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Abelianness of Mumford–Tate Groups Associated to Some Unitary Groups

T. N. VENKATARAMANA
School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai 400 005, India. e-mail: venky@math.tifr.res.in

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Abstract. In this paper, we investigate the action of the $\mathbb{Q}$-cohomology of the compact dual $\widehat{X}$ of a compact Shimura Variety $S(\Gamma)$ on the $\mathbb{Q}$-cohomology of $S(\Gamma)$ under a cup product. We use this to split the cohomology of $S(\Gamma)$ into a direct sum of (not necessarily irreducible) $\mathbb{Q}$-Hodge structures. As an application, we prove that for the class of arithmetic subgroups of the unitary groups $\mathrm{U}(p, q)$ arising from Hermitian forms over CM fields, the Mumford–Tate groups associated to certain *holomorphic* cohomology classes on $S(\Gamma)$ are Abelian. As another application, we show that all classes of Hodge type (1,1) in H^2 of unitary four-folds associated to the group $\mathrm{U}(2, 2)$ are algebraic.

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0. Introduction

Let X be a Hermitian symmetric domain of noncompact type, G (the identity component of) the group of holomorphic automorphisms of X and Γ a cocompact, neat, arithmetic congruence subgroup of G. It is well known ([Baily-Borel]) that $S(\Gamma) = \Gamma\backslash X$ is a smooth projective variety (a connected component of a 'Shimura Variety'). One has the 'Matsushima Formula' for the (de Rham) cohomology of $S(\Gamma)$ with $\mathbb{C}$ coefficients:

$$H^i(S(\Gamma), \mathbb{C}) = \bigoplus_{\pi(C)=0} m(\pi)\ \mathrm{Hom}_K(\overset{i}{\wedge}\,\mathfrak{p}, \pi). \tag{1}$$

We explain the notation: let $\mathfrak{g}$ be the complexified Lie algebra of G, K a maximal compact subgroup of G, $\mathbf{k}$ the complexified Lie algebra of K. One has the Cartan decomposition $\mathfrak{g} = \mathbf{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the complexified tangent space to X at $K (X = G/K)$. Let $\mathfrak{p}^+$ and $\mathfrak{p}^-$ denote respectively the holomorphic and anti-holomorphic tangent spaces to X at K, write $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$. Let C be the Casimir of $\mathfrak{g}$ (an element of the universal enveloping algebra of $\mathfrak{g}$), π a unitary irreducible representation of $G, m(\pi)$ the (finite) multiplicity with which π occurs in $L^2[\Gamma\backslash G]$

(the space of functions on $\Gamma\backslash G$ which are square integrable with respect to a Haar measure on $\Gamma\backslash G$). On the space of K-finite vectors in π, the operator C acts by a scalar, which we denote by $\pi(C)$. In (1), π runs through those unitary representations such that $\pi(C) = 0$.

On the other hand, one may consider the *singular* cohomology of $S(\Gamma)$ with coefficients in $\mathbb{Q}$ and (by de Rham's theorem) one has

$$H^i(S(\Gamma), \mathbb{C}) = H^i_{\text{sing}}(S(\Gamma), \mathbb{Q}) \otimes \mathbb{C}.$$

If π is as above, denote by $\mathrm{Hod}^i(\pi)$ the smallest $\mathbb{Q}$-subspace of $H^i_{\text{sing}}(S(\Gamma), \mathbb{Q})$ whose complexification (under the identification (1)) contains the component $m(\pi)\mathrm{Hom}_K(\overset{i}{\wedge}\, \mathfrak{p}, \pi) \widehat{=def=} H^i(\pi, \Gamma)$. If π is nontrivial, $\pi(C) = 0$, and $i = \inf\{j : \mathrm{Hom}_K(\overset{j}{\wedge}\, \mathfrak{p}, \pi) \neq 0\}$ we will refer to $H^i(\pi, \Gamma)$ as the space of *strongly primitive* cohomology classes of type π in $H^i(S(\Gamma), \mathbb{C})$. These are ([Vog-Zuc], (6.19)) actually primitive classes in the Hodge decomposition of $H^i(S(\Gamma), \mathbb{C})$.

(0.1) The representations occurring in the Matsushima formula have been classified ([Par 1], [Kum], [Vog-Zuc]). Suppose that $\pi = A_{\mathbf{q}}$ is associated to the θ-stable parabolic subalgebra $\mathbf{q}$ of $\mathfrak{g}$ (for details see (1.1)), $\mathbf{u}$ the nil-radical of $\mathbf{q}$, and let $i = \dim(\mathbf{u} \cap \mathfrak{p})$. Let $\mathbf{l}$ be the Levi part of $\mathbf{q}$ chosen as in (1.1), L the subgroup of G whose complex tangent space at the identity is $\mathbf{l}$. Denote by $E(G, L)$ the smallest K-stable subspace of $\overset{*}{\wedge}\, \mathfrak{p}^+$ (the full exterior algebra of $\mathfrak{p}^+$) containing $\overset{*}{\wedge}\, (\mathbf{l} \cap \mathfrak{p}^+)$. We first prove

THEOREM 1. *If $\pi = A_{\mathbf{q}}$ and $\pi' = A_{\mathbf{q}'}$ are two $(\mathfrak{g}, K)$-modules as above, such that $\pi = A_{\mathbf{q}}$ has strongly primitive cohomology in degree i and if $E(G, L)$ is not contained in $E(G, L')$ (L as above and L' chosen similarly for $\mathbf{q}'$), then* $\mathrm{Hod}^i(\pi) \cap \mathrm{Hod}^i(\pi') = (0)$. *The same conclusion holds if both π and π' have strongly primitive cohomology in degree i and $E(G, L) \neq E(G, L')$.*

Remark. Suppose E and F are disjoint complex subspaces of $W \otimes \mathbb{C}$ where W is a $\mathbb{Q}$-vector space. Suppose E_1 and F_1 are the smallest $\mathbb{Q}$-subspaces of W whose complexification contains respectively E and F. It is not always true that E_1 and F_1 are disjoint. Therefore the conclusion of Theorem 1 is special to the situation at hand.

We now explain the main ideas of the proof of Theorem 1.

If $\widehat{X}$ is the compact dual of the symmetric domain X, then the cohomology ring $\mathcal{A} = H^\bullet(\widehat{X}, \mathbb{Q})$ acts on the cohomology ring $H^\bullet(S(\Gamma), \mathbb{Q})$ by cup product. A Theorem of Kostant describes the cohomology of the compact dual $\widehat{X}$ in terms of 'Schubert cells'. We use this description and reinterpret the aforementioned action of $\mathcal{A} = H^\bullet(\widehat{X}, \mathbb{Q})$ in terms of continuous cohomology (see Lemmas (1.4) and (1.5)).

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This description allows us (essentially) to split the cohomology group $H^\bullet(S(\Gamma), \mathbb{Q})$ into disjoint $\mathbb{Q}$-Hodge structures W (these W need not be *irreducible* $\mathbb{Q}$-Hodge structures) such that for distinct W, the annihilators of W in the algebra $\mathcal{A}$ are distinct. Under the assumptions of Theorem 1, we observe (using Lemma (1.5)) that the annihilators of $H^i(\Gamma, \pi)$ and $H^i(\Gamma, \pi')$ are distinct. This allows us to conclude that $\mathrm{Hod}^{\mathrm{i}}(\pi)$ and $\mathrm{Hod}^{\mathrm{i}}(\pi')$ are disjoint (see (1.6) for the details of the proof).

(0.2) Now we specialise to the case $G = \mathrm{U}(p, q)$ (note that at the beginning of the introduction, we had assumed that G was centreless, but we may add a compact centre to G and keep the other hypotheses on Γ; the statements that follow are unchanged). Assume that

$$2 \leqslant p \leqslant q, \quad \text{put} \quad n = p + q, \quad K = \mathrm{U}(p) \times \mathrm{U}(q).$$

Fix a totally imaginary quadratic extension E of a totally real number field F and an n-dimensional E-vector space with a Hermitian form h (see (2.3) for details) and let $G = \mathrm{U}(h)$ be the unitary group of h : G is an algebraic group over F. We choose h so that (here $d = \mathrm{degree}(F/\mathbb{Q})$)

$$G(F \otimes_{\mathbb{Q}} \mathbb{R}) = U(p, q) \times U(p+q)^{d-1}.$$

Let $\Gamma \subset G(F)$ be a neat, congruence arithmetic subgroup. Assume that $d > 1$ (if $d > 1$ then Γ is cocompact). By a slight abuse of notation, we will write $G = \mathrm{U}(p, q)$.

Consider the modules $A_{\mathbf{q}}$. Assume that $\mathbf{u} \cap \mathfrak{p} = \mathbf{u} \cap \mathfrak{p}^+$ (one says then that $A_{\mathbf{q}}$ is *holomorphic*). Then, in the notation of (2.1), $\mathbf{q} = \mathbf{q}(r, s)$ for some r, s, with $0 \leqslant r \leqslant p, \quad 0 \leqslant s \leqslant q$.

THEOREM 2. *Assume either that $s = 0$ and $1 \leqslant r < p/2 (i = rq)$ or that $r = 0, 1 \leqslant s < q/2 (i = sp)$. Then* $\mathrm{Hod}^i(A_{\mathbf{q}})$ *has Abelian Mumford–Tate Group, for the class of Arithmetic groups Γ considered above.*

Remark. In the special case when $(p, q) \neq (2, 2)$ and $r = 0, s = 1$ (then the degree $ps = p$), this is already proved in [Clo-Ven] (see Theorem (6.2) of [Clo-Ven]). One of the proofs of this in [Clo-Ven] was based on Lemma (5.8) (ibid.), which says that if $Z \subset X \otimes Y$, and X, Y, Z are irreducible pure Hodge structures of strictly positive weight and nonnegative Hodge types, and if the only Hodge types of Z are holomorphic and anti-holomorphic, then the Mumford–Tate groups of X, Y, Z are all Abelian.

Theorem 2 will be proved in Section 2. By using Theorem 1 (and certain computations of [Clo-Ven]) we will first show (if $\mathbf{q} = \mathbf{q}(0, s)$ in Theorem 2 and $2s < q$) that the only Hodge types of the Hodge structure $\mathrm{Hod}^{ps}(A_{\mathbf{q}})$ are of the form $(pu, p(s-u))$ for $0 \leqslant u \leqslant s$. This is done in Lemma (2.2). Next, suppose Z is an irreducible $\mathbb{Q}$-Hodge structure in $\mathrm{Hod}^{ps}(A_{\mathbf{q}})$. We will show (by using a result of [Clo-Ven]), that there exists a product $G_1 \times G_2 \subset G = \mathrm{U}(p,q)$ such that $G_1 = U(a,s)$ and $G_2 = \mathrm{U}(b,s)$ $(a+b=p)$ with the property that if $\Gamma_i = G_i \cap \Gamma$ $(i=1,2)$, then $Z \subset H^{as}(S(\Gamma_1)) \otimes H^{bs}(S(\Gamma_2))$. By a generalisation of Lemma (5.8) of [Clo-Ven] mentioned in the preceding paragraph, we will then conclude that the Mumford–Tate group of Z is Abelian (see Equation (9) of Section 2).

