APPLYING THE PROPORTIONAL HAZARD PREMIUM CALCULATION PRINCIPLE

BY

MARIA DE LOURDES CENTENO AND JOÃO ANDRADE E SILVA

ABSTRACT

In this paper we discuss the application of the proportional hazard premium calculation principle.

In Section 2 we propose a method to calculate the proportional hazard premium of a compound risk when the severity distribution is subexponential.

In Section 3 we use the empirical distribution to calculate the premium when the proportional hazard principle is applied, which leads to a systematic underestimation of the premium. After studying the bias of the premium calculated using this non-parametric approach we take advantage of the bootstrap technique with subsampling to reduce it.

1. INTRODUCTION

As it is well known the proportional hazard premium principle (PH premium principle), introduced by Wang (1995), satisfies properties which make of it a very attractive premium from the theoretical point of view, see e.g. Wang (1996) and Andrade e Silva and Centeno (1998). However its use depends on the complete knowledge of the distribution of the aggregate claims amount. In practical terms this means that one must fit a distribution to the data set or use the empirical distribution function. When the aggregate claims amount follows a compound distribution the calculation of the PH premium based on the parametric approach can raise some problems. This will be discussed in the following section.

The non-parametric approach is more appealing, from the practical point of view, but can lead to a significant underestimation of the premium. In subsection 3.1 we calculate the bias of the premium based on the empirical distribution for some distributions, namely the exponential, Pareto and uniform. We also study the rate of convergency of the bias in the exponential case.

In subsection 3.2 we show how to use the bootstrap technique with subsampling to reduce the bias of the estimator of the premium based on the empirical distribution. We also perform two simulation studies to give some insight to this technique.

2. APPLYING THE PH TRANSFORM TO COMPOUND DISTRIBUTIONS IN THE PARAMETRIC MODEL

Let Y, the aggregate claims amount, be a nonnegative random variable, with distribution function $F(y)$. Let $S(y) = 1 - F(y)$ be the survival function. The PH premium principle assigns to the distribution $F(y)$ the premium

$$\pi_\rho = \int_0^\infty (S(y))^{1/\rho} dy, \tag{1}$$

where ρ, with $\rho \geq 1$, is *risk aversion index*.

When using the collective risk model, Y is equal to $X_1 + ... + X_N$, where N is the number of claims occurred in a given period and $\{X_i\}_{i=1,2,...}$ are the individual claims amounts which are assumed to be nonnegative i.i.d. random variables and independent of N. In this case Y has a compound distribution and the company often estimates claim frequency and claim severity separately. Various numerical techniques are available to calculate the compound distribution, being Panjer's recursion formula the most well known. When using the recursion formula we will have to stop somewhere the calculations. When the individual claims amount has distribution G with unlimited support, i.e. $G(x) < 1$, for all $x > 0$, and it is very skewed, the choice of the value where to stop the calculations to obtain a reasonable estimate of the premium is a critical aspect of the model. When X follows a heavy tail distribution $\int_t^\infty (S(y))^{1/\rho} dy$ can be of an expressive size even for large values of t. In this case, when the probability generating function of N is analytic in the neighbourhood of 1 (which happens both for the Poisson and the negative binomial case) and G is subexponential then (see Embrechts *et al.* (1997), pp. 45-46)

$$1 - F(x) \sim E[N](1 - G(x)), \quad x \longrightarrow \infty. \tag{2}$$

Hence for heavy tail severity distributions we will approximate the premium by

$$\pi_\rho^* = \int_0^t (S^*(y))^{1/\rho} dy + (E[N])^{1/\rho} \int_t^\infty (1 - G(y))^{1/\rho} dy, \tag{3}$$

where $S^*(y)$ is the approximation to $S(y)$ obtained by Panjer's recursion formula, and t is a suitable high value.

Example 1. *Suppose that an actuary has arrived to the following estimates for a risk: the number of claims greater than a given observation point $d = 100K$ is Poisson distributed with mean $\lambda_d = 6$ and the size of each claim greater than d is Pareto with parameter $\alpha = 1.647$. Let us consider an excess of loss reinsurance*

treaty. For a layer l xs m, with m ≥ d and l ≤ +∞, the aggregate claims amount are compound Poisson with expected number of claims $\lambda = (d/m)^\alpha \lambda_d$ and severity distribution

$$G(x) = \begin{cases} 1 - \left(\frac{m}{m+x}\right)^\alpha & \text{if } 0 \leq x < l \\ 1 & \text{if } x \geq l. \end{cases} \tag{4}$$

Using $1/\rho = 0.9025$, Table 1 shows the PH transform premium, as percentage of the subject earned premium, $SEP = \$10,000K$, associate to the following layers:

1. $\$400K$ xs $\$100K$
2. $\$500K$ xs $\$500K$
3. $\$900K$ xs $\$100K$
4. the excess over $\$1,000K$

This example for the limited layers was considered by Wang (1998). For the calculation of the premium of the unlimited layer we have used (3) with $t = \$10^5K$, for which the difference between the aggregate survival function $S^*(t)$ and $\lambda(1 - G(t))$, is 2.3×10^{-7}. The step used in the arithmetization of the severity distribution to perform the recursion was $h = \$1K$.

The reason to chose such a high value for t in formula (3) is related to the very small value of α. If instead of $\$10^5K$ we had used $\$10^4K$ for t, the result of applying formula (3) with $\rho = 1$ would be 2.086% of SEP, compared to a theoretical value of 2.090%. Using $t = \$10^5K$ the two terms in the right hand side of (3) are 1.9848% and 0.1055% for $\rho = 1$ and 3.0298% and 0.3580% for $1/\rho = 0.9025$. Note that the relative weight of the second term increases with ρ and is far from negligible in spite of the high values of t.

