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COHEN-MACAULAY EDGE IDEAL WHOSE HEIGHT IS HALF OF THE NUMBER OF VERTICES

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Abstract. We consider a class of graphs G such that the height of the edge ideal $I(G)$ is half of the number $\sharp V(G)$ of the vertices. We give Cohen-Macaulay criteria for such graphs.

§0. Introduction

In this article, a *graph* means a simple graph without loops and multiple edges. Let G be a graph with the vertex set $V(G) = \{x_1, \dots, x_n\}$ and with the edge set $E(G)$. Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K. The *edge ideal* $I(G)$, associated to G, is the ideal of S generated by the set of all square-free monomials $x_i x_j$ so that x_i is adjacent to x_j. For this ideal, the following theorem is known.

THEOREM 0.1 (see [5]). *Suppose that G is an unmixed graph without isolated vertices. Then we have* $2\,\mathrm{height}\, I(G) \geq \sharp V(G)$.

In this article, we treat the class of graphs for which the above equality holds; that is, we consider an unmixed graph without isolated vertex with $2\,\mathrm{height}\, I(G) = \sharp V(G)$. Such a class of graphs is rich, because it includes all the unmixed bipartite graphs and all the grafted graphs. Herzog and Hibi [8] gave beautiful theorems on Cohen-Macaulay edge ideals of bipartite graphs. Our purpose in this article is to generalize their results for our class of graphs.

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It is known that a graph G in our class has a perfect matching (see [6, Remark 2.2]). We may assume that

$$(*) \qquad V(G) = X \cup Y, \qquad X \cap Y = \emptyset,$$

where $X = \{x_1, \dots, x_n\}$ is a minimal vertex cover of G and where $Y = \{y_1, \dots, y_n\}$ is a maximal independent set of G such that $\{x_1y_1, \dots, x_ny_n\} \subset E(G)$.

Hence, $\{x_1 - y_1, \dots, x_n - y_n\}$ is a system of parameters of $S/I(G)$. In Sections 3 and 4, using assumption (*), we give the following characterization of Cohen-Macaulayness, which is similar to the case of bipartite graphs (see [8, Corollary 3.5]).

THEOREM 0.2. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and with* $\operatorname{height} I(G) = n$. *Then the following conditions are equivalent.*

(1) *G is Cohen-Macaulay.*
(2) *$\Delta(G)$ is strongly connected.*
(3) *There is a unique perfect matching in G.*
(4) *G is shellable.*

Note that it includes equivalence between Cohen-Macaulayness and shellability as in the bipartite graphs (see [3]).

We also have a Cohen-Macaulay criterion which is similar to that of Herzog and Hibi [8, Theorem 3.4].

THEOREM 0.3. *Let G be a graph with $2n$ vertices, which are not isolated, and with* $\operatorname{height} I(G) = n$. *We assume conditions (*) and*

$$(**) \qquad x_iy_j \in E(G) \text{ implies } i \leq j.$$

Then the following conditions are equivalent.

(1) *G is Cohen-Macaulay.*
(2) *G is unmixed.*
(3) *The following conditions hold:*
   (i) *if $z_ix_j, y_jx_k \in E(G)$, then $z_ix_k \in E(G)$ for distinct i, j, k and for $z_i \in \{x_i, y_i\}$;*
   (ii) *if $x_iy_j \in E(G)$, then $x_ix_j \notin E(G)$.*

Although in Herzog and Hibi [8] Alexander duality plays an important role in their proof, we give a direct and elementary proof without it. The

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Herzog-Hibi criterion for bipartite graphs is discussed by other authors in the literature that give alternative proofs for it (see [7], [12]).

In Section 5, we introduce a new class of graphs which we call *B-grafted graphs*. They are a generalization of grafted graphs introduced by Faridi [4]. If G is an unmixed B-grafted graph, then we have $2\,\mathrm{height}\,I(G) = \sharp V(G)$. Hence, applying our main result, we show the following.

THEOREM 0.4. *The B-grafted graph $G(H_0; B_1, \ldots, B_p)$ is Cohen-Macaulay (resp., unmixed) if and only if every bipartite graph B_i is Cohen-Macaulay (resp., unmixed) for $i = 1, \ldots, p$.*

See Sections 1 and 5 for undefined concepts and notation.