(0.3) We will use Theorems 1 and 2 to prove

THEOREM 3. *Let $G = U(2,2)$ and Γ any neat cocompact (arithmetic) subgroup. Then $H^2(S(\Gamma),\mathbb{Q}) = X \oplus Y$ where X and Y are 2 $\mathbb{Q}$-Hodge structures such that $X \otimes \mathbb{C} = X^{2,0} \oplus X^{0,2}$, $Y \otimes \mathbb{C} = Y^{1,1}$ (Moreover Y consists only of algebraic classes). Furthermore, if Γ is specialised to be a congruence arithmetic subgroup of the type considered in Theorem 2 then the Mumford–Tate group of $H^2(S(\Gamma),\mathbb{Q})$ is Abelian.*

Remark. D. Blasius and the referee have remarked that in view of (a Theorem of Faltings on) the existence of Hodge–Tate decomposition for the e'tale cohomology of smooth projective varieties defined over number fields, Theorem 3 implies that if Γ is *any* cocompact neat arithmetic subgroup of $\mathrm{U}(2,2)$, then the *Tate classes in $H^2_{\mathrm{et}}(S(\Gamma),\mathbb{Q}_l)$ are all algebraic*. Indeed, by Theorem 3, the only Hodge–Tate types of the quotient of $H^2_{\mathrm{et}}(S(\Gamma)$ by the span $Y \otimes \mathbb{Q}_l$ of algebraic cycles, are (2,0) and (0,2), and hence there are no Tate classes outside $Y \otimes \mathbb{Q}_l$.

Theorem 3 is proved in Section 3. There exist (see (3.1)) three parabolic subalgebras $\mathbf{q}_i (1 \leqslant i \leqslant 3)$ of $\mathfrak{g} = \mathbf{u}(2,2) \otimes \mathbb{C}$ such that all the holomorphic cohomology of the Shimura variety $S(\Gamma)$ in degree 2 is of type $A_{\mathbf{q}_i}$ with $i = 1,2$ and such that all the cohomology of $S(\Gamma)$ of type (1,1) in the degree 2 is of type $A_{\mathbf{q}_3}$ (or else comes from the compact dual $\widehat{X}$). We will show, by using Theorem 1, that for $i = 1,2$,

$$\mathrm{Hod}^2(\mathrm{A}_{\mathbf{q}_3}) \cap \mathrm{Hod}^2(\mathrm{A}_{\mathbf{q}_i}) = 0.$$

This essentially yields Theorem 3.

Additional Remarks. (1) The analogue of Theorem 3 (viz. that if $i = \mathbb{R}$-rank (G), G is simple, $i \geqslant 2$, then $H^i(S(\Gamma),\mathbb{Q})$ has Abelian Mumford–Tate group) is proved for many classical groups G and for all $\mathrm{U}(p,q)$, $2 \leqslant p \leqslant q$, $q > 2$ in [Clo-Ven]. The case of $\mathrm{U}(2,2)$ was excluded there.

(2) The Abelianness of the Mumford–Tate groups arising in Theorem 2 is predicted by certain conjectures of Langlands, Arthur and Kottwitz (see [Bla-Rog]). That these conjectures imply Theorem 2 can be seen from the discussion in (4.2) and (4.3.A) of [Clo-Van].

(3) Note that in Theorem 2 the module $A_{\mathbf{q}}$ contributes to cohomology in degree rq (or sp) which is less than $pq/2$ and the latter is half the complex dimension of the Shimura Variety $S(\Gamma)$. The conjectures alluded to in the previous remark seem

to imply that in degrees less than half the complex dimension of these Shimura Varieties, *all* the cohomology should be Abelian (e.g. should have Abelian Mumford–Tate group).

1. Cup Products with Schubert Cycles and Proof of Theorem 1.

(1.1) DEFINITIONS AND NOTATION. Let X, G, K be as in the introduction. Let T be a maximal torus in K. The space $X = G/K$ is Hermitian symmetric. Therefore T is a maximal torus in G. Fix a Borel subalgebra $\mathbf{b}_K$ of $\mathbf{k}$ containing $\mathbf{t} = \mathrm{Lie}(T) \otimes \mathbb{C}$. Write $\Phi(\mathbf{b}_K, \mathbf{t})$, $\Phi(\mathbf{k}, \mathbf{t})$, $\Phi(\mathfrak{p}, \mathbf{t})$, respectively, for the roots of $\mathbf{t}$ occurring in $\mathbf{b}_K, \mathbf{k}$ and $\mathfrak{p}$. Let $\mathfrak{g}_0$ be the real Lie algebra of $G(\mathbb{R})$. Define $\mathbf{k}_0$ and $\mathbf{t}_0$ similarly.

Fix $X \in iLie(T)$. Then $\alpha(X) \in \mathbb{R}$ for all $\alpha \in \Phi(\mathfrak{g}, \mathbf{t})$. Let

$$\mathbf{q} = \mathbf{t} \bigoplus_{\alpha(X) \geqslant 0} \mathfrak{g}_\alpha, \qquad \mathbf{l} = \mathbf{t} \bigoplus_{\alpha(X)=0} \mathfrak{g}_\alpha,$$

$$\mathbf{u} = \bigoplus_{\alpha(X)>0} \mathfrak{g}_\alpha, \qquad \mathbf{u}^- = \bigoplus_{\alpha(X)<0} \mathfrak{g}_\alpha.$$

Clearly $\mathbf{q}, \mathbf{l}, \mathbf{u}$ are all θ-stable (where θ is the Cartan Involution on $\mathfrak{g}$ with respect to $\mathbf{k}$ (i.e. θ is 1 on $\mathbf{k}$ and -1 on $\mathfrak{p}$)). Let $i = \dim(\mathbf{u} \cap \mathfrak{p})$, $V(\mathbf{q})$ the K-submodule of $\overset{i}{\wedge}\, \mathfrak{p}$ generated by the line $e(\mathbf{q}) = \overset{i}{\wedge}\, (\mathbf{u} \cap \mathfrak{p})$ (it is easy to see that $V(\mathbf{q})$ is irreducible).

Then by [Vog-Zuc] (and by [Vogl]) there exists a unique (unitary) $(\mathfrak{g}_0, K)$-module $\pi = A_\mathbf{q}$ such that $\pi(C) = 0$ and

$$H^i(\mathfrak{g}, K, \pi) = \mathrm{Hom}_K(\overset{i}{\wedge}\, \mathfrak{p}, \pi) = \mathrm{Hom}_K(V(\mathbf{q}), \pi) = \mathbb{C}. \tag{2}$$

Let $\wedge \mathfrak{p}^+$ denote the exterior algebra on the holomorphic part $\mathfrak{p}^+$ of $\mathfrak{p}$. The subalgebra $\mathbf{l}$ of $\mathfrak{g}$ is θ-stable and is defined over $\mathbb{R}$, i.e. there exists a subalgebra $\mathbf{l}_0$ of $\mathfrak{g}_0 = \mathrm{Lie}(G)$ such that $\mathbf{l}_0 \otimes_\mathbb{R} \mathbb{C} = \mathbf{l}$. Let L be the subgroup of G with Lie algebra $\mathbf{l}_0$. Then it is easily seen that $L/L \cap K$ is a Hermitian symmetric subdomain of G/K. We may form the subspace $E(G, L)$ of $\wedge \mathfrak{p}^+$, which is by definition, the K-span of the exterior algebra $\wedge(\mathbf{l} \cap \mathfrak{p}^+)$ in $\wedge \mathfrak{p}^+$.

Further, by [Vog-Zuc], section 6, the *full* cohomology of $A_\mathbf{q}$ is described as follows. The group $K \cap L$ is the maximal compact subgroup of L. Let $(\wedge(\mathbf{l} \cap \mathfrak{p}))^{K \cap L}$ be its space of invariants in the exterior algebra of $\mathbf{l} \cap \mathfrak{p}$. Given $\xi \in (\overset{m}{\wedge}\, (\mathbf{l} \cap \mathfrak{p}))^{K \cap L}$, form the vector $e(\mathbf{q}) \wedge \xi \in \overset{i+m}{\wedge}\, \mathfrak{p}$. Then it is easy to see that the K-span $V(\mathbf{q}, \xi)$ of this vector (is irreducible and) is isomorphic to $V(\mathbf{q})$. Moreover, the isotypical component of $V(\mathbf{q})$ in $\overset{i+m}{\wedge}\, \mathfrak{p}$ is the sum of $V(\mathbf{q}, \xi)$, the sum over all the ξ as above. One also has

$$H^{i+m}(\mathfrak{g}, K, \pi) = \mathrm{Hom}_K(\overset{i+m}{\wedge}\, \mathfrak{p}, \pi) = \sum_\xi \mathrm{Hom}_K(V(\mathbf{q}, \xi), \pi) = \mathbb{C}^{d_m} \tag{2'}$$

where d_m is the dimension of the space $(\overset{m}{\wedge}\, (\mathfrak{p} \cap \mathbf{l}))^{L \cap K}$ of $L \cap K$-invariants in $\overset{m}{\wedge}\, (\mathfrak{p} \cap \mathbf{l})$.

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Remark. Suppose that the compact dual of $X_L = L/L \cap K$ is denoted $\widehat{X_L}$. Suppose that the restriction map $H^*(\widehat{X}) \to H^*(\widehat{X_L})$ is *surjective*. Then, the previous paragraph says that all the cohomology of $\pi = A_{\mathbf{q}}$ is obtained from the one-dimensional cohomology $H^i(\pi)$ by wedging with classes from $H^*(\widehat{X})$ (note that by definition $H^i(\pi)$ consists of strongly primitive classes). It is not always true that the above restriction map is surjective. But we will see in (2.2) that if $\mathbf{q} = \mathbf{q}(0, s)$ as in Section 2, then it is indeed surjective.

Let V_0 be a $\mathbb{Q}$-vector space, and $V = V_0 \otimes_{\mathbb{Q}} \mathbb{C}$. We say that a complex subspace W of V is *defined over* $\mathbb{Q}$ if there exists a $\mathbb{Q}$-subspace W_0 of V_0 such that $W = W_0 \otimes_{\mathbb{Q}} \mathbb{C}$.