3. APPLYING THE PH TRANSFORM TO THE EMPIRICAL DISTRIBUTION

3.1. The bias of the premium

Let $F_n(y)$ be the empirical distribution function of Y based on a random sample of size n, $(Y_1,...,Y_n)$, let $S_n(y)$ be the corresponding empirical survival function and

TABLE 1

PH-TRANSFORM PREMIUMS

Layer l xs m	Pure Premium as percentage of SEP	π_ρ^* as percentage of SEP
$\$400K$ xs $\$100K$	6.000%	6.384%
$\$500K$ xs $\$500K$	1.183%	1.408%
$\$900K$ xs $\$100K$	7.183%	7.742%
xs $\$1,000K$	2.090%	3.388%

$$\hat{\pi}_\rho = \int_0^\infty (S_n(y))^{1/\rho} \, dy, \tag{5}$$

the premium estimator. Although $S_n(y)$ is an unbiased estimator of $S(y), \hat{\pi}_\rho$ given by (5) is a biased estimator of π_ρ, since

$$E[\hat{\pi}_\rho] = \int_0^\infty E[(S_n(y))^{1/\rho}] \, dy \leq \int_0^\infty [E(S_n(y))]^{1/\rho} \, dy = \pi_\rho$$

by Jensen's inequality, with the inequality being strict unless $\rho = 1$.

Let $Y_{k:n}$ be the k-th order statistic in a sample of size n. Then

$$S_n(y) = \begin{cases} 1 & y < Y_{1:n} \\ (n-k)/n, & Y_{k:n} \leq y < Y_{k+1:n}, \ k = 1, \dots, n-1 \\ 0 & y \geq Y_{n:n}, \end{cases}$$

which implies that

$$\hat{\pi}_\rho = \sum_{k=0}^{n-1} \left(\frac{n-k}{n}\right)^{1/\rho} (Y_{k+1:n} - Y_{k:n}), \tag{6}$$

where $Y_{0:n} \equiv 0$.

Let $F_{k:n}(y)$ be the distribution function of the k-th order statistic, i.e. $F_{k:n}(y) = \Pr\{Y_{k:n} \leq y\}$. Given that

$$F_{k:n}(y) = \sum_{r=k}^{n} \binom{n}{r} (F(y))^r (1 - F(y))^{n-r},$$

then

$$1 - F_{k:n}(y) = 1 - F_{k+1:n}(y) - \binom{n}{k} (F(y))^k (1 - F(y))^{n-k}$$

and integrating we obtain

$$E[Y_{k+1:n} - Y_{k:n}] = \int_0^\infty \binom{n}{k} (F(y))^k (1 - F(y))^{n-k} \, dy. \tag{7}$$

Then, using (6) and (7) we get

$$\begin{aligned} E[\hat{\pi}_\rho] &= \sum_{k=0}^{n-1} \left(\frac{n-k}{n}\right)^{1/\rho} \binom{n}{k} \int_0^\infty (F(y))^k (S(y))^{n-k} \, dy \\ &= \sum_{k=1}^{n} \left(\frac{n-k+1}{n}\right)^{1/\rho} \binom{n}{k-1} \int_0^\infty (F(y))^{k-1} (S(y))^{n-k+1} \, dy \end{aligned} \tag{8}$$

which is, when F is absolutely continuous for $x > 0$, equivalent to

$$E[\hat{\pi}_\rho] = \sum_{k=1}^{n} \left(\frac{n-k+1}{n}\right)^{1/\rho} \binom{n}{k-1} \int_0^1 (1-x)^{k-1} x^{n-k+1} \frac{1}{f[S^{-1}(x)]} \, dx, \tag{9}$$

where $f(x) = F'(x)$.

Expressions (8) and (9) can be used to calculate the bias of $\hat{\pi}_\rho$

$$B(\hat{\pi}_\rho) = E[\hat{\pi}_\rho] - \pi_\rho.$$

For some distributions (not in the compound case) the bias is easily calculated, as it is the case when Y is exponential, Pareto or uniform, as can be seen in the following examples.

Example 2. *Let Y be exponential distributed, i.e. $S(y) = \exp(-\theta y)$, $y > 0$ $(\theta > 0)$. In this case $f[S^{-1}(x)] = \theta x$, and using (9) we get, after some calculations,*

$$E[\hat{\pi}_\rho] = n^{-1/\rho}\theta^{-1}\sum_{k=1}^{n}k^{1/\rho-1}$$

and

$$B(\hat{\pi}_\rho) = \left(n^{-1/\rho}\sum_{k=1}^{n}k^{1/\rho-1} - \rho\right)\theta^{-1}. \tag{10}$$

Example 3. *Let Y be Pareto distributed, i.e. $S(y) = \left(\frac{\beta}{\beta+y}\right)^\alpha$, $y > 0$ $(\alpha, \beta > 0)$. In this case $f[S^{-1}(x)] = \alpha\beta^{-1}x^{1+1/\alpha}$, and using (9) we get,*

$$E[\hat{\pi}_\rho] = \beta\alpha^{-1}n^{-1/\rho}n!\sum_{k=1}^{n}k^{1/\rho}\frac{\Gamma(k-1/\alpha)}{k!\,\Gamma(n+1-1/\alpha)}$$

and

$$B(\hat{\pi}_\rho) = \beta\left[\frac{n!}{\alpha n^{1/\rho}}\sum_{k=1}^{n}k^{1/\rho}\frac{\Gamma(k-1/\alpha)}{k!\,\Gamma(n+1-1/\alpha)} - \frac{\rho}{\alpha-\rho}\right]. \tag{11}$$

Example 4. *When Y is uniformly distributed in $(0,1)$ we get*

$$E[\hat{\pi}_\rho] = \frac{1}{(n+1)n^{1/\rho}}\sum_{k=1}^{n}k^{1/\rho}$$

and

$$B(\hat{\pi}_\rho) = \frac{1}{(n+1)n^{1/\rho}}\sum_{k=1}^{n}k^{1/\rho} - \frac{\rho}{\rho+1}.$$

For the exponential distribution it is possible to relate the order of the bias with the sample size, as stated in the result that follows. We were not able of developing similar results for other distributions, but we are strongly convinced that the order of the bias is smaller for the uniform and bigger for the Pareto.