§1. Preliminaries

In this section, we recall some concepts and a notation on graphs and on simplicial complexes that we use in the article.

Let G be a graph with the vertex set $V(G) = \{x_1, \ldots, x_n\}$ and with the edge set $E(G)$. The *induced subgraph* $G|_W$ by $W \subset V(G)$ is defined by

$$G|_W = \big(W, \{e \in E(G); e \subset W\}\big).$$

For $W \subset V(G)$, we denote $G|_{V(G)\setminus W}$ by $G - W$. For $F \subset E(G)$, we denote $(V(G), E(G) \setminus F)$ by $G - F$. For a family F of two-element subsets of $V(G)$, we denote $(V(G), E(G) \cup F)$ by $G + F$.

A subset $C \subset V(G)$ is a *vertex cover* of G if every edge of G is incident with at least one vertex in C. A vertex cover C of G is called *minimal* if there is no proper subset of C which is a vertex cover of G. A subset A of $V(G)$ is called an *independent set* of G if no two vertices of A are adjacent. An independent set A of G is *maximal* if there exists no independent set which properly includes A. Observe that C is a minimal vertex cover of G if and only if $V(G) \setminus C$ is a maximal independent set of G. And also note that $\mathrm{height}\,I(G)$ is equal to the smallest number $\sharp C$ of vertices among all the minimal vertex covers C of G. A graph G is called *unmixed* if all the minimal vertex covers of G have the same number of elements. A graph G is called *Cohen-Macaulay* if $S/I(G)$ is a Cohen-Macaulay ring, where $S = K[x_1, \ldots, x_n]$ is a polynomial ring for a field K.

Finally, a subgraph H of a graph G with $V(G) = V(H)$ is called a *perfect matching* if every connected component of H is a 2-complete graph.

See [2] and [13] for detailed information on this subject.

Set $V = \{x_1, \ldots, x_n\}$. A *simplicial complex* Δ on the vertex set V is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for all $x_i \in V$ and (ii) $F \in \Delta$ and $G \subseteq F$ imply $G \in \Delta$. An element $F \in \Delta$ is called a *face* of Δ. For $F \subset V$, we define the *dimension* of F by $\dim F = \sharp F - 1$, where $\sharp F$ is the cardinality of the set F. A maximal face of Δ with respect to inclusion is called a *facet* of Δ. If all facets of Δ have the same dimension, then Δ is called *pure*.

A pure simplicial complex Δ is called *shellable* if the facets of Δ can be given a linear order $F_1, \ldots, F_m$ such that for all $1 \leq j < i \leq m$, there exist some $v \in F_i \setminus F_j$ and some $k \in \{1, \ldots, i-1\}$ with $F_i \setminus F_k = \{v\}$.

Moreover, a pure simplicial complex Δ is *strongly connected* if for every two facets F and G of Δ there is a sequence of facets $F = F_0, F_1, \ldots, F_m = G$ such that $\dim(F_i \cap F_{i+1}) = \dim \Delta - 1$ for each $i = 0, \ldots, m-1$.

If G is a graph, we define the *complementary simplicial complex* of G by

$$\Delta(G) = \big\{A \subseteq V(G) : A \text{ is an independent set in } G\big\}.$$

Observe that $\Delta(G)$ is the Stanley-Reisner simplicial complex of $I(G)$.

A graph G is called *shellable* if $\Delta(G)$ is a shellable simplicial complex.

§2. Unmixedness

In this section, we survey unmixed graphs whose edge ideals have the height that is half of the number of vertices.

LEMMA 2.1. *Let G be an unmixed graph with nonisolated $2n$ vertices and with* height $I(G) = n$. *Then G has a perfect matching.*

This fact is written in [6, Remark 2.2]. By the lemma for an unmixed graph G with $2n$ vertices, which are not isolated, and with height $I(G) = n$, we may assume that

$$(*) \qquad V(G) = X \cup Y, \qquad X \cap Y = \emptyset,$$

where $X = \{x_1, \ldots, x_n\}$ is a minimal vertex cover of G and where $Y = \{y_1, \ldots, y_n\}$ is a maximal independent set of G such that $\{x_1y_1, \ldots, x_ny_n\} \subset E(G)$.

