Canad. J. Math. Vol. 75 (3), 2023 pp. 828–853
http://dx.doi.org/10.4153/S0008414X22000190

Existence of stationary vortex sheets for the 2D incompressible Euler equation

Daomin Cao, Guolin Qin, and Changjun Zou

Abstract. We construct a new type of planar Euler flows with localized vorticity. Let $\kappa_i \neq 0$, $i = 1, \dots, m$, be m arbitrarily fixed constants. For any given nondegenerate critical point $\mathbf{x}_0 = (x_{0,1}, \dots, x_{0,m})$ of the Kirchhoff–Routh function defined on Ω^m corresponding to $(\kappa_1, \dots, \kappa_m)$, we construct a family of stationary planar flows with vortex sheets that have large vorticity amplitude and concentrate on curves perturbed from small circles centered near $x_{0,i}$, $i = 1, \dots, m$. The proof is accomplished via the implicit function theorem with suitable choice of function spaces.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded or unbounded domain. We consider the stationary Euler equation

$$\begin{cases} \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P, & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} \cdot \nu = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathbf{u} = (u_1, u_2)$ is the velocity field, P is the scalar pressure, and ν is the outward unit normal of $\partial\Omega$.

In a planar flow, the vorticity is defined as the third component of the curl of the velocity field $(u_1, u_2, 0)$, namely, $\omega = \partial_1 u_2 - \partial_2 u_1$. Taking the curl of the first equation in (1.1), we find that ω satisfies the following vorticity equation:

$$\mathbf{u} \cdot \nabla \omega = 0 \quad \text{in } \Omega. \tag{1.2}$$

The velocity is recovered by the Biot–Savart law

$$\mathbf{u} = \nabla^{\perp} (-\Delta)^{-1} \omega,$$

where $(x_1, x_2)^{\perp} = (x_2, -x_1)$ and the operator $(-\Delta)^{-1}$ is given by

$$(-\Delta)^{-1} \omega(x) = \int_{\Omega} G(x - y) \omega(y) dy.$$

Received by the editors January 11, 2022; revised April 21, 2022; accepted May 2, 2022.
Published online on Cambridge Core May 5, 2022.
This work was supported by the NNSF of China Grant (No. 11831009).
AMS subject classification: **76B47**, 35Q31, 76B03.
Keywords: Euler equation, vortex sheets, non-degenerate, the Birkhoff–Rott operator, implicit function theorem.

https://doi.org/10.4153/S0008414X22000190 Published online by Cambridge University Press

Here, $G(x,y)$ is the Green function of $-\Delta$ in Ω with zero Dirichlet data. So $G(x,y)$ takes the form $G(x,y)=\frac{1}{2\pi}\ln\frac{1}{|x-y|}-H(x,y)$ with $H(x,y)$ the harmonic part of $G(x,y)$. We denote $\psi=(-\Delta)^{-1}\omega$ to be the stream function, then the velocity field can be derived by $\mathbf{u}=\nabla^{\perp}\psi$.

In the last century, the two-dimensional Euler equation has been intensively studied, and the global well-posedness of the vorticity equation with initial data in $L^1\cap L^\infty$ was proved by Yudovich in the classical paper [45]. However, many physical phenomena possess strong and irregular fluctuations, such as fluids with small viscosity, where flows tend to separate from rigid walls and sharp corners [5, 37]. To model this phenomenon mathematically, the most natural way is to think of a solution to the Euler equation, in which the velocity changes sign discontinuously across a stream line. This discontinuity induces vorticity concentrated on a curve, which is only a measure rather than a bounded function.

A velocity discontinuity in an inviscid 2D flow is called a vortex sheet, whose vorticity concentrates as a measure along a curve. Suppose that ω is a weak solution to the Euler equation concentrated on a finite number of closed curves Γ_i parameterized by $z_i(\theta)$. Namely, for any test function $\phi\in C_c^\infty(\Omega)$, ω is a measure such that

$$\int_\Omega \phi(x)d\omega(x)=\sum_i\int\gamma_i(\alpha)\phi(z_i(\alpha))|z_i'(\alpha)|d\alpha,$$

where $\gamma_i(\alpha)$ is the vorticity strength at $z_i(\alpha)$. Then, the equation of the sheet can be derived by the Birkhoff–Rott operator in a domain [6, 23, 34, 37, 41]

(1.3)
$$BR(z,\gamma)(x):=\frac{1}{2\pi}P.V.\int\frac{(x-z(\alpha))^{\perp}}{|x-z(\alpha)|^2}\gamma(\alpha)|z'(\alpha)|d\alpha+\int\nabla^{\perp}H(x,z(\alpha))\gamma(\alpha)|z'(\alpha)|d\alpha,$$

where $P.V.$ stands for Cauchy principal value of an integral. Equation (1.3) yields the motion of the sheet

$$\mathbf{u}(z_i(\theta))=-BR(z_i(\theta)) \tag{1.4}$$

with $BR(z_i(\theta)):=-\sum_j BR(z_j,\gamma_j)(z_i(\theta))$.