Let $G_{\mathbb{C}}$ be the algebraic group of adjoint type with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_0 \otimes \mathbb{C}$ (as G is the connected group of automorphisms of X, $G_{\mathbb{C}}$ is in fact the complexification of G). Let $P_{\mathbb{C}}^-$ denote the parabolic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{p}^- \oplus \mathbf{k}$, B_G the Borel subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{p}^- \oplus \mathbf{b}_K$. Then the *compact dual* $\widehat{X} = G_{\mathbb{C}}/P_{\mathbb{C}}^-$ of X is also Hermitian symmetric and the inclusion $G/K = G/G \cap P_{\mathbb{C}}^- \subset G_{\mathbb{C}}/P_{\mathbb{C}}^-$ realises X as a bounded domain in $\widehat{X}$ (see [Borel 1], Section 5). Let $\mathfrak{g}_u = \mathbf{k}_0 \oplus i\mathfrak{p}_0$ be the (real) subalgebra of the complex Lie algebra $G_{\mathbb{C}}$ and G_u the (compact) subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_u$. Then $\widehat{X} = G_u/K$. The restriction of the Killing form κ on $\mathfrak{g}_u$ restricted to $i\mathfrak{p}_0$ is negative definite and K-invariant; we thus get a G_u-invariant metric on $\widehat{X}$ by translation. With respect to this metric, the space of Harmonic forms on $\widehat{X}$, denoted $H^*(\widehat{X}, \mathbb{C})$ is (see [Kostant 1])

$$\mathrm{Hom}_K(\overset{*}{\wedge} (i\mathfrak{p}_0) \otimes \mathbb{C}, \mathbb{C}) = \mathrm{Hom}_K(\wedge \mathfrak{p}, \mathbb{C}).$$

If Γ is as in the introduction, we have an inclusion

$$j : H^*(\widehat{X}, \mathbb{C}) = \mathrm{Hom}_K(\wedge \mathfrak{p}, \mathbb{C}) \hookrightarrow \mathrm{Hom}_K(\wedge \mathfrak{p}, C^\infty(\Gamma \backslash G)(0)) = H^*(S(\Gamma), \mathbb{C})$$

(the last equality being a version of the Matsushima formula (1); here $C^\infty(\Gamma \backslash G)(0))$ denotes the space of smooth functions on $\Gamma \backslash G$ which are annihilated by the Casimir of G.

It follows from the Hirzebruch Proportionality principle (see [Borel 2], Corollary (7.3)) that

$$j(H^*(\widehat{X}, \mathbb{Q})) \subset H^*(S(\Gamma), \mathbb{Q}). \tag{3}$$

Note, moreover, that $(\mathfrak{p}^+)^* = \mathfrak{p}^-$. Further, there is a torus $S = \mathbb{G}_m \subset T_{\mathbb{C}}$ such that $K_{\mathbb{C}} = Z_{G_{\mathbb{C}}}(S)$ and $z \in S$ acts by z on $\mathfrak{p}^-$ and z^{-1} on $\mathfrak{p}^-$ (see [Helgason], the chapter on Hermitian symmetric domains). Therefore, the S-invariants in $\wedge \mathfrak{p} = \wedge \mathfrak{p}^+ \otimes \wedge \mathfrak{p}^-$ are sums of $\overset{m}{\wedge} \mathfrak{p}^+ \otimes \overset{m}{\wedge} \mathfrak{p}^-$ $(m \leqslant \dim \mathfrak{p}^+)$ and we have

$$\mathrm{Hom}_K(\wedge \mathfrak{p}^+ \otimes \wedge \mathfrak{p}^-, \mathbb{C}) = \bigoplus_{m=0}^{\dim(\widehat{X})} \mathrm{Hom}_{K_{\mathbb{C}}}(\overset{m}{\wedge} \mathfrak{p}^+, \overset{m}{\wedge} \mathfrak{p}^+) = \bigoplus_{m=0}^{\dim \widehat{X}} H^{2m}(\widehat{X}, \mathbb{C}). \tag{4}$$

We denote by $\widehat{X}_L = L_u/L_u \cap K$, the compact dual of $X_L = L/L \cap K$. Here $\mathbf{l}_u = \mathbf{l} \cap \mathbf{k}_0 \oplus i(\mathbf{l} \cap \mathfrak{p}_0)$, and L_u the subgroup of G_u with Lie algebra $\mathbf{l}_u$.

The space $\widehat{X} = G_{\mathbb{C}}/P_{\mathbb{C}}^{-}$ has a cellular decomposition whose cells (the *Schubert Cells*) generate the homology group of $\widehat{X}$ over $\mathbb{Q}$. The Schubert Cells are parametrised by the coset space $W = W(G_{\mathbb{C}}, T_{\mathbb{C}})/W(K_{\mathbb{C}}, T_{\mathbb{C}})$ where $W(G_{\mathbb{C}}, T_{\mathbb{C}})$ (resp. $W(K_{\mathbb{C}}, T_{\mathbb{C}})$) is the Weyl group of $G_{\mathbb{C}}$ (resp. $K_{\mathbb{C}}$) with respect to $T_{\mathbb{C}}$. If $w \in W$, let $X_w = B_G w P_{\mathbb{C}}^{-}$ be the corresponding Schubert Cell. As in [Kostant 1], we choose representatives w of least length and [ibid., Remark following (5.13)] in a coset class, there is exactly one element of this length. Given X_w (w of least length), denote by $\lambda_w \in H^{2i}(\widehat{X}, \mathbb{Q})$ its Poincaré dual. If $d = \dim(\widehat{X})$, then $i = d$-length of (w). From (4), we may think of λ_w as an element of $\mathrm{Hom}_{K_{\mathbb{C}}}(\overset{i}{\wedge}\, \mathfrak{p}^+, \overset{i}{\wedge}\, \mathfrak{p}^+)$. We now recall some results of [Kostant 2].

(1.2) A THEOREM OF KOSTANT. (I) *As a representation of* K, $\overset{i}{\wedge}\, \mathfrak{p}^+$ *is a direct sum of irreducible representations, each occurring exactly* once. *Write* Y_i *for the set of irreducible representations of* K *occurring in* $\overset{i}{\wedge}\, \mathfrak{p}^+$ *and write*

$$\overset{i}{\wedge}\, \mathfrak{p}^+ = \bigoplus_{E \in Y_i} E. \tag{5}$$

(II) *The Poincaré dual* λ_w *of the Schubert cell* X_w, *where* w *is of length* $d - i$ *(see the end of (1.1) for notation), is a nonzero scalar multiple, for some* $E \in Y_i$, *of* $\lambda_E \in \mathrm{Hom}_K(E, E) \subset \mathrm{Hom}_K(\overset{i}{\wedge}\, \mathfrak{p}^+, \overset{i}{\wedge}\, \mathfrak{p}^+)$*; here* λ_E *is the identity transformation in* $\mathrm{Hom}_K(E, E)$. *Moreover, for every* $E \in Y_i$, *the corresponding* λ_E *is the Poincaré dual of* X_w *for some* w. *In particular,* $\mathrm{Hom}_K(E, E) \subset H^{2i}(\widehat{X}, \mathbb{Q}) \otimes \mathbb{C}$ *is defined over* $\mathbb{Q}$.

As a consequence of the inclusion (3) and part (II) of Kostant's Theorem, we see that the space

$$\{x \in H^*(S(\Gamma), \mathbb{C});\ j(\lambda_E) \wedge x = 0\} \tag{6}$$

is defined over $\mathbb{Q}$. Here $\wedge$ denotes the cup product.

The action of the torus T on the space $\mathfrak{p}$ is completely reducible and we have a decomposition $\mathfrak{p} = \mathbf{u} \cap \mathfrak{p} \oplus \mathbf{l} \cap \mathfrak{p} \oplus \mathbf{u}^- \cap \mathfrak{p}$ as T-modules. Given a vector space W denote by W^* its dual. We have then a decomposition $\mathfrak{p}^* = (\mathbf{u} \cap \mathfrak{p})^* \oplus (\mathbf{l} \cap \mathfrak{p})^* \oplus (\mathbf{u}^- \cap \mathfrak{p})^*$, again of T-modules. We also note that the element $X \in \mathbf{t}$ acts by strictly negative (resp.positive) eigenvalues on $(\mathbf{u} \cap \mathfrak{p})^*$ (resp. on $(\mathbf{u}^- \cap \mathfrak{p})^*$) and by zero eigenvalue on $(\mathbf{l} \cap \mathfrak{p})^*$.

Before starting on the proof of Theorem 1, we prove a few lemmas.

(1.3) LEMMA. *Let* $(\mathbf{u}, \mathbf{q}, T)$ *be as in (1.1) and let* $e(\mathbf{q})^* = \overset{i}{\wedge}\, (\mathbf{u} \cap \mathfrak{p})^*$. *Consider the restriction map* $B : (\wedge \mathfrak{p}^*)^T \to (\wedge(\mathbf{l} \cap \mathfrak{p})^*)^T$ *and the cup-product* $A : (\wedge \mathfrak{p}^*)^T \to \wedge \mathfrak{p}^*$ *given by* $y \mapsto y \wedge e(\mathbf{q})^*$. *Then the kernels of* A *and* B *are the same.*

Proof. As $i = \dim(\mathbf{u} \cap \mathfrak{p})$, it follows that $e(\mathbf{q})^*$ is a line. Therefore $y \wedge e(\mathbf{q})^* = 0$ if and only if $y \in (\mathbf{u} \cap \mathfrak{p})^* \wedge (\wedge \mathfrak{p})^*$ for any $y \in \wedge \mathfrak{p}^*$, and in particular for any $y \in (\wedge \mathfrak{p}^*)^T$. Let $\mathbf{q} = \mathbf{q}(X)$. Then the eigenvalues of X on $\mathbf{u} \cap \mathfrak{p}$ are positive (by

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the definition of $\mathbf{u}$) whereas X has *strictly* negative eigenvalues on $(\mathbf{u} \cap \mathfrak{p})^*$. Moreover, on $\mathbf{l}, X$ has all eigenvalues 0 (by the definition of $\mathbf{l}$). Hence, if $y \in (\wedge \mathfrak{p}^*)^T \cap (\mathbf{u} \cap \mathfrak{p})^* \wedge (\wedge \mathfrak{p}^*)$, then y cannot lie in $\overset{j}{\wedge} (\mathbf{u} \cap \mathfrak{p})^* \otimes (\wedge (\mathbf{l} \cap \mathfrak{p})^*)(j > 0)$, i.e. $\mathrm{Ker}(A) = (\wedge \mathfrak{p}^*)^T \cap ((\mathbf{u} \cap \mathfrak{p})^* \wedge (\wedge \mathfrak{p}^*))$ =T-invariants in $(\mathbf{u} \cap \mathfrak{p})^* \wedge (\mathbf{u}^- \cap \mathfrak{p})^* \wedge (\wedge \mathfrak{p}^*)$.