For the remainder of this article, set $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ for a field K, and $I(G)$ is an ideal of S. By Lemma 2.1, we have the following ring-theoretic properties of $S/I(G)$.

COROLLARY 2.2. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *We assume condition (*). Then,*

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(i) *each minimal prime ideal of* $I(G)$ *is of the form*

$$(x_{i_1}, \dots, x_{i_k}, y_{i_{k+1}}, \dots, y_{i_n}),$$

where $\{i_1, \dots, i_n\} = \{1, \dots, n\}$*;*

(ii) $\{x_1 - y_1, \dots, x_n - y_n\}$ *is a system of parameters of* $S/I(G)$*.*

For later use we give a characterization of the unmixedness for our graphs, that is, a more detailed description, but a special case of a more general result (see [10, Theorem 2.9] and see [14, Theorem 1.1] for the bipartite case).

PROPOSITION 2.3. *Let* G *be a graph with* $2n$ *vertices, which are not isolated, and with* height $I(G) = n$*. We assume condition (*). Then* G *is unmixed if and only if the following conditions hold.*

(i) *If* $z_i x_j,\ y_j x_k \in E(G)$*, then* $z_i x_k \in E(G)$ *for distinct* i*,* j*,* k *and for* $z_i \in \{x_i, y_i\}$*.*

(ii) *If* $x_i y_j \in E(G)$*, then* $x_i x_j \notin E(G)$*.*

§3. Cohen-Macaulayness

In this section, we give combinatorial characterizations of Cohen-Macaulay graphs whose edge ideals have the height that is half of the number of vertices.

First, we introduce an operator that allows us to construct a new graph. Let G be a graph with $2n$ vertices, which are not isolated, and with height $I(G) = n$. We assume condition (*).

For any $i \in [n] := \{1, \dots, n\}$, set

$$E_i := \big\{k \in [n] : x_k y_i \in E(G)\big\} \setminus \{i\},$$

and define the graph $O_i(G)$ by

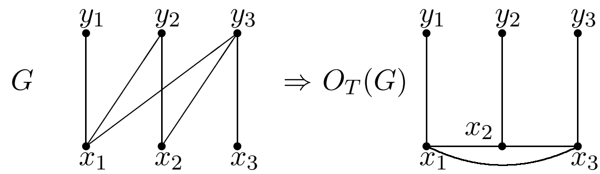
$$O_i(G) := G - \{x_k y_i : k \in E_i\} + \{x_k x_i : k \in E_i\}.$$

Then, for every nonempty subset $T := \{i_1, \dots, i_\ell\}$ of the set $[n]$, we define

$$O_T(G) = O_{i_1} O_{i_2} \cdots O_{i_\ell}(G).$$

Moreover, if $T = \emptyset$, then we set $O_T(G) = G$. Note that $O_T(G)$ is a graph with $2n$ vertices, which are not isolated, and with height $I(G) = n$, satisfying condition (*).

EXAMPLE 3.1. Let $T = \{2,3\}$; then



The next proposition shows that the Cohen-Macaulayness of G can be checked by the unmixedness of all the deformations $O_T(G)$ of G.

PROPOSITION 3.2. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *We assume condition (*). Then the following conditions are equivalent.*

(1) *G is Cohen-Macaulay.*
(2) *$O_T(G)$ is Cohen-Macaulay for every subset T of $[n]$.*
(3) *$O_T(G)$ is unmixed for every subset T of $[n]$.*

Proof. Set $S = K[x_1,\dots,x_n,y_1,\dots,y_n]$, set $S_k = K[x_1,\dots,x_n,y_{k+1},\dots,y_n]$, and set $G_k = O_{T_k}(G)|_{X\cup\{y_{k+1},\dots,y_n\}}$.

