$ or 7. □

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Let $\Gamma_{g,n}$ be the Teichmüller group of $(S, x_1, \ldots, x_n)$, as defined in the introduction. We have the injective homomorphism

$$\Gamma_{g,n} \to \mathrm{Out}^+(\Pi_{g,n}),$$

whose image can be described as the subgroup of $\mathrm{Out}^+(\Pi_{g,n})$ which sends every δ_j to a conjugate of δ_j, and $\mathrm{Out}^+(\Pi_{g,n})$ is the index 2 subgroup of $\mathrm{Out}(\Pi_{g,n})$ inducing the identity on $H_2(S)$. It is clear that the subgroups $V^k\Pi_{g,n}$ are geometric (but not characteristic if $n \geqslant 2$). The corresponding geometric subgroup of $\Gamma_{g,n}$ will be denoted by

$$\Gamma_{g,n}^{k,l} := \mathrm{Ker}(\Gamma_{g,n} \to \mathrm{Out}(\Pi_{g,n}/V^{k,l}\Pi_{g,n})).$$

If $n = 0$, we drop n from the notation. Denote by $\mathcal{M}_{g,n}^{k,l}$ the moduli stack of curves with the geometric level structure defined by $G = \Pi_{g,n}/V^{k,l}\Pi_{g,n}$. Proceeding as in [3] or in [18], it is not difficult to see that it is actually defined over $\mathrm{Spec}(\mathbf{Z}[1/\#G])$. It is representable over an algebraic closed field if $k \geqslant 2$ and $l \geqslant 3$. Let us observe that the levels $\Gamma_{g,n}^{2,l}$ are just the Abelian levels $\Gamma_{g,n}^{(l)}$, defined in Section 1.

From Theorem 2.1 and the previous lemma it follows.

PROPOSITION 2.6. *If and $l \geqslant 3$, then the structural morphism $\overline{\mathcal{M}_{g,n}^{k,l}} \to \mathrm{Spec}(\mathbf{Z}[1/l])$ is smooth if*

- $k = 3, 5, 6$ *or* 7, l *is odd and* $n = 1$;
- $k = 4$ *and* l *is odd*;
- $g = 2$.

Furthermore the geometric levels which arise in the boundary components of reducible curves are of the same type.

Proof. To prove that the complex algebraic variety $\overline{M_{g,n}^{k,l}}$ is smooth, we have only to apply Lemma 2.5 to the smooth geometric levels on $\overline{\mathcal{M}_g}$ defined in [18], and then successively to the ones generated in the boundary of them. By [18, Prop. 2.3.6], we can then conclude that the same statement holds for the corresponding stack over $\mathrm{Spec}(\mathbf{Z}[1/l])$. □

Remark 2.7. In case $k = 3, 5, 6$ or 7 we need the restriction to the case $n \leqslant 1$ since the induction, mentioned in the above proof, does not work. Namely, one starts with $n = 0$, this induces $m = 0$. Then we have proven the case $n = 1$. In the next step m can be either 0 or 1. In the first case we get nothing new, in the second case we have $m = n$ so our argument does not apply (cf. Lemma 2.5).

This in particular extends the theorem of Looijenga on the existence of smooth Galois covers for $\mathcal{M}_g$ (see [13]) to the n-pointed case.

An explicit description of the monodromy along the boundary of $\overline{M_{g,n}^{k,l}}$, in case it is smooth, can be deduced from the one given for $\overline{M_g^{k,l}}$ in [18, Thm. 3.1.3].

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To fix notations, let us recall how the monodromy representation is defined. Let $(C, x_1, \dots, x_n)$ be a complex stable n-pointed curve of genus g with singular points $P_1, \dots, P_s$. Let $\Gamma = \Gamma(C)$ be its dual graph; an edge for each point P_j, a vertex for an irreducible component of C. Let $\pi: (\mathcal{C}, C) \to (B, 0)$ be a local universal deformation of C, where $B \subset \mathbf{C}^{3g-3+n}$ is a polydisc neighbourhood of 0. The coordinates z_i are chosen such that $z_j = 0$, $1 \leqslant j \leqslant s$ parametrizes curves where the singular point P_j subsists. The discriminant locus $\Delta \subset B$ of π is thus given by $z_1 \dots z_s = 0$. Put $U = B\backslash\Delta$, let $x \in U$ and choose $y \in \mathcal{C}_x = \pi^{-1}(x)$. The fundamental group of U is an Abelian group, freely generated by simple loops around the divisors $z_j = 0$, thus naturally isomorphic to the free Abelian group on the edges of Γ, i.e., $\pi_1(U, x) \cong \bigoplus_{e\in\mathrm{Edges}(\Gamma)} Ze$. This provides us with the monodromy representation