Significant efforts have been made in mathematical study of the theory of vortex sheet. In the elegant paper [20], Delort proved global existence of weak solutions with an initial L^2_{loc} velocity and a positive measure vorticity. Later, the proof was simplified by Majda [36]. Duchon and Robert [21] established global existence for a class of initial data concentrated closed to a line. Existence in different setting of vortex sheet with a distinguished sign was also obtained in [22, 42]. For vorticity without a definite sign, only partial results on the existence are known under some additional assumptions [35, 43, 44]. Note that uniqueness for such solutions still remains open.

On the other hand, blow up may occur in the motion of vortex sheet. Indeed, singular formulation was conjectured by Birkhoff [6], and by Birkhoff and Fisher [7]. In [39], Moore showed the possibility that the curvature blows up in finite time even though the initial data are analytic. Moore's result was also supported by numerical study [30]. Ill-posedness for vortex sheet problem in the space H^s with $s>\frac{3}{2}$ was obtained by Caflisch and Orellana [9]. These results demonstrate that the study of vortex sheet is extremely delicate, and hence exact solutions, in particular relative equilibria, are of great importance since their structures persist for long time.

Nevertheless, very few relative equilibria are known. For the vortex sheets in $\mathbb{R}^2$, except for circles and lines, the only nontrivial examples include: uniformly rotating segment [4], in which the vorticity is supported on a segment of length $2a$ with density

$$\gamma(x) = \Omega_r \sqrt{a^2 - x^2}, \qquad \text{for } x \in [-a, a]$$

and angular velocity Ω_r. A generalization of the rotating segment is the Protas–Sakajo class [40], which is made out of segments rotating about a common center of rotation with endpoints at the vertices of a regular polygon. Recently, a new class of vortex sheet was obtained in [24] via degenerate bifurcation from rotating circles. Note that the existence of nontrivial steady vortex sheet in $\mathbb{R}^2$ is not apparent in view of the rigidity results obtained in [23], where the authors showed for uniformly rotating vortex sheets with angular velocity $\Omega_r \le 0$ and strength $\gamma > 0$, only trivial solutions exist.

In a domain $\Omega \subsetneq \mathbb{R}^2$, as far as we know, there seems no nontrivial stationary vortex sheet is known so far. The purpose of the present paper is to construct a family of stationary vortex sheets for a domain (bounded or not), whenever the Kirchhoff–Routh function possesses nondegenerate critical points.

For any given integer $m > 0$, and m real numbers $\kappa_1, \kappa_2, \ldots, \kappa_m$, define the Kirchhoff–Routh function on $\Omega^m = \{\mathbf{x} = (x_1, x_2, \ldots, x_m) \mid x_i \in \Omega, \text{for } i = 1, \ldots, m\}$ as follows:

$$\mathcal{W}_m(x_1, x_2, \ldots, x_m) = -\sum_{i \neq j}^{m} \kappa_i \kappa_j G(x_i, x_j) + \sum_{i=1}^{m} \kappa_i^2 H(x_i, x_i). \tag{1.5}$$

It is known that the location of m-point vortices with strength κ_i $(i = 1, \ldots, m)$ in Ω must be a critical point of $\mathcal{W}_m$ (see, e.g., [31, 32]). Results on the existence and nondegeneracy of critical points for $\mathcal{W}_m$ can be found in [2, 3]. In [25], it was proved that if Ω is convex, then there is no critical point of $\mathcal{W}_m$ in Ω^m with $m \ge 2$ and $\kappa_i > 0$ for all $i = 1, \ldots, m$. Let us point out that although the nondegeneracy of critical points for the Kirchhoff–Routh functions in a general domain is not an easy issue, it is true for most of the domains, as proved in [1, 38]. On the other hand, in [8], it was shown that in a convex domain, $\mathcal{W}_1$ has a unique critical point, which is also nondegenerate. In a recent paper [12], the first author, Yan, and Yu obtained some existence and results on the nondegeneracy of critical points of the Kirchhoff–Routh function for unbounded domains.