Again, $y \in \mathrm{Ker}(B)$ if and only if y, thought of as an element of $\wedge \mathfrak{p}^* = \wedge (\mathbf{u} \cap \mathfrak{p})^* \otimes \wedge (\mathbf{u}^- \cap \mathfrak{p})^* \otimes \wedge (\mathbf{l} \cap \mathfrak{p})^*$, does not have a pure $\wedge (\mathbf{l} \cap \mathfrak{p})^*$ term i.e. $\mathrm{Ker}(B)$ is the space of invariants in $(\mathbf{u} \cap \mathfrak{p})^* \wedge (\mathbf{u}^- \cap \mathfrak{p})^* \wedge (\wedge \mathfrak{p}^*)$ under the action of T. This completes the proof. □

(1.4) LEMMA. *Fix* $z \in (\wedge \mathfrak{p}^*)^K$ *and let* $x \in H^i(A_{\mathbf{q}}, \Gamma) - (0)$ *(here* $i = \dim(\mathbf{u} \cap \mathfrak{p})$ *and hence* x *is strongly primitive). Then*

$$z \wedge x = 0 \Leftrightarrow z \in \mathrm{Ker}\colon H^*(\widehat{X}, \mathbb{C}) \xrightarrow{\mathrm{Res}} H^*(\widehat{X}_L, \mathbb{C})$$

where Res is the restriction map.

Proof. Now $x \in \mathrm{Hom}_K(V(\mathbf{q}), \mathcal{C}^\infty(\Gamma \backslash G)(0))$, with $V(\mathbf{q})$ as in Section (1). Therefore $z \wedge x = 0 \Leftrightarrow z \wedge V(\mathbf{q})^* = 0$. Since $V(\mathbf{q})^*$ is generated by $e(\mathbf{q})^*$ as a K-module, and z is K-invariant, it follows that $z \wedge V(\mathbf{q})^* = 0 \Leftrightarrow z \wedge e(\mathbf{q})^* = 0$. This is equivalent to the statement that $A(z) = 0$, A as in Lemma (1.3). By the Lemma (1.3), $A(z) = 0$ if and only if $B(z) = 0$. But $B(z) = 0$ if and only if z is in the kernel of the restriction map

$$\mathrm{Hom}_K(\wedge \mathfrak{p}, \mathbb{C}) \to \mathrm{Hom}_{K \cap L}(\wedge \mathfrak{p}_L^*, \mathbb{C})$$

($\mathfrak{p}_L = \mathbf{l} \cap \mathfrak{p}$). This yields the Lemma.

NOTATION. Let H_u be any semisimple subgroup of G_u such that $H_u / H_u \cap K$ is a Hermitian symmetric subspace of G_u/K. Let $\mathfrak{p}_H^+ = \mathfrak{p}^+ \cap$ (Lie $(H_u) \otimes \mathbb{C}$) and let $E(G, H)$ denote the K-span of $\wedge \mathfrak{p}_H^+$ in $\wedge \mathfrak{p}^+$. For a given integer m similarly write $E(G, H, m)$ for the K-span of $\overset{m}{\wedge} \mathfrak{p}_H^+$ in $\overset{m}{\wedge} \mathfrak{p}^+$. There is a K-invariant metric on $\mathfrak{p}^+$ induced by the restriction of the negative of the Killing form κ on $\mathfrak{p}_0$ given by $u \mapsto -\kappa(u, \overline{u})$ $(u \in \mathfrak{p}^+)$ whence there is a K-invariant metric on $\wedge \mathfrak{p}^+$ (and in particular on $\overset{m}{\wedge} \mathfrak{p}^+$). Let $E(G, H)^\perp$ (resp. $E(G, H, m)^\perp$) denote the perpendicular to $E(G, H)$ (resp. to $E(G, H, m)$) in $\wedge \mathfrak{p}^+$ (resp. in $\overset{m}{\wedge} \mathfrak{p}^+$) with respect to this metric.

Consider the Matsushima formula

$$H^*(S(\Gamma), \mathbb{C}) = \mathrm{Hom_K}(\wedge \mathfrak{p}, \mathrm{C}^\infty(\Gamma \backslash \mathrm{G})(0)).$$

The K-invariant metric on $\wedge \mathfrak{p}$ and the L^2 metric on $C^\infty(\Gamma \backslash G)$ yields a metric on the right-hand side of the Matsushima formula. Let L be the Lefschetz class of the smooth projective variety $S(\Gamma)$ which arises from the Killing form on G (i.e. the restriction of the Killing form to $\mathbf{p}^+ \otimes \mathbf{p}^-$ [where $\mathbf{p}^+$ is the holomorphic tangent space to G/K at the identity coset] defines, by translation, a

G-invariant closed form of type (1,1) on G/K and, hence, a cohomology class on $S(\Gamma)$; it is proportional to a rational class L). Assume that $S(\Gamma)$ has complex dimension d.

On $H^i(S(\Gamma), \mathbb{Q})$ we may define a pairing $\langle\alpha, \beta\rangle = \alpha \wedge *\beta$ (the pairing takes values in the top-dimensional cohomology of $S(\Gamma)$ which may be identified with $\mathbb{Q}$). Here, $*$ denotes the star operator on the cohomology group of $S(\Gamma)$ associated to the metric coming from the Killing form as above. Note that the star operator is defined over $\mathbb{Q}$ (see [Kleiman], Section 4) in the sense that it takes rational vectors into rational vectors. Note also (ibid. Section 4) that if $\beta \in H^i_{\text{prim}}(S(\Gamma), \mathbb{Q})$ then $*\beta = \beta \wedge L^{d-i} \in H^{2d-i}(S(\Gamma), \mathbb{Q})$. Here, the notion of primitive classes is as in [Kleiman] (namely the orthogonal complement of classes which come from wedging with L). As we observed before, the strongly primitive classes are indeed primitive ([Vog-Zuc], Section 6).

The metric on the right-hand side of the Matsushima formula, restricted to $H^i(S(\Gamma), \mathbb{Q})$, is proportional to this pairing, as may be readily checked. Note that $\alpha \wedge *\beta \in H^{2d}(S(\Gamma), \mathbb{C})$ is equal to the inner product (α', β') of α' and β' with respect to the metric on the right-hand side (here α, β are in $H^i(S(\Gamma), \mathbb{C})$ and α' and β' are their images under the Matsushima isomorphism) in the right-hand side of the Matsushima formula. The formula $\alpha \wedge *\beta = (\alpha', \beta')$ is immediate from the definition of $*$.

LEMMA. *With the foregoing notation, the kernel of the restriction map*

$$\mathrm{Res} : H^*(G_u/K, \mathbb{C}) \to H^*(H_u/H_{\mathbf{u}} \cap K, \mathbb{C})$$

contains the 'Schubert cell' λ_E if and only if $E \subset E(G, H)^{\perp}$.

Proof. Suppose $\lambda_E = \omega$ and $\mathrm{Res}(\omega) = 0$, where $\omega \in H^{2m}(\widehat{X}, \mathbb{C})$. Now the space $H^{2m}(\widehat{X}, \mathbb{C}) \subset \mathrm{Hom}(\overset{m}{\wedge}\, \mathbf{p}^+ \otimes \overset{m}{\wedge}\, \mathbf{p}^-, \mathbb{C})$. View ω as a homomorphism in the latter space. Then, by the definition of λ_E, $\omega(x \otimes y)$ (with $x \in \overset{m}{\wedge}\, \mathbf{p}^+$ and $y \in \overset{m}{\wedge}\, \mathbf{p}^-$) is obtained by first projecting $x \in \overset{m}{\wedge}\, \mathbf{p}^+$ to E (and similarly, projecting $y \in \overset{m}{\wedge}\, \mathbf{p}^-$ to the complex conjugate $\overline{E}$ of E) and then evaluating the resulting element of $E \otimes \overline{E}$ under the unique (upto scalar multiples) K invariant linear form on the space $E \otimes \overline{E}$.

Now, being of type (m, m), the class ω vanishes on $\overset{2m}{\wedge}\, \mathbf{p}_H$ if and only if it vanishes on the component $\overset{m}{\wedge}\, \mathbf{p}_H^+ \otimes \overset{m}{\wedge}\, \mathbf{p}_H^-$. The linear form ω lives only on the space $E \otimes \overline{E}$ and is positive on vectors of the form $z \otimes \overline{z}$ for $z \in E$. Consequently, ω vanishes on $\overset{2m}{\wedge}\, \mathbf{p}_H$ if and only if the projection to E of $\overset{m}{\wedge}\, \mathbf{p}_H^+$ vanishes.

Since the projection map to E from $\overset{m}{\wedge}\, \mathbf{p}^+$ is K equivariant, this is equivalent to saying that the projection of $E(G, H, m)$ to E is zero. By the Multiplicity One Theorem of Kostant, this is equivalent to the assertion $E \subset E(G, H, m)^{\perp}$. This completes the proof. $\square$

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(1.6) We now commence the proof of Theorem 1. With the notation of (1.1) and (1.2), let $E \in Y_i$. Then by (6),

$$\mathrm{Ker}(\lambda_E) = \{x \in H^*(S(\Gamma), \mathbb{C}); j(\lambda_E) \wedge x = 0\}$$

is defined over $\mathbb{Q}$. Now let $x \in H^i(A_{\mathbf{q}}, \Gamma)$ be a strongly primitive class of type $A_{\mathbf{q}}$. We seek $\{E \in Y_i; j(\lambda_E) \wedge x = 0\}$. From Lemmas (1.4) and (1.3), this is the set $\{E \in Y_i; \mathrm{Res}_{\widehat{X_L}}(\lambda_E) = 0\}$. From Lemma (1.5) we thus have:

$$H^i(A_{\mathbf{q}}, \Gamma) \subset \bigcap_{E \subset E(G,L)^{\perp}} \mathrm{Ker}(\lambda_E)$$

(where $\mathrm{Ker}(\lambda_E) = \{u \in H^*(S(\Gamma)); j(\lambda_E) \wedge u = 0\}$). From (6), we obtain that $\mathrm{Ker}(\lambda_E)$ is defined over $\mathbb{Q}$. Hence

$$\mathrm{Hod}^i(A_{\mathbf{q}}) \otimes \mathbb{C} \subset \bigcap_{E \subset E(G,L)^{\perp}} \mathrm{Ker}(\lambda_E).$$

Suppose $E(G, L')$ does not contain $E(G, L)$ (as is assumed in Theorem 1). Then there exists some $E \subset E(G, L')^{\perp} - E(G, L)^{\perp}$ i.e. $E \subset E(G, L) \cap E(G, L')^{\perp}$. Hence wedging with λ_E is nonzero on $\mathrm{Hod}^i(A_{\mathbf{q}})$. But the equation (2′) shows that wedging with λ_E is 0 on $H^i(A_{\mathbf{q}'}, \Gamma)$ (in the equation (2′), we replace i by $i' = \dim(\mathbf{u}(\mathbf{q}') \cap \mathfrak{p})$ and m by $i - i'$). To see this first observe that $j(\lambda_E) \wedge e(\mathbf{q}')^* = 0$ because of Lemmas (1.5) and (1.3). Therefore $e(\mathbf{q}')^* \wedge (\xi)^* \wedge j(\lambda_E) = 0$ for all $(\xi)^* \in (\wedge(\mathbf{l} \cap \mathfrak{p})^*)^{L \cap K}$ (see (2′)). Thus wedging by λ_E is zero on $\mathrm{Hod}^i(A_{\mathbf{q}'})$ as well.