Result. When Y *follows an exponential distribution* $|B(\hat{\pi}_\rho)|$ *converges to zero at the same rate as* $n^{-1/\rho}$.

Proof. Noticing that

$$\rho = n^{-1/\rho} \sum_{k=1}^{n} \int_{0}^{1} (k-x)^{1/\rho-1} dx,$$

and using (10) we have

$$\left| B(\hat{\pi}_{\rho}) \right| = \theta^{-1} n^{-1/\rho} \sum_{k=1}^{n} \int_{0}^{1} \left[(k-x)^{1/\rho-1} - k^{1/\rho-1} \right] dx.$$

As $(k-x)^{1/\rho-1}$ is an increasing function of x for $k \geq 1$ and $0 < x < 1$, we have that

$$\sum_{k=1}^{n} \int_{0}^{1} \left[(k-x)^{1/\rho-1} - k^{1/\rho-1} \right] dx \leq \int_{0}^{1} \left[(1-x)^{1/\rho-1} - 1 \right] dx$$

$$+ \sum_{k=2}^{n} \left[(k-1)^{1/\rho-1} - k^{1/\rho-1} \right] = \rho - n^{1/\rho-1} \leq \rho.$$

On the other hand as $\int_{0}^{1} [(k-x)^{1/\rho-1} - k^{1/\rho-1}] dx$ is strictly positive we can conclude that the left hand side of (12) converges to a positive value A as n goes to infinity. Consequently

$$\lim_{n \to \infty} n^{1/\rho} |B(\hat{\pi}_{\rho})| = \theta^{-1} A.$$

3.2. Correcting the bias via bootstrapping

As we have seen, the distortion of the empirical distribution leads to an under-estimation of the premiums. We use the bootstrap technique, see e.g. Efron and Tibshirani (1993), to correct, at least partially, the bias of that estimator.

The bootstrap technique consists, as it is well known, on the resampling of the original data set. The bootstrap estimator of the bias of $\hat{\pi}_{\rho}$, is

$$\widehat{B(\hat{\pi}_{\rho})} = \hat{\pi}_{\rho}^{*} - \hat{\pi}_{\rho}$$

with

$$\hat{\pi}_{\rho}^{*} = \frac{1}{M} \sum_{b=1}^{M} \hat{\pi}_{\rho}^{*b}$$

where M is the number of bootstrap samples and $\hat{\pi}_{\rho}^{*b}$ is calculated using the b-th bootstrap sample. Hence the bootstrap estimator of the premium π_{ρ} is

$$\hat{\hat{\pi}}_{\rho} = \hat{\pi}_{\rho} - \widehat{B(\hat{\pi}_{\rho})}. \tag{13}$$

Usually the resampling is made with replacement and using bootstrap samples of size n (i.e. the same size as the original sample). However in this case, as
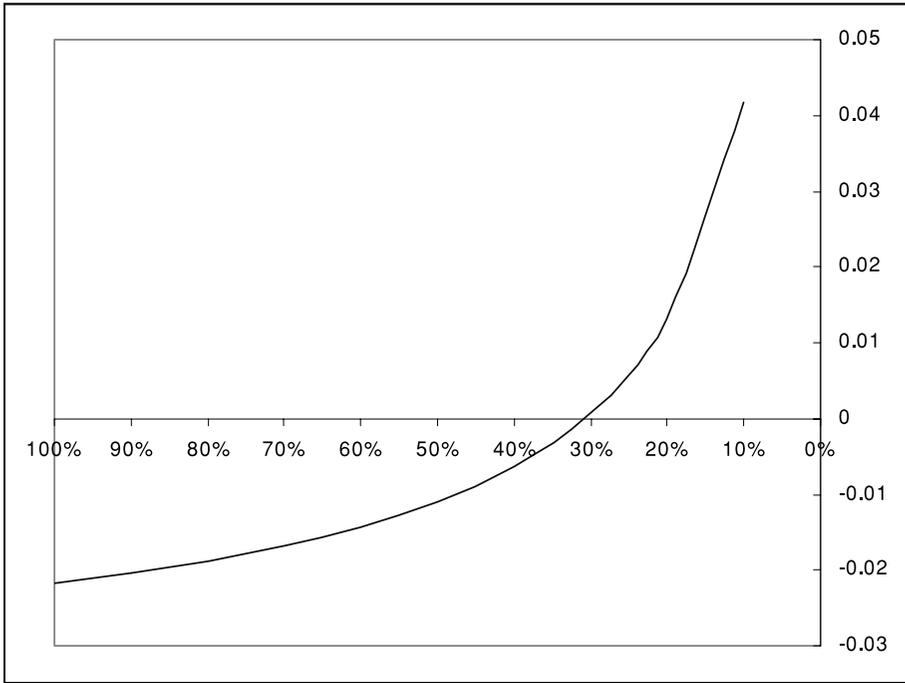
FIGURE 1: $\overline{B\left(\hat{\hat{\pi}}_\rho\right)}$ as function of n'/n

$B(\hat{\pi}_\rho)$ is always negative, we can improve, in principle, the results by using bootstrap samples of size $n' < n$. The reason for the bias to be negative is the underestimation of the right tail of the distorted distribution. The use of a subsampling in the bootstrap method intends to replicate that behaviour and consequently to obtain a better bias correction than the one obtained with the full sample.

Figure 1 shows how the bias of the corrected estimator given by (3) varies with the sample size n' (measured as a percentage of n). The figure has been drawn using a simulation approach (see simulation 1 below) for a Pareto distribution with $\alpha = 4$, sample size of 500 observations, $\rho = 1.2$ and using 2000 replicas and 2000 bootstraps samples for each replica. The shape of the curve is very similar for all the other situations: it starts, when $n' = n$, with a negative value for the mean of the observed bias of the bootstrap premiums, which we denote $\overline{B(\hat{\hat{\pi}}_\rho)}$, then there is an optimal value of n' where $\overline{B(\hat{\hat{\pi}}_\rho)}$, is almost zero, and becomes positive when n' is very small. The question is how to choose n', independently of the distribution.