Giving a nondegenerate critical point $\mathbf{x}_0 = (x_{0,1}, \ldots, x_{0,m}) \in \Omega^m$ of $\mathcal{W}_m$, for ε small, we will construct a branch of vortex sheets concentrated on a finite number of closed curves Γ_i. Moreover, each Γ_i is the perturbation of a small circle with radius ε centered at some point $x_{\varepsilon,\tau,i} \in \Omega$ close to $x_{0,i}$, and the vorticity $\omega\big|_{\Gamma_i}$ satisfies $\int_0^{2\pi} \gamma_i(\alpha)|z_i'(\alpha)|d\alpha \approx \kappa_i$. This result shows the rich diversity of stationary vortex sheet solutions despite that the well-posedness is not fully understood.

Our main theorem is as follows.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a domain (may be unbounded), and let $\kappa_i \neq 0$ $(i = 1, \ldots, m)$ be m given numbers. Suppose that $\mathbf{x}_0 = (x_{0,1}, \ldots, x_{0,m}) \in \Omega^m$ with $x_{0,i} \neq x_{0,j}$, for $i \neq j$, is an isolated critical point of $\mathcal{W}_m$ defined by (1.5) satisfying the nondegeneracy* *condition:* $\deg(\nabla \mathcal{W}_m, \mathbf{x}_0) \neq 0$. *Then, there are* $\varepsilon_0 > 0$ *and* $\tau_0 > 0$, *such that for all* $0 < \varepsilon < \varepsilon_0$ *and* $-\tau_0 < \tau < \tau_0$, *there exists a stationary vortex sheet* $\omega_{\varepsilon,\tau}$ *possessing the following properties:*

(i) $\omega_{\varepsilon,\tau} = \sum_{i=1}^{m} \gamma_i \delta_{\Gamma_i}$ *concentrates on a finite number of closed curves* Γ_i *with strength* γ_i. *Moreover, it holds that* $\gamma_i = \frac{\kappa_i + O(\varepsilon)}{2\pi\varepsilon}$ *and each* Γ_i *is a perturbation of a small circle with radius* ε *and centered at some point* $x_{\varepsilon,\tau,i} \in \Omega$ *satisfying* $|x_{\varepsilon,\tau,i} - x_{0,i}| = O(\varepsilon)$.

(ii) *As* $\varepsilon \to 0^+$, *one has in the sense of measure*

$$\omega_{\varepsilon,\tau} \to \sum_{i=1}^{m} \kappa_i \delta(x - x_{0,i}) \ \text{weakly},$$

where $\delta(x - x_{0,i})$ *is the Dirac delta function concentrated at the point* $x_{0,i}$.

(iii) *For any* $i = 1, \ldots, m$, *the interior of* Γ_i *is convex.*

Remark 1.2 Our result does not rely on the sign of κ_i, which is essential in the global existence of the initial problem as mentioned above. As we shall see in Section 5, the parameter τ stands for the projection on the kernel of the linearized operator, and it slightly affects the shape of Γ_i. More precisely, Γ_i in Theorem 1.1 takes the form $\Gamma_i = \{x_{\varepsilon,\tau,i} + \varepsilon(1 + \varepsilon(f_{\varepsilon,\tau,i}(\theta) + \tau f_{0,i}(\theta)))(\cos\theta, \sin\theta) \mid \theta \in [0, 2\pi)\}$ for some $f_{\varepsilon,\tau,i}$ depending on ε and τ and some fixed $f_{0,i}$ in the kernel.

Remark 1.3 For simplicity, all the scales of Γ_i $(i = 1, \ldots, m)$ are chosen to be of the same order. However, this is not necessary, and one may construct vortex sheet concentrated on Γ_i with the scale of ε_i $(i = 1, \ldots, m)$ different from each other.

Remark 1.4 Fixing $\tau \in (-\tau_0, \tau_0)$ in Theorem 1.1, say $\tau = 0$, we obtain a family of solutions with vortex sheet ω_ε parameterized only by ε, which is of special interest since it is closely related to the classical problem: regularization of point vortices for the Euler equation. This means justifying the weak formulation for point vortex solutions of the incompressible Euler equations by approximating these solutions with more regular solutions. In fact, the vortex sheets obtained in Theorem 1.1 belong to the space $H^{-1}(\Omega)$, whereas the point vortices solution belongs to $H^{-1-\sigma}(\Omega)$ for any $\sigma > 0$, which is more singular than a vortex sheet. Thus, our result can be regarded as a desingularization of point vortices by vortex sheet in some way. For more literature on the desingularization problem, we refer to [10, 11, 17, 28, 33] and the references therein.