(1) $\Longrightarrow$ (2). By relabeling, we may assume that $T = [k]$. Let G be a Cohen-Macaulay graph. Then

$$S/\big(I(G) + (x_1 - y_1,\dots,x_k - y_k)\big) \simeq S_k/\big(I(G_k) + (x_1^2,\dots,x_k^2)\big)$$

is Cohen-Macaulay. Since the polarization preserves Cohen-Macaulayness,

$$S/\big(I(G_k) + (x_1^2,\dots,x_k^2)\big)^{\text{pol}} = S/\big(I(G_k) + (x_1y_1,\dots,x_ky_k)\big) = S/I\big(O_T(G)\big)$$

is Cohen-Macaulay, where $(x_1^2,\dots,x_k^2)^{\text{pol}}$ stands for the polarization of $(x_1^2,\dots,x_k^2)$. See [11] for basic properties of polarization.

(2) $\Longrightarrow$ (3). Every Cohen-Macaulay ideal is unmixed (see [1]).

(3) $\Longrightarrow$ (1). Suppose that G is not Cohen-Macaulay. We want to prove that there exists a subset $T \subset [n]$ such that $O_T(G)$ is not unmixed. Since G is not Cohen-Macaulay, the sequence $\{x_i - y_i : 1 \le i \le n\}$ is not a regular sequence of $S/I(G)$. Hence, there exists $k \ge 1$ such that $\{x_i - y_i : i \in [k-1]\}$ is a regular sequence of $S/I(G)$ and $x_k - y_k$ is not regular on the ring $R := S_{k-1}/(I(G_{k-1}) + (x_1^2,\dots,x_{k-1}^2)) \simeq S/(I(G) + (x_1 - y_1,\dots,x_{k-1} - y_{k-1}))$. Set $J = I(G_{k-1}) + (x_1^2,\dots,x_{k-1}^2)$. Since $x_k - y_k$ is not regular on R, then

$$x_k - y_k \in \bigcup_{P\in\operatorname{Ass} R} P,$$

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and there exists an associated prime ideal $\widetilde{P}$ of J such that $x_k - y_k \in \widetilde{P}$. Since $x_k \in \widetilde{P}$ or $y_k \in \widetilde{P}$, we have $x_k, y_k \in \widetilde{P}$. Hence, height $\widetilde{P} > n$. Hence, R is not unmixed. Therefore, $S/(I(G_{k-1}) + (x_1^2, \ldots, x_{k-1}^2))^{\mathrm{pol}} \simeq S/I(O_{T_{k-1}}(G))$ is not unmixed. ∎

For distinct $i_1, i_2, \ldots, i_r \in [n]$, we denote by $C_{i_1 i_2 \cdots i_r}$ the cycle C with

$$V(C) = \{x_{i_1}, y_{i_1}, x_{i_2}, \ldots, x_{i_r}, y_{i_r}\}$$

and

$$E(C) = \{x_{i_1}y_{i_1}, y_{i_1}x_{i_2}, x_{i_2}y_{i_2}, \ldots, y_{i_r}x_{i_r}, y_{i_r}x_{i_1}\}.$$

PROPOSITION 3.3. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *We assume condition (*). Then the following conditions are equivalent.*

(1) *The subset $\{x_1y_1, x_2y_2, \ldots, x_ny_n\}$ of $E(G)$ is a unique perfect matching in G.*
(2) *The cycle C_{ij} is not a subgraph of G for any $i < j$.*
(3) *For any $r \geq 2$, the cycle $C_{i_1 i_2 \cdots i_r}$ is not a subgraph of G for any subset $\{i_1, i_2, \ldots, i_r\} \subset [n]$ of cardinality r.*

Proof. (1) $\Longrightarrow$ (2). Suppose that C_{ij} is a subgraph of G. Then we have two perfect matchings in G:

$$\{x_1y_1, x_2y_2, \ldots, x_ny_n\},$$

$$\{x_1y_1, x_2y_2, \ldots, x_{i-1}y_{i-1}, x_iy_j, x_jy_i, x_{i+1}y_{i+1}, \ldots, x_ny_n\}.$$

(2) $\Longrightarrow$ (3). We proceed by induction on r.

For $r = 2$ there is nothing to prove. Assume that $r > 2$, and suppose that $C_{i_1 i_2 \cdots i_r}$ is a subgraph of G. Since $y_{i_{r-1}}x_{i_r}, y_{i_r}x_{i_1} \in E(G)$, we have $y_{i_{r-1}}x_{i_1} \in E(G)$ by Proposition 2.3. Hence, $C_{i_1 i_2 \cdots i_{r-1}}$ is a subgraph of G, which is a contradiction with the inductive hypothesis.