Consider

$$\mathrm{Ker}(\lambda_E) = \{x \in \oplus H^i(A_{\mathbf{q}''}, \Gamma); j(\lambda_E) \wedge x = 0\}.$$

Here the summation is over $A_{\mathbf{q}''}$ which contribute to cohomology in degree i (in particular, $\dim(\mathbf{u}(\mathbf{q}'') \cap \mathfrak{p}) \leqslant i$). Write $x = \sum x(\mathbf{q}'')$ accordingly. Let $W(\mathbf{q}'')$ be the isotypical component of $V(\mathbf{q}'')$ in $\overset{i}{\wedge}\mathfrak{p}$. Thus $j(\lambda_E) \wedge x = 0 \Leftrightarrow \lambda_E \wedge (\sum W(\mathbf{q}'')) = 0$. The K-invariance of λ_E and the independence of the $W(\mathbf{q}'')$ now imply that $\lambda_E \wedge x = 0 \Leftrightarrow \lambda_E \wedge x(\mathbf{q}'') = 0$ for all $\mathbf{q}''$ with $x(\mathbf{q}'') \neq 0$. We have thus proved that

$$\mathrm{Ker}(\lambda_E) = \oplus(\mathrm{Ker}(\lambda_E) \cap H^i(A_{\mathbf{q}''}, \Gamma)) \qquad (*)$$

where the sum is over all $\mathbf{q}''$ with $\lambda_E \wedge w = 0$ for some *nonzero* element $w \in W(\mathbf{q}'')$.

Note that if the isotypical component $W(\mathbf{q}'')$ is irreducible, then the K invariance of λ_E implies that if $\lambda_E \wedge w \neq 0$ for *some* nonzero $w \in W(\mathbf{q}'')$ then $\lambda_E \wedge w \neq 0$ for *all* nonzero $w \in W(\mathbf{q}'')$.

Now $V(\mathbf{q})$ occurs with multiplicity one in $\overset{i}{\wedge}\mathfrak{p}$ (this follows, for example, from (2)). As $E \subset E(G, L)$, it now follows, from Lemmas (1.5), and (1.4), and from the previous paragraph, that wedging with λ_E is injective on $H^i(A_{\mathbf{q}}, \Gamma)$. Thus, $\mathbf{q}$ is not one of the $\mathbf{q}''$ which occur in the decomposition $(*)$. Hence $H^i(A_{\mathbf{q}}, \Gamma)$ is orthogonal to $H^i(A_{\mathbf{q}''}, \Gamma)$ for all $\mathbf{q}''$ as in $(*)$. In particular, $H^i(A_{\mathbf{q}}, \Gamma) \subset \mathrm{Ker}(\lambda_E)^{\perp}$.

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But we know that if $V \subset H^i(S(\Gamma), \mathbb{C})$ is defined over $\mathbb{Q}$, then $V^\perp$ is also defined over $\mathbb{Q}$: if $\alpha, \beta \in H^i(S(\Gamma), \mathbb{Q})$, then the $\mathbb{Q}$-linear pairing

$$\langle \alpha, \beta \rangle = \alpha \wedge *\beta \in H^{2d}(S(\Gamma), \mathbb{Q})$$

is the metric defined in the paragraph preceding Lemma (1.5). Therefore $\mathrm{Ker}(\lambda_E)^\perp$ is defined over $\mathbb{Q}$ and therefore: $\mathrm{Hod}^i(A_{\mathbf{q}}) \subset \mathrm{Ker}(\lambda_E)^\perp$. In particular

$$\mathrm{Hod}^i(A_{\mathbf{q}}) \cap Hod^i(A_{\mathbf{q}'}) = (0).$$

This proves Theorem 1.

(1.7) *Remark*. We have proved a stronger statement, namely that for the natural inner product (defined over $\mathbb{Q}$) on the cohomology of $S(\Gamma)$, the spaces $\mathrm{Hod}^{\mathrm{i}}(\mathrm{A}_{\mathbf{q}})$ and $\mathrm{Hod}^{\mathrm{i}}(\mathrm{A}_{\mathbf{q}'})$ are actually orthogonal. We are grateful to the referee for pointing this out and also for indicating a simpler argument to prove the weaker statement of Theorem 1.

2. Proof of Theorem 2

2.1. DEFINITIONS AND NOTATION

Let $p, q \geqslant 1$ be integers, put $n = p + q$. On $\mathbb{C}^n$ fix the standard basis $\epsilon_1, \cdots, \epsilon_n$. With respect to this basis, denote $z \in \mathbb{C}^n, z = \sum_i z_i \epsilon_i$ by $z = (z_1, \cdots, z_n)$. Consider the Hermitian form $h_\infty : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ such that

$$h_\infty(z, w) = \lambda_1 z_1 \overline{w}_1 + \cdots + \lambda_p z_p \overline{w}_p + \mu_1 z_{p+1} \overline{w}_{p+1} + \cdots + \mu_q z_n \overline{w}_n$$

where $\lambda_1, \cdots, \lambda_p$ are real numbers, all strictly greater than 0, and $\mu_1, \cdots, \mu_q$ are real numbers, all strictly less than 0. The subgroup of $\mathrm{GL}_n(\mathbb{C})$ preserving this Hermitian form is denoted $\mathrm{U}(p, q)$. We will assume from now on that $2 \leqslant p \leqslant q$. Let $K = \mathrm{U}(p) \times \mathrm{U}(q)$, T = Diagonals in K. Now $\mathbf{t} = \mathrm{Lie}(T) \otimes \mathbb{C}$ is the set of diagonals in $M_n(\mathbb{C})$; we will view elements of it as n complex numbers $Y = (y_1, \cdots, y_n) \in \mathbf{t}$. Then $i\,\mathbf{t}_0 = i\,\mathrm{Lie}(T) = \{(x_1, \cdots, x_p, y_1, \cdots, y_q); x_i, y_i \in \mathbb{R}\}$. Choose $B_K \subset K_{\mathbb{C}} = \mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C})$ to be the subgroup of matrices of $K_{\mathbb{C}}$ which are upper triangular in $\mathrm{GL}_p(\mathbb{C})$ and lower triangular in $\mathrm{GL}_q(\mathbb{C})$ and set $b_K = \mathrm{Lie}(B_K)$. Then

$$\Phi(\mathbf{b}_K, \mathbf{t}) = \{x_i - x_j : 1 \leqslant i < j \leqslant p\} \cup \{y_j - y_i; 1 \leqslant i < j \leqslant q\}.$$

If $X \in i\,\mathbf{t}_0$ is such that $\alpha(X) \geqslant 0$ for $\alpha \in \Phi(\mathbf{b}_K, \mathbf{t})$, then it satisfies the inequalities

$$x_1 \geqslant \cdots \geqslant x_p, y_q \geqslant y_{q-1} \geqslant \cdots \geqslant y_1.$$

Consider first the parabolic subalgebras $\mathbf{q} = \mathbf{q}(X)$ which contribute to *holomorphic* cohomology i.e. $\mathbf{u} \cap \mathfrak{p}^- = 0$. Then $x_i - y_j \geqslant 0$ for all $i \leqslant p, j \leqslant q$. As in [Clo-Ven], (3A1) we fix $0 \leqslant r \leqslant p, 0 \leqslant s \leqslant q$ such that the inequalities

$$: x_1 \geqslant \cdots \geqslant x_r > x_{r+1} = \cdots = x_p = y_q = \cdots = y_{s+1} > y_s \geqslant \cdots \geqslant y_1, \qquad (*)_{r,s}$$

hold . Write $\mathbf{q}(r,s)$ for $\mathbf{q}(X)$ if X satisfies $(*)_{r,s}$. *We consider only those* $\mathbf{q}(X)$ *for which* $rs = 0$.

If $r = 0$, then

$$X = (\underbrace{a, \cdots, a}_{p-\text{times}}, y_1, \cdots, y_s, \underbrace{a, \cdots, a}_{(q-s)-\text{times}}), \quad \text{and} \quad y_1 \leqslant \cdots \leqslant y_s < a.$$

Let $\mathbf{q} = \mathbf{q}(X), \mathbf{l} = \mathbf{l}(\mathbf{q})$. Then

$$L/L \cap K = \frac{U(p, q-s)}{U(p) \times U(q-s)}, \qquad L_u/L_{\mathbf{u}} \cap K = \frac{U(p+q-s)}{U(p) \times U(p)}.$$

Further, $\dim(\mathbf{u} \cap \mathfrak{p}) = \dim(\mathbf{u} \cap \mathfrak{p}^+) = ps$.

Suppose $\mathbf{q}' = \mathbf{q}(X')$ be another parabolic subalgebra as in (1.1), such that $A_{\mathbf{q}'}$ has strongly primitive (but not necessarily holomorphic) cohomology in degree ps (i.e. $\dim(\mathbf{u}(\mathbf{q}') \cap \mathfrak{p}) = ps$). As before, write $X' = (a'_1, \cdots, a'_p, b'_1, \cdots, b'_q)$. Let $\mathbf{u}' = \mathbf{u}(\mathbf{q}'), \mathbf{l}' = \mathbf{l}(\mathbf{q}'), L' = L(\mathbf{q}')$. Define the partition $I_1 \coprod I_2 \coprod \cdots \coprod I_\ell$ of $\{1, \cdots, p\}$ by the conditions

(1) $a'_i = a'_j$ for all $i, j \in I_\mu \quad (\mu = 1, 2, \cdots, \ell)$,
(2) $a'_i > a'_j$ for all $i \in I_\mu, j \in I_{\mu+1} \quad (\mu, 1, 2, \cdots, \ell - 1)$.