Next, we shall sketch the basic ideas used to prove the main result. Thanks to Lemma 2.1 in [23], we are able to formulate the conditions that the Birkhoff–Rott integral (1.3) satisfies for a stationary vortex sheet into a system of $2m$ coupled integrodifferential equations $\mathcal{F}_{i,1}(\varepsilon, \mathbf{x}, \boldsymbol{f}, \boldsymbol{g}) = 0$ and $\mathcal{F}_{i,2}(\varepsilon, \mathbf{x}, \boldsymbol{f}, \boldsymbol{g}) = 0$, $i = 1, \ldots, m$. We expect that the case $(\varepsilon, \mathbf{x}, \boldsymbol{f}, \boldsymbol{g}) = (0, \mathbf{x}_0, 0, 0)$ corresponds to the point vortices and hence $(0, \mathbf{x}_0, 0, 0)$ is a solution to $\mathcal{F}_{i,1} = 0$ and $\mathcal{F}_{i,2} = 0$ provided that $\mathbf{x}_0$ is a critical point of $\mathcal{W}_m$. Therefore, the first step is to extend $\mathcal{F}_{i,1}$ and $\mathcal{F}_{i,2}$ such that $\varepsilon \le 0$ is allowed. Then, one can verify that $\mathcal{F}_{i,1}(0, \mathbf{x}_0, 0, 0) = \mathcal{F}_{i,2}(0, \mathbf{x}_0, 0, 0) = 0$ does hold when $\mathbf{x}_0$ is a critical point of $\mathcal{W}_m$, and hence we obtain a trivial solution. To apply the implicit function theorem at the solution $(0, \mathbf{x}_0, 0, 0)$, the Gateaux derivative of $\mathcal{F} := (\mathcal{F}_{1,1}, \mathcal{F}_{1,2}, \ldots, \mathcal{F}_{m,1}, \mathcal{F}_{m,2})$ should be an isomorphism, which unfortunately fails. Detailed calculations show that $D\mathcal{F}$ has a $2m$-dimensional kernel $\prod_{i=1}^{m}\{(a\cos(\theta)+b\sin(\theta),\kappa_i(a\cos(\theta)+b\sin(\theta)))\mid(a,b)\in\mathbb{R}^2\}$. Hence, we have to consider the equations in quotient spaces and impose the conditions $-\kappa_i\int\mathcal{F}_{i,1}\sin(\theta)d\theta=\int\mathcal{F}_{i,2}\cos(\theta)d\theta$ and $\kappa_i\int\mathcal{F}_{i,1}\cos(\theta)d\theta=\int\mathcal{F}_{i,2}\sin(\theta)d\theta$ for all $i=1,\dots,m$. Although these conditions seem to be complicated, we successfully convert them into a concise equation $\nabla\mathcal{W}_m(\mathbf{x})=O(\varepsilon)$, which is solvable near $\mathbf{x}_0$ due to the nondegeneracy of $\nabla\mathcal{W}_m$ at $\mathbf{x}_0$. Finally, we can apply the implicit function theorem to obtain the existence. The convexity of the interior of Γ_i follows from calculating the curvature directly. We point out that our procedure of proving Theorem 1.1 borrows the idea of Lyapunov–Schmidt reduction and local bifurcation theory.

The ideas and methods introduced in the present paper may be widely applied to a variety of situations and other models. For example, one may consider an ideal fluid with an irrotational background flow $u_0=\nabla^{\perp}\psi_0$, where ψ_0 is a given harmonic function. In this case, the Kirchhoff–Routh function is given by (see [12])

$$\mathcal{W}_{m,\psi_0}(x_1,x_2,\dots,x_m)=-\sum_{i\neq j}^{m}\kappa_i\kappa_jG(x_i,x_j)+\sum_{i=1}^{m}\kappa_i^2H(x_i,x_i)+2\sum_{i=1}^{m}\kappa_i\psi_0(x_i).$$

Although $\mathcal{W}_{m,\psi_0}$ is slightly different from $\mathcal{W}_m$ given by (1.5), we believe that our method can be modified to construct vortex sheets near critical points of $\mathcal{W}_{m,\psi_0}$ in this situation. In addition, for domains with some symmetry properties, such as the unit disk or the half-space, one may modify our method by considering in function spaces with certain symmetries to construct solutions with vortex sheets near degenerate critical points of the Kirchhoff–Routh function.