(3) $\Longrightarrow$ (1). Suppose that there exists another perfect matching:

$$\{x_1y_{i_1}, x_2y_{i_2}, \ldots, x_ny_{i_n}\} \subset E(G).$$

Then we define a permutation σ by

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}.$$

Then σ can be decomposed as $\sigma = \prod \sigma_i$, where each σ_i is a cycle of σ. Since σ is not an identity permutation, for some i the cycle σ_i is of the form $(j_1 j_2 \cdots j_r)$ with $r \geq 2$. Then we have that $C_{j_r j_{r-1} \cdots j_1}$ is a subgraph of G. ∎

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Now we give characterizations of Cohen-Macaulayness, which is analogous to the corresponding result for bipartite graphs (see [8, Corollary 3.5]).

THEOREM 3.4. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$ *satisfying condition (*). Then the following conditions are equivalent.*

(1) *G is Cohen-Macaulay.*
(2) *$\Delta(G)$ is strongly connected.*
(3) *The cycle C_{ij} is not a subgraph of G for any $i < j$.*

Proof. (1) $\Longrightarrow$ (2). This is well known.

(2) $\Longrightarrow$ (3). Assume that C_{ij} is a subgraph of G for some $i < j$. Let F be a facet of $\Delta(G)$ such that $x_i \in F$. Since $x_i y_j \in E(G)$, we have $y_j \notin F$, and by the unmixedness of G it follows that $x_j \in F$. Hence, $\{x_i, x_j\} \subset F$. Let F' be a facet of $\Delta(G)$ such that $\{y_i, y_j\} \subset F'$.

We show that there does not exist a chain of facets of $\Delta(G)$ such that

$$F = F_0, F_1, \ldots, F_m = F', \quad \text{with } \sharp(F_i \cap F_{i+1}) = n-1 \text{ for } i = 1, \ldots, m-1.$$

Every facet $H \in \Delta(G)$ is one of the following forms:

$$H = \{z_1, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_{j-1}, x_j, z_{j+1}, \ldots, z_n\}$$

or

$$H = \{z_1, \ldots, z_{i-1}, y_i, z_{i+1}, \ldots, z_{j-1}, y_j, z_{j+1}, \ldots, z_n\},$$

where $z_k \in \{x_k, y_k\}$, since $\{x_i y_i, x_j y_j, x_i y_j, x_j y_i\} \subset E(G)$. Hence, it is impossible to find such a chain. Hence, $\Delta(G)$ is not strongly connected.

(3) $\Longrightarrow$ (1). In order to prove the statement by Proposition 3.2, it is sufficient to verify that $O_T(G)$ is unmixed for every subset T of $[n]$. In contrast, suppose that there exists $T \subset [n]$ such that $G' := O_T(G)$ is not unmixed. By Proposition 2.3, one of the following cases occurs:

(i.a) there exist distinct $i, j, k \in [n]$ such that $x_i x_j, y_j x_k \in E(G')$ but $x_i x_k \notin E(G')$;
(i.b) there exist distinct $i, j, k \in [n]$ such that $y_i x_j, y_j x_k \in E(G')$ but $y_i x_k \notin E(G')$;
(ii) there exist distinct $i, j \in [n]$ such that $x_i y_j, x_i x_j \in E(G')$.

In case (i.a), since $j \notin T$, we have $y_j x_k \in E(G)$. Moreover, since $j \notin T$, $x_i x_j \in E(G')$ implies that

(i.aa) $x_i x_j \in E(G)$

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or

(i.ab) $y_i x_j \in E(G)$ and $i \in T$.

In subcase (i.aa), we have $x_i x_k \in E(G)$ by Proposition 2.3. Hence, $x_i x_k \in E(G')$. This contradicts $x_i x_k \notin E(G')$.

In subcase (i.ab), we have $y_i x_k \in E(G)$ by Proposition 2.3 with $i \in T$. Hence, $x_i x_k \in E(G')$. This contradicts $x_i x_k \notin E(G')$.