$$\rho: \pi_1(U, x) \to \mathrm{Out}(\pi_1(\mathcal{C}_x\backslash\{x_1, \dots, x_n\}, y)).$$

The points P_j determine nontrivial distinct isotopy classes of circles on $\mathcal{C}_x\backslash x_1, \dots, x_n$, which have pairwise disjoint representatives c_j.

In particular we get an induced representation in the automorphism group, modulo inner automorphisms, of the quotient of the fundamental group by the invariant subgroup defining the level. As we saw in Section 2 this representation is equivalent to the explicit description of a small neighbourhood of a point $P \in \overline{\mathcal{M}^{\lambda}_{g,n}}$ as a Galois cover of a neighbourhood of the point $[C] \in \overline{\mathcal{M}_{g,n}}$. More precisely $(\rho^{-1}(\Gamma^{\lambda}_{g,n})) = l_1{\cdot}\mathbf{Z}\oplus l_2{\cdot}\mathbf{Z}\oplus \cdots \oplus l_s{\cdot}\mathbf{Z}$ if and only if the Galois cover $\overline{\mathcal{M}^{\lambda}_{g,n}} \to \overline{\mathcal{M}_{g,n}}$ is locally equivalent, in the aforementioned neighbourhoods, to the cover of polydiscs

$$\begin{aligned} &B^{3g-3+n} \to B^{3g-3+n},\\ &(z_1, \dots, z_{3g-3+n}) \mapsto (z_1^{l_1}, \dots, z_s^{l_s}, z_{s+1}, \dots, z_{3g-3+n}). \end{aligned}$$

We want to obtain the coefficients of the monodromy at a point $P' \in \overline{\mathcal{M}^{\lambda_1}_{g',n+1}}$ from those at $P \in \overline{\mathcal{M}^{\lambda}_{g,n}}$.

In order to do this we assume for simplicity that C is union of two components C_1 and C_2, with C_1 of genus g', C_2 smooth of genus g'', and $g' + g'' = g$. The point P' will then be lying over $[C_1] \in \overline{\mathcal{M}_{g',n+1}}$. We assume furthermore that $z_1 = 0$ is a local equation for the divisor $\overline{\mathcal{M}_{g',n+1}} \times \overline{\mathcal{M}_{g'',1}} \subset \overline{\mathcal{M}_{g,n}}$, and that the projection $\overline{\mathcal{M}_{g',n+1}} \times \overline{\mathcal{M}_{g'',1}} \to \overline{\mathcal{M}_{g',n+1}}$ is given in our local coordinates by $(0, z_2, \dots, z_{3g-3+n}) \mapsto (z_2, \dots, z_{3g'-3+n+2})$.

With these assumptions the Galois cover $\overline{\mathcal{M}^{\lambda_1}_{g',n+1}} \to \overline{\mathcal{M}_{g',n+1}}$ will be equivalent, near the points P' and $[C_1]$, to the cover of polydiscs

$$\begin{aligned} &B^{3g'-3+n+1} \to B^{3g'-3+n+1},\\ &(z_2, \dots, z_{3g'-3+n+2}) \mapsto (z_2^{l_2}, \dots, z_s^{l_s}, \dots, z_{3g'-3+n+2}), \end{aligned}$$

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hence the kernel of the corresponding monodromy representation is

$$\operatorname{Ker} \rho^{\lambda_1}_{g',n+1} = l_2 \cdot \mathbf{Z} \oplus \cdots \oplus l_s \cdot \mathbf{Z}.$$