We would like to make a brief remark on the approach of constructing vortex patches via the contour dynamics equation, which shares a similar spirit as the construction of vortex sheet we consider in the present paper. Many celebrated contributions have been made with the contour dynamics equation method in recent years (see, e.g., [13–16, 18, 19, 26–29] and the references therein). However, since a vortex patch is actually a bounded function, the contour dynamics equation is more regular than the equations of a vortex sheet. Hence, more effort is needed in the process of our proof.

This paper is organized as follows. In Section 2, we derive the equations that the Birkhoff–Rott integral satisfies for a stationary vortex sheet and define the function spaces which will be used later. In Section 3, we extend the functionals and show their C^1 regularity. Section 4 is devoted to study the linearization operators, where we prove that the derivative is an isomorphism in quotient spaces. In Section 5, we choose $\mathbf{x}$ properly such that the range of our functional belongs to the quotient spaces and apply the implicit function theorem to prove Theorem 1.1.

2 Formulation and functional setting

Since ω is a stationary sheet, using Lemma 2.1 in [23], we derive the following equations that the BR equation (1.4) and γ_i satisfy.

$$BR(z_i(\theta))\cdot\mathbf{n}(z_i(\theta))=0, \tag{2.1}$$

where $\mathbf{n}(z_i(\theta))$ is the unit normal vector of Γ_i at $z_i(\theta)$, and

$$BR(z_i(\theta)) \cdot \mathbf{s}(z_i(\theta)) \frac{\gamma_i(\theta)}{|z_i'(\theta)|} = C, \tag{2.2}$$

where $\mathbf{s}(z_i(\theta))$ is the unit tangential vector. Note that (2.2) can be rewritten as

$$(I - P_0)\left[BR(z_i(\theta)) \cdot \mathbf{s}(z_i(\theta)) \frac{\gamma_i(\theta)}{|z_i'(\theta)|}\right] = 0, \tag{2.3}$$

where P_0 is the projection to the mean value defined by $P_0 f := \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$. Since $\mathbf{x}_0 \in \Omega^m$, we can take $r_0 > 0$ sufficiently small such that $B_{r_0}(x_{0,i}) \subset \Omega$ for all $i = 1, \ldots, m$, where $B_{r_0}(x_{0,i})$ is the ball with radius r_0 and centered at $x_{0,i}$. We aim to construct vortex sheets localized near $\mathbf{x}_0$. Thus, for $\varepsilon > 0$ small, we assume that $z_i (i = 1, \ldots, m)$ are of the following form:

$$z_i(\theta) = x_i + \varepsilon R_i(\theta)(\cos\theta, \sin\theta)$$

with $R_i(\theta) = 1 + \varepsilon f_i(\theta)$, and $x_i \in B_{r_0}(x_{0,i})$ to be chosen later. We also assume

$$\gamma_i(\alpha) = \frac{\kappa_i + \varepsilon g_i(\alpha)}{2\pi|z_i'(\alpha)|}.$$

We end this section by introducing some notations and definitions that will be used in this paper and reformulating equations (2.1) and (2.3). Denote the mean value of integral of g on the unit circle by

$$⨍ g(\tau)d\tau := \frac{1}{2\pi}\int_0^{2\pi} g(\tau)d\tau$$

and set

$$A(\theta,\alpha) := 4\sin^2\left(\frac{\theta-\alpha}{2}\right), \quad A_{ij} = |x_i - x_j|^2,$$

$$B(f,\theta,\alpha) := 4(f(\theta)+f(\alpha))\sin^2\left(\frac{\theta-\alpha}{2}\right) + \varepsilon\left((f(\theta)-f(\alpha))^2 + 4f(\theta)f(\alpha)\sin^2\left(\frac{\theta-\alpha}{2}\right)\right),$$

$$\begin{aligned} B_{ij}(\theta,\alpha) = {} & 2(x_i - x_j)\cdot((\cos\theta,\sin\theta) - (\cos\alpha,\sin\alpha)) + 2\varepsilon(x_i - x_j)\cdot\big(f_i(\theta)(\cos\theta,\sin\theta) \\ & - f_j(\theta)(\cos\alpha,\sin\alpha)\big) + \varepsilon((1+\varepsilon f_i(\theta))(\cos\theta,\sin\theta) - (1+\varepsilon f_j(\alpha))(\cos\alpha,\sin\alpha))^2. \end{aligned}$$