Let $F = \{j; 1 \leqslant j \leqslant q; \quad b'_j \neq a'_i$ for *any* i, with $1 \leqslant i \leqslant p\}$. Define the partition $J_1 \coprod \cdots \coprod J_\ell$ of $\{1, 2, \cdots, q\} - F$ by the conditions

$$J_\mu = \{j; 1 \leqslant j \leqslant q \text{ and } b'_j = a'_i \text{ for all } i \in I_\mu\}.$$

Let $i_\mu = \mathrm{Card}(I_\mu), j_\mu = \mathrm{Card}(J_\mu)$. Then it is easy to see that

$$L'/L' \cap K = \prod_{\mu=1}^{\ell} \quad \mathrm{U}(i_\mu, j_\mu)/\mathrm{U}(i_\mu) \times \mathrm{U}(j_\mu),$$

$$L'_u/L'_u \cap K = \prod_{\mu=1}^{\ell} \quad \mathrm{U}(i_\mu + j_\mu)/\mathrm{U}(i_\mu) \times \mathrm{U}(j_\mu).$$

Here $\mathrm{U}(i_\mu) \subset \mathrm{U}(p)$ is the subgroup which fixes the elements $\{\epsilon_k; k \notin I_\mu, 1 \leqslant k \leqslant p\}$ and similarly $\mathrm{U}(j_\mu) \subset \mathrm{U}(q)$ fixes $\{\epsilon_{k+p}; k \notin J_\mu, 1 \leqslant k \leqslant q\}$.

(2.2) LEMMA. *With the above notation, suppose that* $E(G, L) \subset E(G, L')$, *that* $A_{\mathbf{q}'}$ *contributes to strongly primitive cohomology in degree* $i = ps$ *and that*

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$r = 0 \leqslant s < q/2$. Then $\ell = 1$ and

$$\dim(\mathbf{u}' \cap \mathfrak{p}^+) = pu, \dim(\mathbf{u}' \cap \mathfrak{p}^-) = p(s-u)$$

for some u with $0 \leqslant u \leqslant s$.

If Γ is any cocompact arithmetic subgroup of $\mathrm{U}(p,q)$ as in the introduction, then the only Hodge types of the Hodge structure $\mathrm{Hod}^i(A_{\mathbf{q}})$ are of the form $(pu, p(s-u))$.

Proof. We use some computations from [Clo-Ven]. Suppose $H/K \cap K$ is a Hermitian symmetric subdomain of the Hermitian symmetric domain G/K, and $E(G, H, r)$ the K-span of $\wedge^r \mathfrak{p}_H^+$ in $\wedge^r \mathfrak{p}^+$. Let $\mathbf{q}_0 \subset \mathfrak{g}$ be a standard θ-stable parabolic subalgebra associated to an element $X_0 \in i\mathfrak{t}_0 : \mathbf{q}_0 = \mathbf{q}(X_0)$ is associated to

$$X_0 = (c_1, \cdots, c_p; d_1, \cdots, d_q)$$

with $c_1 \geqslant \cdots \geqslant c_p; d_q \geqslant \cdots \geqslant d_1$. Suppose that $\dim(\mathfrak{u}_0 \cap \mathfrak{p}) = \dim(\mathfrak{u}_0 \cap \mathfrak{p}^+) = r$. As usual, denote by $V(\mathbf{q}_0)$ the K-span of $\wedge^r(\mathfrak{u}_0 \cap \mathfrak{p}^+)$ in $\wedge^r(\mathfrak{p}^+)$.

According to [Clo-Ven], the space $E(G, H, r)$ contains $V(\mathbf{q}_0)$ in the following cases.

Case (1). Suppose $G = \mathrm{U}(p, q)$ and $H = U(p, b)$ with $b < q$. Here H is the subgroup of G which fixes the last $q - b$ elements of the basis $\epsilon_1, \cdots, \epsilon_n$ $(n = p + q)$. Assume that $pb \geqslant r$. Choose $X_0 = (c_1, \cdots, c_p,\ d_1, \cdots, d_q)$ with $c_1 = \cdots = c_p = d_q = \cdots = d_{m+1} > d_m \geqslant d_{m-1} \geqslant \cdots \geqslant d_1$. In the notation of (2.1), $\mathbf{q}_0 = \mathbf{q}(0, m)$. Note that $r = \dim(\mathfrak{u}_0 \cap \mathfrak{p}^+) = pm \leqslant pb$. By (3.A.5) of [Clo-Ven], $V(\mathbf{q}_0) \subset E(G, H, pm)$.

Case (2). $G = \mathrm{U}(p, q)$ and $H = \mathrm{U}(p_1, q_1) \times \ldots \times \mathrm{U}(p_l, q_l)$ with $\sum p_i \leqslant p$ and $\sum q_i \leqslant q$. Let $\mathbf{q}_0 = \mathbf{q}(0, m)$ be as in Case (1). By (3.A.8) of [Clo-Ven], $E(G, H, pm) \supset V(\mathbf{q}_0)$ if and only if $\sum p_i = p$ and $q_i \geqslant m$ for each i with $1 \leqslant i \leqslant l$.

Now take $H = L$ and $H' = L'$ in the last two paragraphs. Suppose that $m = q - s (= b)$. By Case (1) of the foregoing, $E(G, L, p(q-s)) \supset V(\mathbf{q}_0)$. By the assumptions of the Lemma, $E(G, L', p(q-s)) \supset E(G, L, p(q-s)) \supset V(\mathbf{q}_0)$. By (2) of the last paragraph, this implies that $q_i \geqslant q - s$ for each i, i.e., $l(q-s) \leqslant \sum q_i \leqslant q$. Thus, $s \geqslant q(1 - 1/l)$. As $s < q/2$ by assumption, we get $l = 1$ and $p_1 = p$.

In the notation preceding (2.2) (esp. the inequalities 1 and 2), we get $I_1 = (1, 2, \cdots, p)$ from the fact that $l = 1$. Now $X' = (a_1', \cdots, a_p', b_1', \cdots, b_q')$ is such that $a_1' = \cdots = a_p' = a'$. Now $b_q' \geqslant \ldots \geqslant b_1'$. Write, as before, $J_1 = \{j; 1 \leqslant j \leqslant q, b_j' = a'\}$. Then, $J_1 = \{u+1, u+2, \cdots, q-v\}$ for some u, v with $u \leqslant q - v - 1$. Thus, $\mathbf{u}' \cap \mathfrak{p}^+$ is spanned by the root spaces corresponding the roots $x_i - y_j$, where $j \leqslant u$ and i is arbitrary. The dimension of $\mathbf{u}' \cap \mathfrak{p}^+$ is therefore equal to pu. Similarly, we get $\dim(\mathbf{u}' \cap \mathfrak{p}^-) = pv$. Since $\dim(\mathbf{u}' \cap \mathfrak{p}) = ps$, we obtain that $u + v = s$. This proves the first part of the Lemma.

As to the second part, note that if $\mathbf{q}'$ is any parabolic subalgebra such that $E(G, L)$ is not contained in $E(G, L')$, then by Theorem 1, $\mathrm{Hod}^i(A_{\mathbf{q}'})$ lies in the orthogonal complement to $\mathrm{Hod}^i(A_{\mathbf{q}})$. On the other hand, if $E(G, L) \subset E(G, L')$ then exactly the same proof as in the first part of our Lemma (note that we did not need that $\mathbf{q}'$ had strongly primitive cohomology in degree i to conclude that $l = 1$ and that $p_1 = p$) shows that $L' = \mathrm{U}(p, q_1)$. Thus, the compact symmetric space associated to L' is the Grassmannian $\mathbf{G}_{p,b+p}$ of p planes in $\mathbb{C}^{b+p}$. Now, the compact dual $\widehat{X}$ is $\mathbf{G}_{p,p+q}$.

Now, the restriction map from the cohomology of $\mathbf{G}_{p,p+q}$ to that of $\mathbf{G}_{p,p+b}$ is *surjective* (e.g. see [Par 2]). Therefore, if $E(G, L) \subset E(G, L')$ and ω' is a (not necessarily strongly primitive) cohomology class in degree $i = ps$ of type $A_{\mathbf{q}'}$, then, by the remark following (2′) of (1.1), the class ω' is of the form $\omega' = \omega'' \wedge \alpha$ where ω'' is strongly primitive and $\alpha \in H^k(\widehat{X})$. Hence, $\mathrm{Hod}^i(A_{\mathbf{q}'})$ is contained in the image of wedging with $H^k(\widehat{X})$. However, the classes in $H^i(A_{\mathbf{q}})$ are holomorphic and hence strongly primitive. If $k \geqslant 1$ then the image I_k of wedging with $H^k(\widehat{X})$ cannot contain *any* strongly primitive class. Hence, the orthogonal complement of I_k (is defined over $\mathbb{Q}$ and) contains $H^i(A_{\mathbf{q}})$ and therefore contains $\mathrm{Hod}^i(A_{\mathbf{q}})$.

Therefore, if $\mathrm{Hod}^i(A_{\mathbf{q}'})$ and $\mathrm{Hod}^i(A_{\mathbf{q}})$ are not orthogonal, it means that $E(G, L) \subset E(G, L')$ (Theorem 1) and also that $\mathbf{q}'$ contributes to strongly primitive cohomology in degree $i = ps$. Now we can apply the result of the first part of our Lemma to conclude that the Hodge types of $H^i(A_{\mathbf{q}'}, \Gamma)$ are of the form $(pu, p(s - u))$ for some $u \leqslant s$. This proves the second part of the Lemma.

(2.3) NOTATION. In this section, we specify the kind of discrete subgroups Γ of $G = U(p, q)$ which we will consider in Theorem 2. Let $F/\mathbb{Q}$ be a totally real number field of degree $d \geqslant 2$. Let E/F be a totally imaginary quadratic extension, we denote by $z \mapsto \bar{z}$ the nontrivial automorphism of E fixing F pointwise. Let V be an n-dimensional E-vector space with $n = p + q$, and let h: $V \times V \to E$ be a Hermitian form defined as follows: fix an infinite place of F and let $(\mathbb{R} \simeq) F_\infty$ the resulting completion. Let $\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_q \in F$ be such that $\lambda_1, \cdots, \lambda_p$ are all positive in all the real completions of F; $\mu_1, \cdots, \mu_q$ are all positive in all the real completions of F *except* F_∞ where they are all negative. Fix a basis $e_1, \cdots, e_p; f_1, \cdots, f_q$ of V and if $z = (z_1, \cdots, z_p, z_{p+1}, \cdots, z_n) \in V(E)$, set

$$h(z, z) = \lambda, z_1\bar{z}_1 + \cdots + \lambda_p z_p \bar{z}_p + \mu_1 z_{p+1}\bar{z}_{p+1} + \cdots + \mu_q z_n \bar{z}_n.$$

Then $\mathrm{SU}(h)$, the unitary group of h, is an algebraic group over F; clearly

$$\mathrm{U}(h)(F \otimes_{\mathbb{Q}} \mathbb{R}) = \mathrm{U}(p, q) \times \mathrm{U}(p + q)^{d-1}.$$

Then a congruence subgroup Γ of $\mathrm{U}(h)(F)$ which is neat (when projected to $\mathrm{U}(F_\infty) = \mathrm{U}(p, q)$) gives a cocompact lattice in $\mathrm{U}(p, q)$ verifying the hypothesis of the Introduction. These are the $\Gamma's$ considered in Theorem 2.