In particular it is clear now how to deduce the monodromy along the boundary for $\overline{M^{k,l}_{g,n}}$ from the description of that of $\overline{M^{[k],l}_g}$ given in Theorem 3.1.3 of [18]*, using that the lower central series coincides with the weight filtration in the non-pointed case. Let us keep the notations introduced at the beginning and let us denote moreover by $E(\Gamma)$ (respectively $B(\Gamma)$) the set of all edges (respectively those corresponding to separating bounding simple closed curves) and by $B^{un}(\Gamma)$ (respectively B^{un}_1) those elements of $B(\Gamma)$ which are unmarked (respectively unmarked and of genus 1). As in [18] let us define, for $n, l \in \mathbf{Z}$, $n_l = n/\gcd(l, n)$. We have then

PROPOSITION 2.8. *The kernel of the local monodromy representation for* $\overline{M^{k,l}_{g,n}}$ *over a neighbourhood of* $[C] \in \overline{\mathcal{M}_{g,n}}$ *is*

If $k = 4$ *and* l *is odd* :
$$\rho^{-1}(\Gamma^{k,l}_{g,n}) = \bigoplus_{e \in E(\Gamma)} l \cdot \mathbf{Z}e \oplus \bigoplus_{e \in B(\Gamma)} \mathbf{Z}e.$$
If $k = 5$ *or* 6, l *is odd and* $n = 1$:
$$\rho^{-1}(\Gamma^{k,l}_{g,n}) = \bigoplus_{e \in E(\Gamma)} l \cdot \mathbf{Z}e \oplus \bigoplus_{e \in B^{un}(\Gamma)} \mathbf{Z}e.$$
If $k = 7$, 1 *is odd and* $n = 1$:
$$\rho^{-1}(\Gamma^{k,l}_{g,n}) = \bigoplus_{e \in E(\Gamma)} l \cdot \mathbf{Z}e \oplus \bigoplus_{e \in B^{un}_1(\Gamma)} l_3\mathbf{Z}e.$$

Remark 2.9. In the cases $k = 5$ or 6 and l is odd, or in case k is at least 7 and l is odd or divisible by 4, one can prove that $\overline{M^{k,l}_{g,n}}$ is nonsingular (see [19, Thm. 3.3.3]).

An easy consequence of the previous proposition (combined with the remark) is the following.

COROLLARY 2.10. *For every finite cover* X *of* $\overline{M_{g,n}}(\mathbf{C})$ *which is étale* (*in the orbifold sense*) *over* $M_{g,n}(\mathbf{C})$ *there exists a finite smooth Galois cover of* $\overline{M_{g,n}}(\mathbf{C})$ *dominating* X.

Proof. Let H be the subgroup of $\Gamma_{g,n}$ such that dividing out Teichmüller space by H we obtain $X \times_{\overline{M_{g,n}}} M_{g,n}$. By taking the intersection of all normal subgroups containing H we may suppose that H is normal (and still of finite index).

Let $(S_g, x_1, \ldots, x_n)$ be an oriented closed n-pointed surface of genus g and let D_0 be a Dehn twist around a nonseparating simple closed curve and $D_{i,m}$, for $i =$

* [18, Sec. 5.1] contains a minor error: it is falsely claimed that there always exists a loop α as described. However we can choose α intersecting a minimal number (but at least one) of the edges involved in σ, show that the edges of σ it hits are linearly independent and span a primitive submodule of $H_1(S)$ and proceed as in [18, Sec. 5.1].

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$1, \dots, [g/2]$ and $m = 0, \dots, n$ be a Dehn twist around a simple closed curve which separates S_g into two submanifolds of genus i and $g - i$ carrying m, respectively, $n - m$ of the marked points. Let l_0, respectively, $l_{i,m}$ be the minimal positive integer such that H contains $D_0^{l_0}$, respectively, $D_{i,m}^{l_{i,m}}$ and set l equal to 12 times the lowest common multiple of l_0 and all $l_{i,m}$. Let H_l be the intersection of H with $\Gamma_{g,n}^{7,l}$, this is again a normal subgroup of finite index contained in H.

We claim that, for any integer m and any Dehn twist τ_γ, the group H_l contains τ_γ^m, if and only if $\Gamma_{g,n}^{7,l}$ contains τ_γ^m. The 'only if' part is trivial since H_l is contained in $\Gamma_{g,n}^{7,l}$. Let us prove the other implication. Proposition 3.5 tells us that $\Gamma_{g,n}^{7,l}$ contains τ_γ^m only if either γ bounds an unmarked genus one surface and $l/6$ divides m or γ is a separating simple curve and $l/2$ divides m or we are not in one of the above two cases and l divides m. In all these cases, we see that τ_γ^m is contained in H_l as well.