For $k \geq 3$, we will also frequently use the function spaces given in the following, whose norms are naturally defined as norms of product spaces.

$$X^k = \left\{ g \in H^k \;\middle|\; g(\theta) = \sum_{j=1}^{\infty} a_j\cos(j\theta) + b_j\sin(j\theta) \right\},$$

$$X_i^k := \left\{ (f_1, f_2) \in X^{k+1} \times X^k \;\middle|\; \begin{array}{l} -\kappa_i ⨍ f_1(\theta)\cos(\theta)d\theta = ⨍ f_2(\theta)\cos(\theta)d\theta, \\ -\kappa_i ⨍ f_1(\theta)\sin(\theta)d\theta = ⨍ f_2(\theta)\sin(\theta)d\theta \end{array} \right\},$$

https://doi.org/10.4153/S0008414X22000190 Published online by Cambridge University Press

$$Y_i^k := \left\{ (f_1, f_2) \in X^k \times X^k \;\middle|\; \begin{cases} -\kappa_i \fint f_1(\theta)\sin(\theta)d\theta = \fint f_2(\theta)\cos(\theta)d\theta, \\ \kappa_i \fint f_1(\theta)\cos(\theta)d\theta = \fint f_2(\theta)\sin(\theta)d\theta \end{cases} \right\},$$

$$\mathcal{X}^k := \{(\boldsymbol{f}, \boldsymbol{g}) \mid (f_i, g_i) \in X_i^k, \ i = 1, \dots, m\},$$

$$\mathcal{Y}^k := \{(\boldsymbol{f}, \boldsymbol{g}) \mid (f_i, g_i) \in Y_i^k, \ i = 1, \dots, m\}.$$

For given $\boldsymbol{f} = (f_1, \dots, f_m)$ and $\boldsymbol{g} = (g_1, \dots, g_m)$, denote $\tilde{g}_{i,\varepsilon}(t) = \kappa_i + \varepsilon g_i(t)$. Then, we can reduce equations (2.1) and (2.3) to

(2.4)

$$\begin{aligned}
0 &= \mathcal{F}_{i,1}(\varepsilon, \mathbf{x}, \boldsymbol{f}, \boldsymbol{g}) \\
&:= \frac{1}{\varepsilon} P.V. \fint \frac{R_i(\alpha)\sin(\theta-\alpha)}{A(\theta,\alpha) + \varepsilon B(f_i,\theta,\alpha)} \tilde{g}_{i,\varepsilon}(\alpha) d\alpha + \frac{1}{R_i(\theta)} P.V. \fint \frac{f_i'(\theta) R_i(\alpha)(1-\cos(\theta-\alpha))}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)} \tilde{g}_{i,\varepsilon}(\alpha) d\alpha \\
&+ \frac{1}{R_i(\theta)} P.V. \fint \frac{\varepsilon f_i'(\theta)(f_i(\theta) - f_i(\alpha))}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)} \tilde{g}_{i,\varepsilon}(\alpha) d\alpha \\
&+ \sum_{j \neq i} \frac{1}{R_i(\theta)} \fint \frac{(x_i - x_j) \cdot \big(R_i(\theta)(-\sin\theta, \cos\theta) + \varepsilon f_i'(\theta)(\cos\theta, \sin\theta)\big)}{A_{ij} + \varepsilon B_{ij}(\theta,\alpha)} \tilde{g}_{j,\varepsilon}(\alpha) d\alpha \\
&+ \sum_{j \neq i} \frac{1}{R_i(\theta)} \fint \frac{\varepsilon^2 f_i'(\theta) R_i(\theta) - \varepsilon^2 f_i'(\theta) R_j(\alpha)\cos(\theta-\alpha) + \varepsilon R_i(\theta) R_j(\alpha)\sin(\theta-\alpha)}{A_{ij} + \varepsilon B_{ij}(\theta,\alpha)} \tilde{g}_{i,\varepsilon}(\alpha) d\alpha \\
&- \sum_{j=1}^{m} \frac{2\pi}{R_i(\theta)} \fint \nabla H(z_i(\theta), z_j(\alpha)) \cdot \big(R_i(\theta)(-\sin\theta, \cos\theta) + \varepsilon f_i'(\theta)(\cos\theta, \sin\theta)\big) \tilde{g}_{i,\varepsilon}(\alpha) d\alpha \\
&=: \mathcal{F}_{i,11} + \mathcal{F}_{i,12} + \mathcal{F}_{i,13} + \mathcal{F}_{i,14} + \mathcal{F}_{i,15} + \mathcal{F}_{i,16},
\end{aligned}$$