To give some insight to the problem we performed two simulation studies. In these studies we also use the Jackknife, as an alternative to the bootstrap, to correct the bias. As we will see the Jackknife performs better than full sampling bootstrap, but we can do better by using a suitable level of subsampling.

Simulation 1. This case is based on two families of distributions, the Pareto and the Gamma. In the Pareto case we have considered that the parameter α takes the value 2, 3 or 4, originating what we call Pareto_2, Pareto_3 and Pareto_4 respectively. The other parameter was chosen such that the distribution has mean equal to 1, i.e. we have assumed that the survival function for the Pareto_α is

$$S_\alpha(x) = \left(\frac{\alpha - 1}{x + \alpha - 1}\right)^\alpha, \quad x > 0.$$

For the gamma distribution we have kept the same mean and assumed that the variance was equal to 1 (exponential) or 2. We have considered that the original sample size is either 100, 500 or 1000. Tables 2 and 3 give the bootstrap and Jackknife results for $\rho = 1.2$ and $\rho = 1.15$ respectively, with $M = 2000$ and using 2000 replicas.

As we have mentioned, the bias of the premium based on the empirical distribution (column labelled $B(\hat{\pi}_\rho)$, and calculated using expressions (8), (10) and (11), for the gamma, exponential and Pareto distributions) can be of a considerable size, namely for heavy tail distributions when n is small and ρ is high. For instance for the Pareto_2, with $n = 100$ and $\rho = 1.2$, the absolute value of the bias is 8.4% of the theoretical premium. The average bias of the premiums based on the 2000 replicas (before the bootstrap is performed), labelled $\bar{B}$, is in those cases still a bit far from $B(\hat{\pi}_\rho)$, but it was out of our computing facilities to consider a big enough number of replicas in the simulation study. The following 10 columns show the average bias for the bootstrap premiums. The bootstrap with $n' = n$, only corrects the bias partially, since after that correction we still observe a negative bias in all the situations of our example. As we can see from Tables 2 and 3, the pattern shown in Figure 1 is characteristic of the behaviour of the bias as function of n'/n. The optimal proportion n'/n depends, of course, on the distribution, on n and on ρ. As a rule n'/n should increase with n and decrease with ρ. For instance, for $\rho = 1.2$, a bootstrap with a resampling of $n'/n = 40\%$ performs, in average, better than the bootstrap with full resampling (and certainly better than with no bootstrap), at least for the sample sizes considered in the study. The following column, labelled JN, shows the average bias for the Jackknife premiums, i.e. we use the Jackknife technique instead of bootstrap to correct the bias of the premiums. As we can see the Jackknife performs better than the full sampling bootstrap but, using subsampling, we can obtain a smaller bias with the bootstrap.

The last column of the tables, provides the average bias of the premiums if we have used the maximum likelihood estimators of the parameters. For the Pareto_2, the figures are not presented because, for a large number of samples (20 for $\rho = 1.2$ and 11 for $\rho = 1.15$), we got a maximum likelihood estimate $\hat{\alpha}$ smaller than ρ. In some other situations the estimate was very close to ρ, which would imply an extremely high premium. For the Pareto_3 and Pareto_4 we didn't get any sample with a maximum likelihood estimate smaller than ρ, but we got values not very far from ρ, which explain the positive values of the bias in those cases. The maximum likelihood performs well (in association with

the PH premium calculation principle) if the distribution does not have a very heavy tail, or if the sample size is big enough.

As a conclusion, and for values of ρ around 1.15-1.2 we can say that use of the bootstrap technique with a resampling proportion size of 40% of the original sample size provides good results in general, but when we know that we are dealing with heavy tail loss distributions, we could use a smaller resampling size.

Table 4 is equivalent to Table 2 but with $M = 5000$. As it can be seen, the results are quite similar. Hence $M = 2000$ seems enough.

Simulation 2. The goal of the second simulation study is to analyse the behaviour of our procedures when the aggregate claims amount are generated according to a compound Poisson or negative binomial distributions. For the individual claims amount we have considered the three Pareto's and the exponential distributions of the former example. The expected number of claims was set at 1. As in the previous simulation study $\rho = 1.2$ and $\rho = 1.15$, $M = 2000$, we use 2000 replicas and n is equal to 100, 500 or 1000.

Tables 5 and 6 refer to the compound Poisson case and are similar to tables 2 and 3, with the exceptions of $B(\hat{\pi}_\rho)$ and ML, which are not considered now. Column labelled π_ρ^* was obtained using (3), with t equal to 10000, 926 and 297 when α equal to 2, 3 and 4 respectively, values for which the $\Pr\{X > t\} = 10^{-8}$. For the exponential case we only considered the first term of (3), with $t = 9000$.

As expected with $\lambda = 1$ the premiums associated to the compound distribution are greater than for the corresponding Pareto's.

The results are very similar to the former case. We obtained the same behaviour for the bias as a function of n'/n, and again we would recommend a resampling of $n'/n = 40\%$, but if we knew that we were dealing with heavy tail loss distributions, we could use a smaller resampling size.

The results for the simulations with $M = 5000$ give very similar values, reason why we do not provide them here.

Tables 7 and 8 refer to the negative binomial case, with $\rho = 1.2$ and with the variance of the number of claims equal to 1.05 and 1.2 respectively. As we can see the results lead to the same conclusion.

REFERENCES

ANDRADE E SILVA, J. and CENTENO, M.L. (1998) Comparing risk adjusted premiums from the reinsurance point of view. *ASTIN Bulletin*, **28**, 221-239.

EMBRECHTS, P., KLÜPPELBERG C., MIKOSCH, T. (1997) *Modelling extremal events*. Springer-Verlag, Berlin.

EFRON, B. and TIBSHIRANI, R. (1993) *An Introduction to the Bootstrap*. Chapman and Hall, London.

WANG, S. (1995) Insurance pricing and increased limits ratemaking by proportional hazards transforms. *Insurance: Mathematics and Economics*, **17**, 43-54.