Define X_l to be the normalization of $\overline{M_{g,n}}$ in the function field of the quotient of Teichmüller space by H_l, thus it dominates both X and $\overline{M_g^{7,l}}$. It follows from the local monodromy description as explained in the paragraphs preceding Proposition 3.5 and from what we said above that all ramification indices along all irreducible components of the boundary divisor of X_l coincide with those of $\overline{M_g^{7,l}}$. Thus the covering $X_l \to \overline{M_g^{7,l}}$ is not only étale over the locus parametrizing smooth curves but even generically étale over the boundary. Furthermore, X_l is normal by definition and $\overline{M_g^{7,l}}$ is smooth (by Proposition 3.4, since l is divisible by 4), so purity of branch locus (see [20, Thm. 3.1]) applies and tells us that this cover is actually etale. Thus X_l, being an étale cover of a smooth variety, is smooth. □

3. Simple Connectivity of Some Covers

Fix an oriented closed compact reference surface $S_g = S_{g,0}$ and write $S_{g,1}$ for S_g left out one point. The inclusion $S_{g,1} \hookrightarrow S_g$ induces an isomorphism on homology $H_1(S_{g,1}, \mathbf{Z}) \cong H_1(S_g, \mathbf{Z})$, so we will write H for both of them. We will denote by D_α the (say right-handed) Dehn twist around a simple closed curve α. We will call a Dehn twist *separating* if α is a separating curve (so if we gave α an orientation, its homology class would be what was previously called a bridge). A *bounding pair map* is a homeomorphism of S_g of the form $D_\alpha D_\beta^{-1}$, where α and β are disjoint homologous simple closed curves not in the same isotopy class (their oriented homology classes form a cut pair). Let K_g resp. $K_g(l)$ be the subgroup of the mapping class group Γ_g generated by separating Dehn twists, resp., by these and by lth powers of all Dehn twists. Let $\mathrm{Tor}_{g,i}$ be the Torelli group for i is 0 or 1, which, by Johnson's work (see [7]), is known to be generated by bounding pair maps. It clearly contains separating Dehn twists.

PROPOSITION 3.1. *If $g \geqslant 3, n \geqslant 0$ the groups $\Gamma_{g,n}^{(l)}$ are generated by lth powers of Dehn twists around not separating closed curves and by bounding pair maps.*

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Proof. Consider the following diagram, where i is either 0 or 1.

$$
\begin{array}{ccccccccc}
 & & & & \Gamma^{(l)}_{g,i} & \longrightarrow & \mathrm{Ker}(r_l) & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Tor}_{g,i} & \longrightarrow & \Gamma_{g,i} & \longrightarrow & \mathrm{Sp}(H) & \longrightarrow & 1 \\
 & & & & & & \downarrow{\scriptstyle r_l} & & \\
 & & & & & & \mathrm{Sp}(H_1(S_{g,0}, \mathbf{Z}/l\mathbf{Z})) & &
\end{array}
$$

By a result of Mennicke (see [15, 10 Satz]), the kernel of the morphism induced by reduction modulo l, $\mathrm{Ker}(r_l\colon \mathrm{Aut}(H) \to \mathrm{Aut}(H_1(S_g, \mathbf{Z}/l\mathbf{Z})))$, is generated by lth powers of symplectic transvections, i.e. by images of lth powers of Dehn twists. Combined with Johnson's result, this implies that $\Gamma^{(l)}_{g,i}$ is generated by all bounding pair maps and by all lth powers of Dehn twists. To extend the result to all n let us just observe that the kernel of the natural map $\Gamma^{(l)}_{g,n+1} \to \Gamma^{(l)}_{g,n}$ is equal to that of $\Gamma_{g,n+1} \to \Gamma_{g,n}$, thus it is spanned by bounding pair maps. □

PROPOSITION 3.2. *The group $\Gamma^{(l)}_2$ equals $\Gamma^{6,l}_2$ and is generated by* lth *powers of all Dehn twists and by separating Dehn twists. For $n \geqslant 1$ the groups $\Gamma^{3,l}_{2,n}$ are generated by* lth *powers of Dehn twists, separating Dehn twists and bounding pair maps*.