and

$$0 = \mathcal{F}_{i,2} := (I - P_0)\tilde{\mathcal{F}}_{i,2}, \tag{2.5}$$

where $\tilde{\mathcal{F}}_{i,2}$ is given by

(2.6)

$$\begin{aligned}
\tilde{\mathcal{F}}_{i,2}(\varepsilon, \mathbf{x}, \boldsymbol{f}, \boldsymbol{g}) &:= \frac{\tilde{g}_{i,\varepsilon}(\theta)}{\varepsilon(R_i(\theta)^2 + (R_i'(\theta))^2)} P.V. \fint \frac{\varepsilon f_i'(\theta) R_i(\alpha)\sin(\theta-\alpha)}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)} \tilde{g}_{i,\varepsilon}(\alpha) d\alpha \\
&+ \frac{\tilde{g}_{i,\varepsilon}(\theta)}{\varepsilon(R_i(\theta)^2 + (R_i'(\theta))^2)} P.V. \fint \frac{R_i(\theta) R_i(\alpha)(\cos(\theta-\alpha) - 1)}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)} \tilde{g}_{i,\varepsilon}(\alpha) d\alpha \\
&+ \frac{\tilde{g}_{i,\varepsilon}(\theta)}{R_i(\theta)^2 + (R_i'(\theta))^2} P.V. \fint \frac{R_i(\theta)(f_i(\alpha) - f_i(\theta))}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)} \tilde{g}_{i,\varepsilon}(\alpha) d\alpha \\
&+ \sum_{j \neq i} \frac{\tilde{g}_{i,\varepsilon}(\theta)}{R_i(\theta)^2 + (R_i'(\theta))^2} \fint \frac{(x_i - x_j)^{\perp} \cdot \big(R_i(\theta)(-\sin\theta, \cos\theta) + \varepsilon f_i'(\theta)(\cos\theta, \sin\theta)\big)}{A_{ij} + \varepsilon B_{ij}(\theta,\alpha)} \tilde{g}_{j,\varepsilon}(\alpha) d\alpha \\
&+ \sum_{j \neq i} \frac{\tilde{g}_{i,\varepsilon}(\theta)}{R_i(\theta)^2 + (R_i'(\theta))^2} \fint \frac{-\varepsilon R_i^2(\theta) + \varepsilon^2 f_i'(\theta) R_j(\alpha)\sin(\theta-\alpha) + \varepsilon R_i(\theta) R_j(\alpha)\cos(\theta-\alpha)}{A_{ij} + \varepsilon B_{ij}(\theta,\alpha)} \tilde{g}_{j,\varepsilon}(\alpha) d\alpha
\end{aligned}$$

https://doi.org/10.4153/S0008414X22000190 Published online by Cambridge University Press

$$-\sum_{j=1}^{m}\frac{2\pi\tilde{g}_{i,\varepsilon}(\theta)}{R_i(\theta)^2+(R_i'(\theta))^2}\int \nabla^{\perp}H(z_i(\theta),z_j(\alpha))\cdot[R_i(\theta)(-\sin\theta,\cos\theta)+\varepsilon f_i'(\theta)(\cos\theta,\sin\theta)]$$

$$\times\,\tilde{g}_{i,\varepsilon}(\alpha)d\alpha =: \mathcal{F}_{i,21}+\mathcal{F}_{i,22}+\mathcal{F}_{i,23}+\mathcal{F}_{i,24}+\mathcal{F}_{i,25}+\mathcal{F}_{i,26}.$$

3 Extension and regularity of functionals

To apply the implicit function theorem at $\varepsilon = 0$, we need to extend the functions $\mathcal{F}_{i,1}$ and $\mathcal{F}_{i,2}$ defined in Section 2 to $\varepsilon \le 0$ and check the C^1 regularity.

Let us first show the continuity of these functionals. Letting V be the unit ball centered at origin in $\left(X^{k+1}\times X^k\right)^m$ and $B_{r_0}(\mathbf{x}_0)$ be the ball centered at $\mathbf{x_0}$ in Ω^m with radius r_0, we have the following proposition.