In case (i.b), $y_i x_j, y_j x_k \in E(G')$ implies that $i, j \notin T$. Hence, $y_i x_j, y_j x_k \in E(G)$. Then $y_i x_k \in E(G)$ by Proposition 2.3. Hence, $y_i x_k \in E(G')$. This contradicts $y_i x_k \notin E(G')$.

In case (ii), $x_i y_j \in E(G')$ implies that $j \notin T$. Hence, $x_i y_j \in E(G)$. Moreover, $x_i x_j \in E(G')$ implies that

(ii.a) $y_i x_j \in E(G)$ and $i \in T$

or

(ii.b) $x_i x_j \in E(G)$.

In subcase (ii.a), we have $y_i x_j, y_j x_i \in E(G)$. This contradicts the assumption that C_{ij} is not a subgraph of G.

In subcase (ii.b), we have $x_i x_j, x_i y_j \in E(G)$. Hence, G is not unmixed by Proposition 2.3. This contradicts the assumption that G is unmixed. $\square$

The next lemma is crucial for giving another criterion for the Cohen-Macaulayness of our graphs.

LEMMA 3.5. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *We assume condition (*).*

If G is a Cohen-Macaulay graph, then there exists a suitable simultaneous change of labeling on both $\{x_i\}$ and $\{y_i\}$ (i.e., we relabel $(x_{i_1}, \dots, x_{i_n})$ and $(y_{i_1}, \dots, y_{i_n})$ as $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ at the same time), such that $x_i y_j \in E(G)$ implies that $i \leq j$.

Proof. We can define a partial order $\preceq$ on X by

$$x_i \preceq x_j \quad \text{if and only if } x_i y_j \in E(G).$$

In fact, the reflexivity holds by condition (*), the transitivity holds by unmixedness of G (see Proposition 2.3(i)), and the antisymmetry holds since G contains no cycle C_{ij} for any $i < j$. Take a linear extension of $\preceq$, which we call $\preceq'$. By the linear order $\preceq'$, we have $x_{i_1} \preceq' \cdots \preceq' x_{i_n}$. We relabel them as $x_1 \preceq' \cdots \preceq' x_n$. At the same time, we relabel $y_{i_1}, \dots, y_{i_n}$ as $y_1, \dots, y_n$. Then if $x_i y_j \in E(G)$, $x_i \preceq' x_j$. Hence, $i \leq j$. $\square$

Hence, for a Cohen-Macaulay graph G with $2n$ vertices, which are not isolated, and with height $I(G) = n$ satisfying condition (*), we may assume that

$$(**) \qquad x_i y_j \in E(G) \text{ implies } i \leq j.$$

Now we state another Cohen-Macaulay criterion on our graphs, which is a generalization of Herzog and Hibi ([8, Theorem 3.4]).

THEOREM 3.6. *Let G be a graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *We assume conditions (*) and (**). Then the following conditions are equivalent.*

(1) *G is Cohen-Macaulay.*
(2) *G is unmixed.*
(3) *The following conditions hold:*
  (i) *if $z_i x_j, y_j x_k \in E(G)$, then $z_i x_k \in E(G)$ for distinct i, j, k and for $z_i \in \{x_i, y_i\}$;*
  (ii) *if $x_i y_j \in E(G)$, then $x_i x_j \notin E(G)$.*

Proof. (1) $\Longrightarrow$ (2). This is well known.
(2) $\Longrightarrow$ (1). This follows from Theorem 3.4, since we assume condition (**).
(2) $\Longleftrightarrow$ (3). This follows from Proposition 2.3. ∎

We remark that the equivalence between (1) and (2) in Theorem 3.6 is a special case of [9, Theorem 4.3].

As an easy consequence of the previous results, we obtain the upper bound for the minimal number $\mu(I(G))$ of generators of $I(G)$, as follows.