Proof. Clearly we have $K_2(l) \subset \Gamma^{6,l}_2 \subset \Gamma^{(l)}$. Birman proved that the Torelli group in genus 2 is normally generated by one separating Dehn twist, see [2, Thm. 2]. Arguing as in the proof of the previous proposition, we obtain the desired results. □

PROPOSITION 3.3. *For $g \geqslant 2, l \geqslant 3, n \geqslant 0$, and k an algebraically closed field of characteristic not dividing l, the moduli spaces $\overline{M^{(l)}_{g,n}}(k)$ are simply connected.*

Proof. Clearly it is enough to prove the proposition over the complex numbers. As we saw in Section 1, $\Gamma^{(l)}_{g,n}$ is the fundamental group of $M^{(l)}_{g,n}$. Using the local monodromy description, we can interpret the statements of Proposition 3.1 and 3.2 as saying that this fundamental group is generated by 'small' loops around the divisor at infinity of the compactified variety. Indeed lth powers of Dehn twists around not separating closed curves and Dehn twists around separating curves correspond respectively to simple loops around branches of the divisor of singular irreducible or singular reducible curves in $\overline{M^{(l)}_{g,n}}$, while bounding pair maps correspond to simple loops around the loci where two branches, belonging to the same irreducible component of the divisor of singular irreducible curves, meet.

This easily implies that these compactifications are simply connected. □

PROPOSITION 3.4. *If $g \geqslant 3$ and l odd, the groups $\Gamma^{6,l}_g$ and $\Gamma^{6,l}_{g,1}$ are generated by lth powers of all Dehn twists and by separating Dehn twists*.

Proof. We have to prove that $K_g(l) = \Gamma_g^{6,l}$. The inclusion $K_g(l) \subset \Gamma_g^{6,l}$ follows from [18, Sec. 4.1]. Thus we have $\Gamma_g^{(l)} \supset \Gamma_g^{6,l} \cdot \mathrm{Tor}_g \supset K_g(l) \cdot \mathrm{Tor}_g$. So by Proposition 3.1, we know that $\Gamma_g^{6,l} \cdot \mathrm{Tor}_g$ equals $K_g(l) \cdot \mathrm{Tor}_g$. Thus it suffices to prove that $\Gamma_g^{6,l} \cap \mathrm{Tor}_g$ equals $K_g(l) \cap \mathrm{Tor}_g$, because $\Gamma_g^{6,l}/K_g(l)$ contains $(\Gamma_g^{6,l} \cap \mathrm{Tor}_g)/(K_g(l) \cap \mathrm{Tor}_g)$ as a normal subgroup with quotient $(\Gamma_g^{6,l} \cdot \mathrm{Tor}_g)/(K_g(l) \cdot \mathrm{Tor}_g)$.

To prove that $\Gamma_g^{6,l} \cap \mathrm{Tor}_g \subset K_g(l) \cap \mathrm{Tor}_g$, first note that both sides contain K_g, so again using the trivial inclusion we only have to prove that $(\Gamma_g^{6,l} \cap \mathrm{Tor}_g)/K_g$ equals $(K_g(l) \cap \mathrm{Tor}_g)/K_g$. Note that what we have said so far carries over word for word to the pointed case, to which we switch for a moment.

We will make use of Johnson's results. In [8, Thm. 1 and Sec. 6], he constructs surjective homomorphisms

$$\tau_{g,1} \colon \mathrm{Tor}_{g,1} \to \bigwedge^3 H.$$

Furthermore, in [9, Thm. 6] it is proved that $K_{g,1}$ equals the kernel of $\tau_{g,1}$. Roughly, $\tau_{g,1}$ is obtained as follows: if $\psi = D_\alpha D_\beta^{-1}$ is a bounding pair map, then $\psi(x)x^{-1} \in \pi_1(S_{g,1})^{[2]}$ for any $x \in \pi_1(S_{g,1})$. Consider the element $\psi([\alpha])[\alpha]^{-1}$ modulo $\pi_1(S_{g,1})^{[3]}$. This yields $\mathrm{Tor}_{g,1} \to H \otimes \pi_1(S_{g,1})^{[2]}/\pi_1(S_{g,1})^{[3]} \cong H \otimes \wedge^2 H$. In [8, Sect. 4], Johnson shows that the image of $\tau_{g,1}$ is contained inside the submodule $\wedge^3 H$ of $H \otimes \wedge^2 H$ and he gives an explicit formula for $\tau_{g,1}(\psi)$.