WANG, S. (1996) Premium calculation by transforming the layer premium density, *ASTIN Bulletin*, **26**, 71-92.

WANG, S. (1998) Implementation of PH-transforms in ratemaking, *Proceedings of the Casualty Actuarial Society*, **LXXXV**, 940-979.

MARIA DE LOURDES CENTENO
CEMAPRE, ISEG,
Technical University of Lisbon,
Rua do Quelhas, 2, 1200 781 Lisbon,
Portugal
E-mail: lcenteno@iseg.utl.pt

TABLE 2

BOOTSTRAP RESULTS, 2000 BOOTSTRAPS, $\rho = 1.2$

n	Distribution	π_ρ	$B(\hat{\pi}_\rho)$	$\bar{B}$	$\overline{B(\hat{\hat{\pi}}_\rho)}$ for n'/n equal to										JN	ML
					100%	90%	80%	70%	60%	50%	40%	30%	20%	10%		
100	Pareto$_2$	1.5000	-0.1261	-0.1388	-0.1134	-0.1108	-0.1075	-0.1033	-0.0980	-0.0913	-0.0822	-0.0685	-0.0463	0.0004	-0.0994	—
	Pareto$_3$	1.3333	-0.0458	-0.0508	-0.0361	-0.0345	-0.0325	-0.0300	-0.0268	-0.0227	-0.0168	-0.0080	0.0069	0.0403	-0.0293	0.0492
	Pareto$_4$	1.2857	-0.0296	-0.0330	-0.0217	-0.0204	-0.0188	-0.0169	-0.0143	-0.0110	-0.0063	0.0008	0.0132	0.0418	-0.0169	0.0161
	Gamma	1.2865	-0.0170	-0.0191	-0.0104	-0.0094	-0.0080	-0.0064	-0.0042	-0.0013	0.0029	0.0094	0.0212	0.0499	-0.0068	-0.0036
	Exponential	1.2000	-0.0098	-0.0112	-0.0056	-0.0049	-0.0041	-0.0031	-0.0018	0.0000	0.0025	0.0066	0.0139	0.0324	-0.0039	-0.0013
500	Pareto$_2$	1.5000	-0.0743	-0.0838	-0.0687	-0.0670	-0.0650	-0.0625	-0.0593	-0.0552	-0.0497	-0.0416	-0.0279	0.0011	-0.0605	0.0452
	Pareto$_3$	1.3333	-0.0209	-0.0238	-0.0168	-0.0160	-0.0151	-0.0139	-0.0124	-0.0103	-0.0075	-0.0034	0.0040	0.0208	-0.0137	0.0069
	Pareto$_4$	1.2857	-0.0120	-0.0139	-0.0091	-0.0086	-0.0079	-0.0071	-0.0060	-0.0045	-0.0025	0.0005	0.0060	0.0189	-0.0072	0.0021
	Gamma	1.2865	-0.0051	-0.0064	-0.0040	-0.0037	-0.0032	-0.0027	-0.0020	-0.0009	0.0004	0.0026	0.0067	0.0172	-0.0025	-0.0014
	Exponential	1.2000	-0.0029	-0.0039	-0.0021	-0.0019	-0.0017	-0.0013	-0.0009	-0.0003	0.0005	0.0018	0.0042	0.0105	-0.0017	-0.0010
1000	Pareto$_2$	1.5000	-0.0590	-0.0602	-0.0482	-0.0468	-0.0452	-0.0432	-0.0406	-0.0372	-0.0327	-0.0259	-0.0150	0.0089	-0.0412	0.0224
	Pareto$_3$	1.3333	-0.0149	-0.0148	-0.0099	-0.0093	-0.0086	-0.0077	-0.0066	-0.0051	-0.0031	0.0000	0.0053	0.0177	-0.0075	0.0041
	Pareto$_4$	1.2857	-0.0081	-0.0081	-0.0049	-0.0045	-0.0041	-0.0035	-0.0028	-0.0018	-0.0004	0.0018	0.0056	0.0146	-0.0035	0.0015
	Gamma	1.2865	-0.0030	-0.0034	-0.0022	-0.0020	-0.0017	-0.0014	-0.0010	-0.0004	0.0005	0.0019	0.0044	0.0109	-0.0010	-0.0006
	Exponential	1.2000	-0.0017	-0.0021	-0.0011	-0.0010	-0.0008	-0.0006	-0.0004	0.0000	0.0005	0.0013	0.0027	0.0066	-0.0008	-0.0006