COROLLARY 3.7. *Let G be a graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *Then we have the following.*

(i) *If G is unmixed, then $\mu(I(G)) \leq n^2$.*
(ii) *If G is Cohen-Macaulay, then $\mu(I(G)) \leq (n(n+1))/2$.*

Proof. The statements are consequences of the criteria for unmixedness and for Cohen-Macaulayness given by Proposition 2.3 and Theorem 3.6. ∎

§4. Shellability and Cohen-Macaulay type

In this section, if G is a graph such that $\sharp V(G) = 2n$ and height $I(G) = n$, we show the equivalence between the Cohen-Macaulayness and shellability of G. We also express the Cohen-Macaulay type of $S/I(G)$ in a combinatorial way.

THEOREM 4.1. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *Then G is Cohen-Macaulay if and only if G is shellable.*

Here we give a proof only of the following lemma. The rest of the proof is almost identical to the proof of [3, Theorem 2.9].

LEMMA 4.2. *Let G be a Cohen-Macaulay graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *Then there exists a vertex $v \in V(G)$ such that* $\deg(v) = 1$.

Proof. Since G is Cohen-Macaulay, it is unmixed. By Lemma 2.1, G has a perfect matching. Then we may assume condition (*). Suppose that each $v \in V(G)$ has at least degree 2. Let $i_1, i_2, \ldots$ be a sequence such that $y_{i_1}x_{i_2}, y_{i_2}x_{i_3}, \ldots \in E(G)$ with $i_j \neq i_{j+1}$. Since the cardinality of Y is finite, there must exist integers $s < t$ such that $i_t = i_s$. We may assume that $i_s, i_{s+1}, \ldots, i_{t-1}$ are distinct. This induces that the cycle $C_{i_s i_{s+1} \cdots i_{t-1}}$ is a subgraph of G. Therefore, G is not Cohen-Macaulay by Proposition 3.3 and Theorem 3.4. ∎

Now we express the Cohen-Macaulay type of a graph belonging to our class, imitating the bipartite case (see [13, pp. 184–185]).

LEMMA 4.3. *Let G be a Cohen-Macaulay graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *We assume condition (*). Then*

$$\mathrm{Soc}\big(K[x_1, \ldots, x_n]/(I(O_{[n]}(G)|_X) + (x_1^2, \ldots, x_n^2))\big)$$

is generated by all the monomials $x_{i_1} \cdots x_{i_r}$ such that $\{x_{i_1}, \ldots, x_{i_r}\}$ is a maximal independent set of $O_{[n]}(G)|_X$.

Proof. The ring $A := K[x_1, \ldots, x_n]/(I(O_{[n]}(G)|_X) + (x_1^2, \ldots, x_n^2))$ is spanned as a K-vector space by the image of 1 and the images of the square-free monomials

$$x_{i_1} \cdots x_{i_r}, \quad 1 \leq i_1 < i_2 < \cdots < i_r \leq n \tag{4.1}$$

such that $x_{i_j} x_{i_k} \notin E(O_{[n]}(G)|_X)$, for $j \neq k$; that is, $\{x_{i_1}, \ldots, x_{i_r}\}$ is an independent set of $O_{[n]}(G)|_X$. Since A is an Artinian positively graded algebra,

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Soc $A = (0 :_A A_+)$ is generated by the images of the square-free monomials of form (4.1) such that $\{x_{i_1}, \dots, x_{i_r}\}$ is a maximal independent set of $O_{[n]}(G)|_X$. $\square$

Corollary 4.4. *Let G be a Cohen-Macaulay graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *We assume condition (*). Then we have the following.*

(i) *type $S/I(G) = \sharp\Upsilon(O_{[n]}(G)|_X)$, where $\Upsilon(O_{[n]}(G)|_X)$ is the family of all minimal vertex covers of $O_{[n]}(G)|_X$. In particular, type $S/I(G)$ is independent from the base field K.*

(ii) *G is level if and only if $O_{[n]}(G)|_X$ is unmixed. In particular, the levelness of G is independent from the base field K.*

Proof. Set $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$, and set $S_n = K[x_1, \dots, x_n]$.

(i) Since G is Cohen-Macaulay and since $\{x_1 - y_1, \dots, x_n - y_n\}$ is a regular sequence, we have

$$\begin{aligned}\text{type } S/I(G) &= \dim_K \operatorname{Soc} S/\big(I(G) + (x_1 - y_1, \dots, x_n - y_n)\big)\\ &= \dim_K \operatorname{Soc} S_n/\big(I(O_{[n]}(G)|_X) + (x_1^2, \dots, x_n^2)\big)\\ &= \sharp\Upsilon\big(O_{[n]}(G)|_X\big)\end{aligned}$$

by Lemma 4.3.