We claim that there exist $\binom{2g}{3}$ bounding pair maps ϕ_i such that their images under $\tau_{g,1}$ generate $\wedge^3 H$. Actually, this is precisely what is stated in the first paragraph of the proof of [8, Thm. 1], nl. a genus one bounding pair map is mapped to a generator of a unimodular sublattice and the map $\tau_{g,1} \colon \mathrm{Tor}_{g,1} \to \wedge^3 H(S_{g,1})$ is $\Gamma_{g,1}$-invariant. Let $\phi_i, i \in I, \#I = \binom{2g}{3}$, be such a set and let ψ be an element of $(\Gamma_{g,1}^{6,l} \cap \mathrm{Tor}_g)/K_g$. We write it as $\prod_{i \in I} \phi_i^{l_i}$. The assumption $\psi \in \Gamma_{g,1}^{6,l}$ implies that for any $x \in H$, the element $\psi(x)x^{-1}$ in $\pi_1(S_{g,1})^{[2]}/\pi_1(S_{g,1})^{[3]} \cong \wedge^2 H$ is actually in $\pi_1(S_{g,1})^{[3],l}/\pi_1(S_{g,1})^{[3]}$.

To return to the nonpointed case, we have to replace $\tau_{g,1}$ by

$$\tau_g \colon \mathrm{Tor}_g \to \bigwedge^3 H/([S_g] \wedge H),$$

where $[S_g] \in \wedge^2 H$ is the fundamental class of S_g and the right-hand side is a free **Z**-module of rank $\binom{2g}{3} - 2g$.

From [18, Lem. 6.3] it follows that if l is odd, $\pi_1(S_g)^{[3],l}$ generates inside the free module $\pi_1(S_g)^{[2]}/\pi_1(S_g)^{[3]}$ precisely the submodule generated by all l-fold multiples of all elements; i.e.

$$\pi_1(S_g)^{[3],l}/\pi_1(S_g)^{[3]} \cong l \cdot \bigwedge^2 H/[S_g].$$

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Thus the image $H \otimes \pi_1(S_g)^{[3],l}/\pi_1(S_g)^{[3]} \to \wedge^3 H/([S_g] \wedge H)$ equals $l \cdot \wedge^3 H/([S_g] \wedge H)$. Choose $J \subset I$ such that the $\tau_g(\phi_i)$, $i \in J$ yield a basis of $\wedge^3 H/([S_g] \wedge H)$. The assertion that $\sum_{i \in I} l_i \tau_{g,1}(\phi_i)$ is contained in the submodule $l \cdot \wedge^3 H/([S_g] \wedge H)$ implies that $l | l_i$ for all $i \in I$.

Clearly the same reasoning carries over again to the pointed case, so that the proposition follows.

THEOREM 3.5. *For $g \geqslant 2, l \geqslant 3$ and odd, and k an algebraically closed field of characteristic not dividing l, the moduli spaces $\overline{M_g^{6,l}}(k)$ and $\overline{M_{g,1}^{6,l}}(k)$ are simply connected.*

Proof. The theorem follows from the same kind of arguments used in the proof of Proposition 4.3. □

Ivanov asked whether $H_1(\Gamma) = 0$ for every finite index subgroup Γ of Γ_g, at least when g is sufficiently large (see [6, Question 3.2]). We can now give an affirmative answer to this question in case Γ contains $\Gamma_g^{6,l}$ for some odd l at least 3.