TABLE 3

BOOTSTRAP RESULTS, 2000 BOOTSTRAPS, $\rho = 1.15$

n	Distribution	π_ρ	$B(\hat{\pi}_\rho)$	$\bar{B}$	$\overline{B(\hat{\hat{\pi}}_\rho)}$ for n'/n equal to										JN	ML
					100%	90%	80%	70%	60%	50%	40%	30%	20%	10%		
100	Pareto$_2$	1.3529	-0.0771	-0.0881	-0.0705	-0.0687	-0.0664	-0.0635	-0.0598	-0.0551	-0.0487	-0.0391	-0.0232	0.0106	-0.0614	—
	Pareto$_3$	1.2432	-0.0293	-0.0337	-0.0235	-0.0223	-0.0209	-0.0192	-0.0169	-0.0140	-0.0099	-0.0036	0.0071	0.0314	-0.0191	0.0337
	Pareto$_4$	1.2105	-0.0193	-0.0223	-0.0143	-0.0134	-0.0123	-0.0110	-0.0092	-0.0068	-0.0035	0.0016	0.0105	0.0314	-0.0113	0.0121
	Gamma	1.2137	-0.0115	-0.0135	-0.0075	-0.0067	-0.0057	-0.0046	-0.0030	-0.0010	0.0021	0.0068	0.0153	0.0363	-0.0049	-0.0031
	Exponential	1.1500	-0.0067	-0.0080	-0.0039	-0.0035	-0.0029	-0.0022	-0.0012	0.0000	0.0019	0.0048	0.0102	0.0238	-0.0029	-0.0012
500	Pareto$_2$	1.3529	-0.0429	-0.0507	-0.0408	-0.0398	-0.0384	-0.0368	-0.0347	-0.0319	-0.0283	-0.0229	-0.0137	0.0062	-0.0358	0.0302
	Pareto$_3$	1.2432	-0.0127	-0.0152	-0.0105	-0.0100	-0.0094	-0.0086	-0.0076	-0.0062	-0.0043	-0.0015	0.0035	0.0151	-0.0086	0.0050
	Pareto$_4$	1.2105	-0.0074	-0.0091	-0.0059	-0.0056	-0.0051	-0.0045	-0.0038	-0.0028	-0.0015	0.0006	0.0043	0.0133	-0.0047	0.0014
	Gamma	1.2137	-0.0033	-0.0046	-0.0031	-0.0029	-0.0026	-0.0022	-0.0017	-0.0010	0.0000	0.0015	0.0043	0.0117	-0.0020	-0.0013
	Exponential	1.1500	-0.0019	-0.0028	-0.0016	-0.0015	-0.0013	-0.0011	-0.0008	-0.0004	0.0002	0.0011	0.0027	0.0072	-0.0013	-0.0010
1000	Pareto$_2$	1.3529	-0.0333	-0.0342	-0.0266	-0.0257	-0.0247	-0.0234	-0.0217	-0.0195	-0.0166	-0.0122	-0.0050	0.0110	-0.0223	0.0155
	Pareto$_3$	1.2432	-0.0088	-0.0088	-0.0056	-0.0053	-0.0048	-0.0043	-0.0035	-0.0026	-0.0012	0.0009	0.0044	0.0127	-0.0042	0.0031
	Pareto$_4$	1.2105	-0.0049	-0.0050	-0.0029	-0.0027	-0.0024	-0.0020	-0.0015	-0.0009	0.0001	0.0015	0.0040	0.0102	-0.0021	0.0011
	Gamma	1.2137	-0.0019	-0.0024	-0.0018	-0.0016	-0.0015	-0.0012	-0.0010	-0.0005	0.0001	0.0010	0.0027	0.0072	-0.0008	-0.0007
	Exponential	1.1500	-0.0011	-0.0015	-0.0009	-0.0008	-0.0007	-0.0006	-0.0004	-0.0001	0.0002	0.0008	0.0017	0.0044	-0.0007	-0.0006

TABLE 4

BOOTSTRAP RESULTS, 5000 BOOTSTRAPS, $\rho = 1.2$

n	Distribution	π_ρ	$B(\hat{\pi}_\rho)$	$\bar{B}$	$\overline{B(\hat{\pi}_\rho)}$ for n'/n equal to										JN	ML
					100%	90%	80%	70%	60%	50%	40%	30%	20%	10%		
100	Pareto$_2$	1.5000	-0.1261	-0.1388	-0.1138	-0.1111	-0.1078	-0.1037	-0.0987	-0.0918	-0.0827	-0.0694	-0.0473	-0.0010	-0.0994	—
	Pareto$_3$	1.3333	-0.0458	-0.0508	-0.0363	-0.0347	-0.0328	-0.0303	-0.0272	-0.0230	-0.0172	-0.0085	0.0064	0.0397	-0.0293	0.0337
	Pareto$_4$	1.2857	-0.0296	-0.0330	-0.0218	-0.0206	-0.0190	-0.0171	-0.0147	-0.0113	-0.0067	0.0005	0.0128	0.0413	-0.0169	0.0121
	Gamma	1.2865	-0.0170	-0.0191	-0.0102	-0.0092	-0.0079	-0.0062	-0.0041	-0.0012	0.0030	0.0095	0.0213	0.0500	-0.0068	-0.0031
	Exponential	1.2000	-0.0098	-0.0112	-0.0057	-0.0051	-0.0043	-0.0033	-0.0020	-0.0002	0.0023	0.0064	0.0137	0.0322	-0.0039	-0.0012
500	Pareto$_2$	1.5000	-0.0743	-0.0838	-0.0688	-0.0671	-0.0651	-0.0627	-0.0596	-0.0555	-0.0499	-0.0417	-0.0282	0.0010	-0.0605	0.0302
	Pareto$_3$	1.3333	-0.0209	-0.0238	-0.0168	-0.0161	-0.0151	-0.0140	-0.0125	-0.0104	-0.0076	-0.0034	0.0039	0.0207	-0.0137	0.0050
	Pareto$_4$	1.2857	-0.0120	-0.0139	-0.0091	-0.0086	-0.0079	-0.0071	-0.0060	-0.0046	-0.0026	0.0005	0.0059	0.0189	-0.0072	0.0014
	Gamma	1.2865	-0.0051	-0.0064	-0.0036	-0.0033	-0.0028	-0.0023	-0.0016	-0.0006	0.0008	0.0030	0.0071	0.0177	-0.0025	-0.0013
	Exponential	1.2000	-0.0029	-0.0039	-0.0021	-0.0019	-0.0017	-0.0014	-0.0009	-0.0004	0.0005	0.0018	0.0041	0.0105	-0.0017	-0.0010
1000	Pareto$_2$	1.5000	-0.0590	-0.0602	-0.0480	-0.0467	-0.0451	-0.0430	-0.0405	-0.0371	-0.0326	-0.0259	-0.0150	0.0087	-0.0412	0.0155
	Pareto$_3$	1.3333	-0.0149	-0.0148	-0.0098	-0.0092	-0.0085	-0.0077	-0.0066	-0.0051	-0.0031	0.0000	0.0053	0.0176	-0.0075	0.0031
	Pareto$_4$	1.2857	-0.0081	-0.0081	-0.0049	-0.0045	-0.0040	-0.0034	-0.0027	-0.0017	-0.0004	0.0018	0.0055	0.0145	-0.0035	0.0011
	Gamma	1.2865	-0.0030	-0.0034	-0.0018	-0.0016	-0.0013	-0.0010	-0.0005	0.0000	0.0009	0.0022	0.0047	0.0113	-0.0010	-0.0007
	Exponential	1.2000	-0.0017	-0.0021	-0.0011	-0.0009	-0.0008	-0.0006	-0.0004	0.0000	0.0005	0.0012	0.0027	0.0066	-0.0008	-0.0006