(2.4) We now begin the proof of Theorem 2. Assume that the θ-stable parabolic subalgebra is of the form $\mathbf{q} = \mathbf{q}(0, s)$ with $0 < s < q/2$. Fix $a > 0, b > 0$ with $a + b = p$ (since $p \geqslant 2$, this is possible). Put $q_1 = q_2 = s$. As $2s < q$, we get $q_1 + q_2 q - 1$. Let $G_1 = \mathrm{U}(a, q_1), G_2 = U(b, q_2)$, where G_1 (resp. G_2) is the subgroup of $\mathrm{U}(p, q)$ which fixes the vectors ϵ_j if $a + 1 \leqslant j \leqslant p$ or if $p + q_1 + 1 \leqslant j \leqslant n = p + q$ (resp. the vectors ϵ_j if $1 \leqslant j \leqslant a$, or $p + 1 \leqslant j \leqslant p + q_1$ or $p + q_1 + q_2 + 1 \leqslant j \leqslant n$). Then $\Gamma_i = \Gamma \cap G_i$ is cocompact in $G_i (i = 1, 2)$. By [Clo-Ven] (3A8), $H^{ps,0}(\Gamma, A_{\mathbf{q}})$ injects into a sum of $H^{as,0}(S(\Gamma_1')) \otimes H^{bs,0}(S(\Gamma_2'))$, where Γ_1', Γ_2' are subgroups of G_1, G_2 commensurate to Γ_1, Γ_2.

Note that for any smooth projective variety X, the holomorphic part $H^{i,0}(X)$ of the cohomology lies in the *primitive* part of the cohomology of the Lefschetz decomposition of $H^i(X, \mathbb{C})$ (see [Kleiman] for a definition). This primitive part $H^i_{\mathrm{prim}}(X)$, is defined over $\mathbb{Q}$. Therefore, the last sentence of the foregoing paragraph implies that $\mathrm{Hod}^{\mathrm{ps}}(\mathrm{A}_{\mathbf{q}})$ injects (under the restriction map) into a sum of $H^{as}_{\mathrm{prim}}(S(\Gamma_1')) \otimes H^{bs}_{\mathrm{prim}}(S(\Gamma_2'))$.

Suppose now that $Z \subset \mathrm{Hod}(A_{\mathbf{q}})$ is an *irreducible* $\mathbb{Q}$-Hodge structure. Then $Z^{ps,0} \neq 0$, for, otherwise $Z^{\perp} \otimes \mathbb{C}$ contains $H^{ps,0}(S(\Gamma), \mathbb{C}) \supset H^{ps,0}(A_{\mathbf{q}}, \Gamma)$ and so, $Z^{\perp} \supset \mathrm{Hod}(A_{\mathbf{q}})$ (since $\mathrm{Hod}(A_{\mathbf{q}})$ is the *smallest* $\mathbb{Q}$-Hodge structure whose complex points contain $H^{ps}(A_{\mathbf{q}}, \Gamma)$), but $Z \cap Z^{\perp} = 0$ whence this is impossible.

Now $\mathrm{Hod}^{\mathrm{ps}}(\mathrm{A}_{\mathbf{q}})$ is disjoint from $\mathrm{Hod}^{\mathrm{ps}}(\mathrm{A}_{\mathbf{q}'})$ if $\mathbf{q}'(\neq \mathbf{q})$ contributes to holomorphic cohomology in degree $ps = R$. To see this, suppose $\mathbf{q}' = \mathbf{q}(r', s')$. If $r' = 0$, then the primitive cohomology of $A_{\mathbf{q}'}$ is in degree $ps' = ps$ i.e. $s' = s$ and $\mathbf{q}' = \mathbf{q}$. We may thus suppose that $r \neq 0$. Then $L' = U(p - r', q - s)$ with $p - r' < p$, and $L = U(p, q - s)$ with $q - s > s$. Since $\mathbf{q} = \mathbf{q}(0, s)$, it follows from (3A5) of [Clo-Ven] that $E(G, L, s) \supset V(\mathbf{q})$. But, by (3A5) of [Clo-Ven] again, $E(G, L', s)^{\perp} \supset V(\mathbf{q})$ (since $p - r' < p$). Therefore, $E(G, L, s) \neq E(G, L', s)$. Now, both $A_{\mathbf{q}}$ and $A_{\mathbf{q}'}$ contribute to strongly primitive cohomology in degree $i = ps$. Thus, by Theorem 1, we obtain that $\mathrm{Hod}^{\mathrm{ps}}(\mathrm{A}_{\mathbf{q}})$ is disjoint from $\mathrm{Hod}^{\mathrm{ps}}(\mathrm{A}_{\mathbf{q}'})$.

In conclusion, $Z^{ps,0} \subset H^{ps,0}(A_{\mathbf{q}}, \Gamma)$ and therefore $Z^{ps,0}$ injects into some product $H^{as,0}(S(\Gamma_1)) \otimes H^{bs,0}(S(\Gamma_2))$ by (3 A 8) of [Clo-Ven]. As observed before, the latter is a subspace of $H^{as}_{\mathrm{prim}}(S(\Gamma_1)) \otimes H^{bs}_{\mathrm{prim}}(S(\Gamma_2))$.

From (2.2) and Theorem 1, it follows that if $\pi' = A_{\mathbf{q}'}$ contributes to cohomology in degree ps (and $\pi = A_{\mathbf{q}}$), then the Hodge types of $H^{ps}(A_{\mathbf{q}'}, \Gamma)$ are of the form $(pu, p(s - u))$ unless $\mathrm{Hod}^{\mathrm{ps}}(\pi')$ and $\mathrm{Hod}^{\mathrm{ps}}(\pi)$ are disjoint Hodge structures. By assumption, Z is an irreducible Hodge structure. Therefore, the only Hodge types of Z are of the form $Z^{pu,p(s-u)} (0 \leqslant u \leqslant s)$ and

$$Z^{pu,p(s-u)} \subset \oplus X^{\alpha,\beta} \otimes Y^{\gamma,\delta} \tag{7}$$

with $\alpha + \beta = as, \gamma + \delta = bs, \alpha + \gamma = pu, \beta + \delta = p(s - u)$.

(2.5) LEMMA. $\alpha = au, \gamma = bu$.

Proof. To see this, first note that if $\mathbf{q}'$ is as in (2.1) such that the image of the cohomology group $H^{pu,p(s-u)}(A_{\mathbf{q}'}, \Gamma)$ in $H^{as}_{\mathrm{prim}}(S(\Gamma_1)) \otimes H^{bs}_{\mathrm{prim}}(S(\Gamma_2))$ is nonzero, then the $K_{\mathbb{C}}$-span of $\overset{as}{\wedge}\mathfrak{p}_{G_1}\otimes\overset{bs}{\wedge}\mathfrak{p}_{G_2}$ must contain $V(\mathbf{q}')$. For, otherwise, this $K_{\mathbb{C}}$ -span lies in the orthogonal complement of $V(\mathbf{q}')$. Now, a form ω' on $S(\Gamma)$ of type $A_{\mathbf{q}'}$ restricted to the product of $S(\Gamma_1)$ with $S(\Gamma_2)$ vanishes on the tangent space to the identity coset in the product (since the tangent space is in the orthogonal complement of $V(\mathbf{q}')$). This is true of any $G(\mathbb{Q})$- translate of the form ω' (this translate is a form on some finite covering of $S(\Gamma)$). The density of $G(\mathbb{Q})$ in $G(\mathbb{R})$ (and arguments similar to those in [Clo-Ven], section 1), then imply that the restriction of the *differential form* ω' to the product vanishes. This contradicts our assumption that the image of ω' in $H^{as}_{\text{prim}}(S(\Gamma_1))\otimes H^{bs}_{\text{prim}}(S(\Gamma_2))$ is non-zero.

If $\omega\in Z$, then write it as a sum of forms of type ω' as in the Matsushima decomposition (1). Here each ω' is of type $A_{\mathbf{q}'}$ for some $\mathbf{q}'$. We restrict our attention only on those ω' which are nonzero. Since Z is irreducible, and injects into the cohomology of the product of $S(\Gamma'_1)$ with $S(\Gamma'_2)$, it follows that all these ω' also are nonzero in $H^{as}_{\text{prim}}(S(\Gamma'_1)\otimes H^{bs}_{\text{prim}}(S(\Gamma'_2))$. Therefore, the conclusions of the last paragraph hold for these $\mathbf{q}'$. Note also that ω' is strongly primitive, because of (2.2) (see the last paragraph of the proof of (2.2)).

Let us now prove Lemma (2.5). Let ω be an element of the left hand side of (7). Write ω as a sum of ω' as in the preceding paragraph. Fix a ω' and the corresponding $\mathbf{q}'$. Then,

$$u(x_1+\cdots+x_p)=\sum m_{ij}(x_i-x_j)+u_1x_1+\cdots+u_ax_a+v_1x_{a+1}+\cdots+v_bx_{a+b} \tag{8}$$

with $\sum u_i=\alpha, \sum v_i=\gamma$ and $m_{ij}\geqslant 0$. For, let $\mathbf{q}_1,\mathbf{q}_2$ be parabolic subalgebras of $\mathrm{Lie}(G_1)\otimes\mathbb{C}$ and $\mathrm{Lie}(G_1)\otimes\mathbb{C}$ contributing to cohomology in degrees (as) and (bs) respectively, and let $e(\mathbf{q}_1)$ and $e(\mathbf{q}_2)$ be the analogues of $e(\mathbf{q})$ for the pairs $(G_1,\mathbf{q}_1)$ and $(G_2,\mathbf{q}_2)$. Now $\wedge^{as}(\mathbf{p}_1)$ (respectively $\wedge^{bs}(\mathbf{p}_2)$ contains a subrepresentation (cf. Equation (2$'$)) isomorphic to $V(\mathbf{q}_1)$ (resp. $V(\mathbf{q}_2)$). Let e_1 be the vector in $\wedge^{as}(\mathbf{p}_1)$ corresponding to $e(\mathbf{q}_1)$ in this subrepresentation. Similarly define e_2.