COROLLARY 3.6. *If $g \geqslant 3, l \geqslant 3$ and odd, then every subgroup of Γ_g containing $\Gamma^{6,l}$ has trivial first rational homology.*

Proof. Let V_0 be the normalization of the Satake compactification of $M_g(\mathbf{C})$ in the function field of $\overline{M_g^{6,l}}(\mathbf{C})$ and let $f: \overline{M_g^{6,l}} \to V_0$ be the induced birational morphism. Then V_0 is projective and the codimension of the image of the boundary of $\overline{M_g^{6,l}}$ under f is at least 2. So [16, Thm. 3] yields that the first homology group of $M_g^{6,l}$ is zero, so the same holds for any cover it dominates. □

Remark 3.7. In the proof of the above proposition, we need $g \geqslant 3$ to ensure that the condition in Mumford's theorem is fulfilled, nl. that the codimension of the image of the boundary under f is at least 2. If the genus is two, the dimension of $M_g^{6,l}$ is 3 and, by Theorem 2.3 there are two-dimensional boundary components of the form $\overline{M_{1,1}^{6,l}} \times \overline{M_{1,1}^{6,l}}$. Their images in the Satake compactification remain two-dimensional.

COROLLARY 3.8. *When $g \geqslant 3, l \geqslant 3$ and odd, the Picard group of $M_g^{6,l}$ is finitely generated.*

Proof. This follows from Proposition 3.6 and [5, Thm. 6.3]. (Cf. [6, Question 3.2]). □

In Section 2 we saw that the boundary divisors of $\overline{M_g^{6,l}}$ are smooth for l odd. These divisors are themselves moduli of curves with level structure described in Lemma 2.5. We can extend Theorem 3.5 to these moduli spaces.

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THEOREM 3.9. *For* $n \geqslant 0$ *and* $l \geqslant 3$, *the projective variety* $\overline{M^{6,l}_{g,n}}$ *is smooth and simply connected. Moreover, the natural morphism* $\overline{M^{6,l}_{g,n+1}} \to \overline{M^{6,l}_{g,n}}$ *is a stable curve.*

Proof. We will proceed by induction. The induction start is given by the simple connectivity of $\overline{M^{6,l}_{g}}$ (Theorem 3.5).

So let us assume that $\overline{M^{6,l}_{g,n}}$ is simply connected. The first step consists of proving the following lemma.

LEMMA 3.10. *The natural morphism* $\overline{M^{6,l}_{g,n+1}} \xrightarrow{p} \overline{M^{6,l}_{g,n}}$ *is a fibration in* (*connected*) *stable curves, which are smooth over* $M^{6,l}_{g,n}$.

Proof. We claim that the induced map on fundamental groups $p_*\colon \Gamma^{6,l}_{g,n+1} \to \Gamma^{6,l}_{g,n}$ is surjective. To see this let us consider the level defined by the normal subgroup $p_*(\Gamma^{6,l}_{g,n+1}) < \Gamma_{g,n}$. To compute the ramification at infinity of the corresponding Galois cover $X \to \mathcal{M}_{g,n}$, we just remark that $f \in p_*(\Gamma^{6,l}_{g,n+1})$ if and only if there is a lifting $\tilde{f}$ in $\Gamma_{g,n+1}$ such that $\tilde{f} \in \Gamma^{6,l}_{g,n+1}$.

Let $[C]$ be a point of $\overline{\mathcal{M}_{g,n}}$ for which we choose a representing marked Riemann surface together with a set of disjoint closed curves such that contractings these curves yields $[C]$. We use the notation of Section 2. We claim that the kernel of the local monodromy representation for $p_*(\Gamma^{6,l}_{g,n+1})$ in a suitable neighbourhood of $[C]$ is given by

$$\bigoplus_{e \in E(\Gamma)} l \cdot \mathbf{Z}e \oplus \bigoplus_{e \in B^{un}(\Gamma)} \mathbf{Z}e,$$

i.e. the same as that for $\Gamma^{6,l}_{g,n}$. This can be seen as follows. The inclusion $p_*(\Gamma^{6,l}_{g,n+1}) < \Gamma^{6,l}_{g,n}$ implies the corresponding inclusion for the kernels of the respective monodromy representations. The reverse inclusion follows from the remark we made at the beginning of the proof of this lemma and the local monodromy description for $\Gamma^{6,l}_{g,n+1}$ given in Proposition 2.8.

It follows that $X \to \overline{M^{6,l}_{g,n}}$ is an étale morphism. The simple connectivity of $\overline{M^{6,l}_{g,n}}$ implies that this étale morphism is actually an isomorphism and therefore p_* is surjective.