TABLE 5

BOOTSTRAP RESULTS FOR COMPOUND POISSON DISTRIBUTIONS, $\rho = 1.2$

n	Severity Distribution	π_ρ^*	$\bar{B}$	$\overline{\hat{B}(\hat{\pi}_\rho)}$ for n'/n equal to										JN
				100%	90%	80%	70%	60%	50%	40%	30%	20%	10%	
100	Pareto$_2$	1.5498	-0.1493	-0.1236	-0.1207	-0.1172	-0.1129	-0.1076	-0.1006	-0.0910	-0.0763	-0.0527	-0.0021	-0.0097
	Pareto$_3$	1.3939	-0.0589	-0.0433	-0.0414	-0.0393	-0.0366	-0.0332	-0.0286	-0.0222	-0.0122	0.0047	0.0438	-0.0364
	Pareto$_4$	1.3515	-0.0401	-0.0276	-0.0261	-0.0244	-0.0222	-0.0194	-0.0156	-0.0103	-0.0018	0.0128	0.0477	-0.0227
	Exponential	1.2822	-0.0180	-0.0103	-0.0093	-0.0082	-0.0068	-0.0051	-0.0026	0.0010	0.0068	0.0171	0.0436	-0.0080
500	Pareto$_2$	1.5498	-0.0817	-0.0661	-0.0644	-0.0622	-0.0597	-0.0564	-0.0522	-0.0465	-0.0381	-0.0238	0.0065	-0.0575
	Pareto$_3$	1.3939	-0.0243	-0.0171	-0.0163	-0.0153	-0.0140	-0.0125	-0.0103	-0.0074	-0.0029	0.0051	0.0235	-0.0139
	Pareto$_4$	1.3515	-0.0145	-0.0094	-0.0088	-0.0081	-0.0072	-0.0061	-0.0045	-0.0023	0.0011	0.0072	0.0220	-0.0074
	Exponential	1.2822	-0.0049	-0.0025	-0.0022	-0.0019	-0.0014	-0.0009	-0.0001	0.0011	0.0029	0.0064	0.0153	-0.0019
1000	Pareto$_2$	1.5498	-0.0643	-0.0517	-0.0503	-0.0486	-0.0466	-0.0438	-0.0405	-0.0356	-0.0287	-0.0174	0.0070	-0.0446
	Pareto$_3$	1.3939	-0.0180	-0.0128	-0.0122	-0.0115	-0.0106	-0.0094	-0.0079	-0.0057	-0.0025	0.0031	0.0161	-0.0105
	Pareto$_4$	1.3515	-0.0106	-0.0071	-0.0067	-0.0062	-0.0056	-0.0048	-0.0037	-0.0022	0.0001	0.0041	0.0140	-0.0058
	Exponential	1.2822	-0.0034	-0.0019	-0.0018	-0.0016	-0.0013	-0.0005	0.0002	0.0012	0.0033	0.0086		-0.0016

TABLE 6

BOOTSTRAP RESULTS FOR COMPOUND POISSON DISTRIBUTIONS, $\rho = 1.15$

n	Severity Distribution	π_ρ^*	$\bar{B}$	$\overline{\hat{B}(\hat{\pi}_\rho)}$ for n'/n equal to										JN
				100%	90%	80%	70%	60%	50%	40%	30%	20%	10%	
100	Pareto$_2$	1.3903	−0.0969	−0.0791	−0.0771	−0.0747	−0.0717	−0.0680	−0.0631	−0.0564	−0.0459	−0.0291	0.0074	−0.0701
	Pareto$_3$	1.2887	−0.0405	−0.0296	−0.0283	−0.0268	−0.0249	−0.0225	−0.0193	−0.0148	−0.0076	0.0045	0.0329	−0.0251
	Pareto$_4$	1.2599	−0.0283	−0.0194	−0.0184	−0.0171	−0.0156	−0.0136	−0.0110	−0.0072	−0.0010	0.0094	0.0349	−0.0162
	Exponential	1.2115	−0.0133	−0.0077	−0.0071	−0.0063	−0.0053	−0.0040	−0.0022	0.0003	0.0045	0.0120	0.0315	−0.0062
500	Pareto$_2$	1.3903	−0.0491	−0.0389	−0.0378	−0.0364	−0.0347	−0.0326	−0.0298	−0.0260	−0.0203	−0.0107	0.0101	−0.0336
	Pareto$_3$	1.2887	−0.0156	−0.0108	−0.0103	−0.0097	−0.0088	−0.0078	−0.0063	−0.0043	−0.0013	0.0042	0.0170	−0.0089
	Pareto$_4$	1.2599	−0.0097	−0.0062	−0.0059	−0.0054	−0.0048	−0.0040	−0.0029	−0.0014	0.0009	0.0052	0.0154	−0.0050
	Exponential	1.2115	−0.0036	−0.0019	−0.0017	−0.0015	−0.0012	−0.0008	−0.0003	0.0005	0.0018	0.0043	0.0106	−0.0015
1000	Pareto$_2$	1.3903	−0.0379	−0.0298	−0.0290	−0.0279	−0.0266	−0.0248	−0.0226	−0.0195	−0.0149	−0.0075	0.0089	−0.0256
	Pareto$_3$	1.2887	−0.0116	−0.0082	−0.0078	−0.0074	−0.0067	−0.0060	−0.0050	−0.0036	−0.0014	0.0023	0.0112	−0.0068
	Pareto$_4$	1.2599	−0.0071	−0.0048	−0.0046	−0.0042	−0.0038	−0.0033	−0.0026	−0.0016	−0.0001	0.0027	0.0094	−0.0040
	Exponential	1.2115	−0.0025	−0.0015	−0.0014	−0.0012	−0.0010	−0.0008	−0.0005	−0.0001	0.0007	0.0021	0.0058	−0.0013