(ii) When G is Cohen-Macaulay, G is level if and only if

$$\operatorname{Soc} S/\big(I(G) + (x_1 - y_1, \dots, x_n - y_n)\big)$$

is equigenerated. By Lemma 4.3, it is equivalent that $O_{[n]}(G)|_X$ is unmixed. $\square$

The next result generalizes [8, Corollary 3.6].

Corollary 4.5. *Let G be a Cohen-Macaulay graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *We assume condition (*). Then the following conditions are equivalent.*

(1) *G is Gorenstein.*

(2) *$I(G) = (x_1y_1, \dots, x_ny_n)$.*

(3) *G is a complete intersection.*

Proof. (1) $\Rightarrow$ (2). G is Gorenstein if and only if $S/I(G)$ is Cohen-Macaulay and type $S/I(G) = 1$. Since $1 = \text{type } S/I(G) = \sharp\Upsilon(O_{[n]}(G)|_X)$, it follows that $O_{[n]}(G)|_X$ has a unique minimal vertex cover. Hence, $O_{[n]}(G)|_X$ is isolated n vertices. Hence, $I(G) = (x_1y_1, \dots, x_ny_n)$.

(2) ⇒ (3). This is true from its definition.
(3) ⇒ (1). See [1]. □

§5. B-grafted graph

In this section, we introduce a new class of graphs G with $\sharp V(G) = 2n$ and with height $I(G) = n$, and we study its Cohen-Macaulayness.

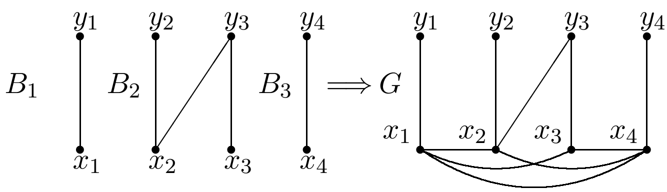
Let H_0 be a graph with the labeled vertices $1, 2, \ldots, p$.

For every $i = 1, \ldots, p$, let B_i be a bipartite graph with labeled partition X_i and Y_i such that $\sharp X_i = \sharp Y_i = n_i$. (We do not give a label to each vertex of B_i, but we distinguish the partition X_i and Y_i.) We assume that B_i has no isolated vertex for every $i = 1, \ldots, p$. We define the graph

$$G = G(H_0; B_1, \ldots, B_p)$$

as follows. The vertex set of G is $V(G) := X \cup Y$, where $X = X_1 \cup \cdots \cup X_p$ and $Y = Y_1 \cup \cdots \cup Y_p$. The edge set $E(G)$ of G is defined by $xy \in E(G)$ if and only if either there exist i, j such that $x \in X_i, y \in X_j$, and $ij \in E(H_0)$ or there exists i such that $x \in X_i, y \in Y_i$, and $xy \in E(B_i)$. We call such a graph G the *B-grafted graph*. Note that X is a minimal vertex cover of G and that Y is a maximal independent set of G. Note also that $\sharp V(G) = 2\left(\sum_{i=1}^{p} n_i\right)$.

EXAMPLE 5.1. Let H_0 be a cycle of length 3. By the following bipartite graphs B_1, B_2, and B_3, we obtain the B-grafted graph G:



REMARK 5.2. If B_i is just a complete graph with two vertices, that is, a complete bipartite graph with $\sharp X_i = \sharp Y_i = 1$ for $i = 1, \ldots, p$, then the B-grafted graph G is called a *grafted graph* in [4].

THEOREM 5.3. *The B-grafted graph $G(H_0; B_1, \ldots, B_p)$ is Cohen-Macaulay (resp., unmixed) if and only if every bipartite graph B_i is Cohen-Macaulay (resp., unmixed) for $i = 1, \ldots, p$.*

Proof. It is clear from Theorem 3.4 (resp., Proposition 2.3). □

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