Consider the projection π' of $E(G,G_1\times G_2,ps)$ to $V(\mathbf{q}')$ and look at the image v' of $(e_1\otimes e_2)$ under π'. After applying an element $\alpha\in U(\mathbf{b}_K)$ to v' we get $e(\mathbf{q}')$ because $e(\mathbf{q}')$ is the unique highest weight vector of $V(\mathbf{q}')$. Since $u(\mathbf{b}_K\cap\mathrm{Lie}(G_1)\otimes\mathbb{C})$ and $u(\mathbf{b}_K\cap Lie(G_2)\otimes\mathbb{C})$ fix the lines e_1 and e_2, by the Poincare-Birkhoff-Witt Theorem, we may assume that

$$\alpha\in u(\mathbf{b}_3),\mathbf{b}_3=\text{span of } E_{ij}(i\leqslant a<j)\subset\mathbf{b}_K.$$

Thus $e(\mathbf{q}')=\alpha(\pi'(e_1\otimes e_2))$ and we may assume that α is a weight vector for T. By comparing the T-weights in this equation, we get (8) (Cf. (3.6) and (3.7) in (3A8) of [Clo-Ven]).

Now (8) shows that $u_i\leqslant u\leqslant v_j$. By switching the roles of G_1 and G_2 (this is possible by conjugating by a permutation matrix in K and, hence, does not affect our conclusions), we obtain that $v_j\leqslant u\leqslant u_i$, whence $u_i=v_j=u$ for all i,j, and so, $\alpha(=\sum u_i)=au,\ \gamma=(\sum v_j)=bu$. This completes the proof of Lemma (2.5).

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(2.6). HODGE STRUCTURES. Before continuing the proof of Theorem 2, we introduce some notation. If V is a $\mathbb{Q}$-Hodge structure, let G_V be its Mumford–Tate group, $\mathfrak{g}_V$ its Lie algebra. Since $\mathfrak{g}_V \subset V \otimes V^*$, $\mathfrak{g}_V$ also has a $\mathbb{Q}$-Hodge structure. If $V \otimes \mathbb{C} = \bigoplus_{r=0}^{m} V^{r,m-r}$, then

$$(V \otimes V^*) \otimes \mathbb{C} = \bigoplus_{m \geqslant r,s \geqslant 0} V^{r,m-r} \otimes (V^*)^{-s,s-m}$$

$$= \bigoplus_{|u| \leqslant m} (V \otimes V^*)^{u,-u}.$$

If $Z \subset X \otimes Y$ is a $\mathbb{Q}$-Hodge structure in a product of $\mathbb{Q}$-Hodge structures X, Y, we have surjections (defined over $\mathbb{Q}$) from $G_{X \oplus Y}$ onto G_X, G_Y and G_Z.

We return to our $Z \subset \mathrm{Hod}^{\mathrm{ps}}(\mathrm{A}_{\mathbf{q}})$. By (2.2) (cf. the discussion preceding the equation (7)), the Hodge types of Z are of the form $(pu, p(s-u))$, whence those of $Z \otimes Z^*$ are

$$(Z)^{pu,p(s-u)} \otimes (Z^*)^{(-ps,-p(s-v))} \subseteq \bigoplus_{|w| \leqslant s} (Z \otimes Z^*)^{pw,-pw}.$$

Hence $\mathfrak{g}_Z \otimes \mathbb{C} = \bigoplus_{|w| \leqslant s} \mathfrak{g}_Z^{(-pw,pw)}$. The surjection $\mathfrak{g}_{X \oplus Y} \to \mathfrak{g}_Z$ is a morphism of Hodge structures, hence $\mathfrak{g}_{X \oplus Y}^{(-pw,pw)}$ maps onto $\mathfrak{g}_Z^{(-pw,pw)}$. Applying $\mathfrak{g}_{X \oplus Y}^{(-pw,pw)}$ to the inclusion (7) we obtain

$$\mathfrak{g}_Z^{(-pw,pw)}(\mathrm{LHS}) \subset \oplus \mathfrak{g}_X^{(-pw,pw)}(X^{\alpha,\beta}) \otimes Y^{\gamma,\delta}$$

$$\oplus X^{\alpha,\beta} \otimes \mathfrak{g}_Y^{-pw,pw}(Y^{\gamma,\delta}). \tag{9}$$

By the Lemma (2.5) proved above, $\alpha = au, \gamma = bu$. On the other hand,

$$\mathfrak{g}_Z^{(-pw,pw)}(Z^{pu,p(s-u)}) \subset Z^{pu-pw,p(s-u)+pw}$$

is contained in a direct sum of $X^{\epsilon,\varphi} \otimes Y^{\psi,\eta}$ such that $\epsilon = a(u-w)$ and $\psi = b(u-w)$, again by the Lemma (2.5). Therefore, we must either have

$$\epsilon = a(u-w) = \alpha - pw = au - pw \quad \text{i.e.} \quad bw = 0 \quad \text{and} \quad w = 0$$

(if the left-hand side of equation (9) projects nontrivially to the first part of the direct sum on the right-hand side of (9)) or we must have

$$\psi = bu - bw = \gamma - pw = bu - pw \quad \text{i.e.} \quad aw = 0 \quad \text{and} \quad w = 0$$

(if the left-hand side of (9) projects nontrivially to the second part of the direct sum on the right-hand side of (9)).

Thus $\mathfrak{g}_Z \otimes \mathbb{C} = \mathfrak{g}_Z^{0,0}$. Therefore $\mathbb{C}^*$ is contained in the centre of G_Z. However, the centre of G_Z is a $\mathbb{Q}$-algebraic group containing $\mathbb{C}^*$; by the definition of the Mumford-

–Tate group, $G_Z \subset$ centre of G_Z i.e. G_Z is Abelian. In conclusion, $\mathrm{Hod}^{ps}(A_{\mathbf{q}})$ has Abelian Mumford–Tate group.

A similar proof shows that if $\mathbf{q} = \mathbf{q}(r, 0)$ with $r < p/2$ then $\mathrm{Hod}^{qr}(A_{\mathbf{q}})$ has Abelian Mumford–Tate group. This completes the proof of Theorem 2.

3. Proof of Theorem 3

(3.1) NOTATION. We keep the notation of (2.1). The group now is $G = \mathrm{U}(2,2)$. Let $\mathbf{q}_1 = \mathbf{q}(0,1)$, $\mathbf{q}_2 = \mathbf{q}(1,0)$ and $\mathbf{q}_3 = \mathbf{q}(X_3)$, $X_3 = (a, b, b, a)$ with $a < b$. Then it is easily checked that $A_{\mathbf{q}_1}, A_{\mathbf{q}_2}, A_{\mathbf{q}_3}$ have primitive cohomologies of type (2, 0), (2, 0) and (1, 1), respectively. We see that if $\mathbf{u}_i = \mathbf{u}(\mathbf{q}_i)(i = 1, 2, 3)$, then

$$\mathbf{u}_1 \cap \mathfrak{p} = \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{u}_2 \cap \mathfrak{p} = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{u}_3 \cap \mathfrak{p} = \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix}.$$

If $\mathbf{l}_i = \mathbf{l}(\mathbf{q}_i)(i = 1, 2, 3)$, then

$$\mathbf{l}_1 \cap \mathfrak{p} = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}, \qquad \mathbf{l}_2 \cap \mathfrak{p} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix},$$

$$\mathbf{l}_3 \cap \mathfrak{p} = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \\ 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}.$$

It is clear that $E(G, L_1, 2)$ contains $V(\mathbf{q}_1)$ but does not contain $V(\mathbf{q}_2)$. Similarly $E(G, L_2, 2)$ contains $V(\mathbf{q}_2)$ but does not contain $V(\mathbf{q}_1)$ (for proofs, see (3A5) [Clo-Ven] for L_1 and (3A6), [ibid] for L_2). Further, $E(G, L_3, 2)$ contains both $V(\mathbf{q}_1)$ and $V(\mathbf{q}_2)$ ((3A8) of [Clo-Ven]).

Set $E_i = V(\mathbf{q}_i)(i = 1, 2)$. If $\lambda_{E_1}, \lambda_{E_2}$ are as in (1.2), then, by Lemma (1.5) we get

$$\mathrm{Ker}_{\mathrm{prim}}(\lambda_{E_1}) \otimes \mathbb{C} = H^2(A_{\mathbf{q}_2}, \Gamma) \oplus \overline{H^2(A_{\mathbf{q}_2}, \Gamma)}$$

$$\mathrm{Ker}_{\mathrm{prim}}(\lambda_{E_2}) \otimes \mathbb{C} = H^2(A_{\mathbf{q}_1}, \Gamma) \oplus \overline{H^2(A_{\mathbf{q}_1}, \Gamma)}.$$

Therefore, $H^2(S(\Gamma), \mathbb{Q})$, for *any* cocompact torsion-free discrete subgroup of $\mathrm{U}(2,2)$,

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has the decomposition into $\mathbb{Q}$-Hodge structures:

$$H^2_{\mathrm{prim}} = \mathrm{Ker}_{\mathrm{prim}}(\lambda_{E_1}) \oplus \mathrm{Ker}_{\mathrm{prim}}(\lambda_{E_2}) \oplus Z$$

where $Z \otimes \mathbb{C} = H^2(A_{\mathfrak{q}_3}, \Gamma) = H^{1,1}(A_{\mathfrak{q}_3}, \Gamma) \subset H^{1,1}(S(\Gamma), \mathbb{C})$. Here, the subscript prim refers to the intersection of the relevant space with the space of primitive classes (i.e. the orthogonal complement to $H^*(S(\Gamma)) \wedge L$ in $H^*(S(\Gamma))$, with respect to the natural inner product on $H^*(S(\Gamma))$). By the Lefschetz theorem on rational classes of type (1,1), we have: Z is spanned by algebraic classes. This proves the first part of Theorem 3, namely, that the classes of type (1,1) lie in the complex points of a rational Hodge structure disjoint from a Hodge structure whose complexification contains all classes of type (2,0) and (0,2).

(3.2) We now prove the second part of Theorem 3. Assume that Γ is a congruence arithmetic subgroup of the kind considered in (2.3). Let $X = \mathrm{Ker}_{\mathrm{prim}}(\lambda_{E_1})$, $Y = \mathrm{Ker}_{\mathrm{prim}}(\lambda_{E_2})$. We have assumed that Γ is of the kind described in (2.3). Let G_1 (resp. G_2) denote the subgroup of $G = U(2,2)$ which leaves the subspace $Ke_2 \oplus Kf_2$ (resp. $Ke_1 \oplus Kf_1$) *pointwise* invariant. Then, by [Clo-Ven], (3A8), any irreducible Hodge substructure X' of X injects into $H^1(S(\Gamma_1)) \otimes H^1(S(\Gamma_2))$ for some congruence subgroups Γ_1 of $G_1 \cap \Gamma$ and Γ_2 of $G_2 \cap \Gamma$. By the Lemma (5.8) of [Clo-Ven] – since $X' \otimes \mathbb{C} = (X')^{(2,0)} \oplus (X')^{0,2}$ – the Mumford–Tate group of X' is Abelian, whence so is that of X. This finishes the proof of Theorem 3.

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