TABLE 7

BOOTSTRAP RESULTS FOR COMPOUND NEGATIVE BINOMIAL DISTRIBUTIONS, VARIANCE = 1.05, ρ = 1.2

n	Severity Distribution	π_ρ^*	$\bar{B}$	$\overline{\hat{B}(\hat{\pi}_\rho)}$ for n'/n equal to										JN
				100%	90%	80%	70%	60%	50%	40%	30%	20%	10%	
100	Pareto$_2$	1.5520	-0.1363	-0.1100	-0.1072	-0.1036	-0.0993	-0.0938	-0.0866	-0.0764	-0.0616	-0.0371	0.0154	-0.0956
	Pareto$_3$	1.3965	-0.0472	-0.0310	-0.0292	-0.0269	-0.0242	-0.0206	-0.0158	-0.0089	0.0013	0.0190	0.0596	-0.0237
	Pareto$_4$	1.3543	-0.0304	-0.0173	-0.0159	-0.0140	-0.0117	-0.0088	-0.0048	0.0009	0.0097	0.0249	0.0611	-0.0121
	Exponential	1.2858	-0.0123	-0.0042	-0.0033	-0.0021	-0.0007	0.0012	0.0038	0.0076	0.0136	0.0245	0.0519	-0.0018
500	Pareto$_2$	1.5520	-0.0859	-0.0705	-0.0688	-0.0668	-0.0644	-0.0613	-0.0572	-0.0515	-0.0428	-0.0289	0.0018	-0.0619
	Pareto$_3$	1.3965	-0.0255	-0.0182	-0.0174	-0.0164	-0.0153	-0.0137	-0.0116	-0.0086	-0.0040	0.0038	0.0223	-0.0150
	Pareto$_4$	1.3543	-0.0155	-0.0103	-0.0097	-0.0090	-0.0082	-0.0070	-0.0055	-0.0033	0.0002	0.0062	0.0210	-0.0083
	Exponential	1.2858	-0.0057	-0.0032	-0.0029	-0.0026	-0.0022	-0.0016	-0.0008	0.0004	0.0023	0.0057	0.0147	-0.0026
1000	Pareto$_2$	1.5520	-0.0604	-0.0472	-0.0459	-0.0443	-0.0422	-0.0395	-0.0360	-0.0315	-0.0248	-0.0134	0.0117	-0.0402
	Pareto$_3$	1.3965	-0.0180	-0.0128	-0.0122	-0.0115	-0.0106	-0.0095	-0.0080	-0.0059	-0.0027	0.0029	0.0159	-0.0106
	Pareto$_4$	1.3543	-0.0108	-0.0073	-0.0069	-0.0064	-0.0058	-0.0050	-0.0040	-0.0026	-0.0003	0.0038	0.0136	-0.0060
	Exponential	1.2858	-0.0040	-0.0025	-0.0023	-0.0021	-0.0019	-0.0015	-0.0011	-0.0004	0.0007	0.0028	0.0081	-0.0022

TABLE 8

BOOTSTRAP RESULTS FOR COMPOUND NEGATIVE BINOMIAL DISTRIBUTIONS, VARIANCE = 1.2, $\rho = 1.2$

n	Severity Distribution	π_ρ^*	$\overline{B}$	$\overline{B(\hat{\pi}_\rho)}$ for n'/n equal to										JN
				100%	90%	80%	70%	60%	50%	40%	30%	20%	10%	
100	Pareto$_2$	1.5582	-0.1262	-0.0995	-0.0967	-0.0932	-0.0888	-0.0832	-0.0758	-0.0656	-0.0510	-0.0260	0.0260	-0.0855
	Pareto$_3$	1.4041	-0.0402	-0.0237	-0.0219	-0.0197	-0.0169	-0.0132	-0.0084	-0.0015	0.0086	0.0268	0.0676	-0.0167
	Pareto$_4$	1.3626	-0.0244	-0.0111	-0.0096	-0.0078	-0.0055	-0.0025	0.0016	0.0073	0.0160	0.0318	0.0684	-0.0060
	Exponential	1.2963	-0.0087	-0.0002	0.0007	0.0019	0.0034	0.0054	0.0081	0.0121	0.0181	0.0296	0.0580	0.0023
500	Pareto$_2$	1.5582	-0.0911	-0.0761	-0.0745	-0.0724	-0.0699	-0.0669	-0.0628	-0.0573	-0.0489	-0.0349	-0.0045	-0.0680
	Pareto$_3$	1.4041	-0.0270	-0.0198	-0.0190	-0.0180	-0.0168	-0.0152	-0.0132	-0.0102	-0.0056	0.0023	0.0211	-0.0168
	Pareto$_4$	1.3626	-0.0161	-0.0110	-0.0104	-0.0097	-0.0087	-0.0076	-0.0061	-0.0038	-0.0004	0.0058	0.0210	-0.0091
	Exponential	1.2963	-0.0055	-0.0029	-0.0026	-0.0022	-0.0017	-0.0011	-0.0003	0.0009	0.0029	0.0065	0.0161	-0.0022
1000	Pareto$_2$	1.5582	-0.0598	-0.0464	-0.0448	-0.0431	-0.0410	-0.0383	-0.0348	-0.0302	-0.0230	-0.0113	0.0131	-0.0392
	Pareto$_3$	1.4041	-0.0186	-0.0133	-0.0127	-0.0120	-0.0111	-0.0100	-0.0085	-0.0064	-0.0032	0.0025	0.0155	-0.0111
	Pareto$_4$	1.3626	-0.0114	-0.0078	-0.0074	-0.0069	-0.0063	-0.0056	-0.0045	-0.0031	-0.0008	0.0034	0.0133	-0.0066
	Exponential	1.2963	-0.0044	-0.0029	-0.0027	-0.0024	-0.0022	-0.0018	-0.0013	-0.0006	0.0005	0.0027	0.0083	-0.0025