Let us now consider the Stein factorization of p

$$\overline{M^{6,l}_{g,n+1}} \xrightarrow{p'} \overline{Y} \xrightarrow{f} \overline{M^{6,l}_{g,n}},$$

where p' has connected fibers and f is finite. We have to show that f is an isomorphism. Put $Y := f^{-1}(M^{6,l}_{g,n})$.

Consider the factorization

$$M^{6,l}_{g,n+1} \xrightarrow{h} \mathcal{C}^{6,l}_{g,n} \xrightarrow{\pi} M^{6,l}_{g,n},$$

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where $\mathcal{C}^{6,l}_{g,n}$ is the universal curve over $M^{6,l}_{g,n}$. We know that π is smooth and h is étale, so p restricted to $M^{6,l}_{g,n+1}$ is a smooth morphism. A smooth morphism is separated in the sense of Definition 6.1.1(b) in [17] and applying Theorem 6.2.1 ibidem, we can conclude that f is étale.

Thus the induced map on fundamental groups $\pi_1(Y) \xrightarrow{f_*} \Gamma^{6,l}_{g,n}$ is an inclusion. Since we have proved above that $p_* = f_* \circ p'_*$ is surjective we have that f_* is surjective too. In conclusion f_* is an isomorphism and, hence, also f. □

Let us finish the proof of the simple connectivity of $\overline{M^{6,l}_{g,n+1}}$, using that $\overline{M^{6,l}_{g,n}}$ is simply connected.

From Lemma 3.10 we have that $M^{6,l}_{g,n+1} \to M^{6,l}_{g,n}$ is a smooth fibration in curves. Let us denote by S the fiber over a point $a \in M^{6,l}_{g,n}$, and by $\tilde{a}$ a point in S. We have the following commutative diagram of fundamental groups

$$\begin{array}{ccccccc}
\pi_1(S,\tilde{a}) & \longrightarrow & \pi_1(M^{6,l}_{g,n+1},\tilde{a}) & \longrightarrow & \pi_1(M^{6,l}_{g,n},a) & \longrightarrow & 1 \\
& \searrow & \downarrow & & \downarrow & & \\
& & \pi_1(\overline{M^{6,l}_{g,n+1}},\tilde{a}) & \longrightarrow & \pi_1(\overline{M^{6,l}_{g,n}},a) & = & 1,
\end{array}$$

where the first row is exact.

The diagram tells us that the only nontrivial generators for $\pi_1(\overline{M^{6,l}_{g,n+1}},\tilde{a})$ come from $\pi_1(S,\tilde{a})$. But the compact surface S is embedded in the family of stable curves $\overline{M^{6,l}_{g,n+1}} \to \overline{M^{6,l}_{g,n}}$ in such a way that every simple loop on S becomes a vanishing loop for some stable curve of the family; this means that the image of $\pi_1(S)$ inside $\pi_1(\overline{M^{6,l}_{g,n+1}})$ is trivial and so $\pi_1(\overline{M^{6,l}_{g,n+1}}) = 1$. □

Reversing the procedure applied in the proof of Theorem 3.9, we can prove

COROLLARY 3.11. *The normal subgroup $\Gamma^{6,l}_{g,n} < \Gamma_{g,n}$, for $l \geqslant 3$ odd, is generated by Dehn twists along simple separating closed curves and lth powers of Dehn twists along not separating simple closed curves.*

Proof. If we take a simple loop $\alpha \in M^{6,l}_{g,n}$ with base point a, we know that it bounds a closed disc D contained in $\overline{M^{6,l}_{g,n}}$. We can assume that D crosses the boundary of $\overline{M^{6,l}_{g,n}}$ normally. The inverse image of D in $M^{6,l}_{g,n}$ is then a closed disc minus a finite number of points. This means that α is homotopic in $M^{6,l}_{g,n}$ to the composition of a finite number of simple loops around the boundary of $\overline{M^{6,l}_{g,n}}$. From the local monodromy representation, we know that these correspond in $\Gamma_{g,n}$ to Dehn twists along simple separating closed curves and lth powers of Dehn twists along not separating simple closed curves. □

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