Proposition 3.1 *The functionals $\mathcal{F}_{i,1}$ and $\mathcal{F}_{i,2}$ can be extended from $(-\varepsilon_0,\varepsilon_0)\times B_{r_0}(\mathbf{x}_0)\times V$ to $X^k\times X^k$ as continuous functionals.*

Proof Throughout the proof, we will frequently use the following Taylor's formula:

$$\frac{1}{(A+B)^{\lambda}}=\frac{1}{A^{\lambda}}-\lambda\int_0^1\frac{B}{(A+tB)^{1+\lambda}}dt. \tag{3.1}$$

Let us consider $\mathcal{F}_{i,1}$ first. We need to prove that $\partial^l\mathcal{F}_{i,1}\in L^2$ for $l=0,1,\dots,k$. For the first term

$$\mathcal{F}_{i,11} := \frac{1}{\varepsilon}P.V.\int\frac{(1+\varepsilon f_i(\alpha))\sin(\theta-\alpha)}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}(\kappa_i+\varepsilon g_i(\alpha))d\alpha,$$

since $R_i(x)=1+\varepsilon f_i(x)$, the possible singularity caused by $\varepsilon=0$ may occur only when we take zeroth-order derivative of $\mathcal{F}_{i,11}$. Thus, we first show that $\mathcal{F}_{i,11}\in L^2$. We decompose the kernel into two parts

$$\frac{1}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}=\frac{1}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}\cdot\frac{1}{1+2\varepsilon f_i(\theta)+\varepsilon^2(f_i(\theta)^2+f_i'(\theta)^2)}+\mathcal{K}_R, \tag{3.2}$$

where $\mathcal{K}_R := \frac{1}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}-\frac{1}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}\cdot\frac{1}{1+2\varepsilon f_i(\theta)+\varepsilon^2(f_i(\theta)^2+f_i'(\theta)^2)}$ is more regular than $\frac{1}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}$. Indeed, by using (3.1), we calculate

$$\begin{aligned}
&\sin(\theta-\alpha)\,\mathcal{K}_R\\
&=\frac{\sin(\theta-\alpha)}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}-\frac{\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}\cdot\frac{1}{1+2\varepsilon f_i(\theta)+\varepsilon^2(f_i(\theta)^2+f_i'(\theta)^2)}\\
&=\frac{\sin(\theta-\alpha)}{1+2\varepsilon f_i(\theta)+\varepsilon^2(f_i(\theta)^2+f_i'(\theta)^2)}
\end{aligned}$$

https://doi.org/10.4153/S0008414X22000190 Published online by Cambridge University Press

$$\times \frac{\left(1+2\varepsilon f_i(\theta)+\varepsilon^2(f_i(\theta)^2+f_i'(\theta)^2)\right)4\sin^2\left(\frac{\theta-\alpha}{2}\right)-A(\theta,\alpha)-\varepsilon B(f_i,\theta,\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)(A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha))}$$

$$= \frac{\varepsilon\sin(\theta-\alpha)}{1+2\varepsilon f_i(\theta)+\varepsilon^2(f_i(\theta)^2+f_i'(\theta)^2)}$$

$$\times \frac{f_i(\theta)-f_i(\alpha)+\varepsilon f_i(\theta)(f_i(\theta)-f_i(\alpha))+\varepsilon\frac{4f_i'(\theta)^2\sin^2\left(\frac{\theta-\alpha}{2}\right)-(f_i(\theta)-f_i(\alpha))^2}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}$$

$$= \varepsilon\left(\frac{(f_i(\theta)-f_i(\alpha))\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}+O(\varepsilon)\right),$$

where the constant in $O(\varepsilon)$ depends on $\|f\|_{W^{2,\infty}} \le C\|f\|_{H^3}$. This implies

$$|\sin(\theta-\alpha)\mathcal{K}_R| \le C\varepsilon.$$

Then, it is easy to see that

$$\frac{1}{\varepsilon}\fint \mathcal{K}_R\sin(\theta-\alpha)(1+\varepsilon f_i(\alpha))(\kappa_i+\varepsilon g_i(\alpha))d\alpha$$

$$= \fint \frac{(1+\varepsilon f_i(\alpha))(\kappa_i+\varepsilon g_i(\alpha))(f_i(\theta)-f_i(\alpha))\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}+O(\varepsilon)$$

$$= \fint \frac{\kappa_i(f_i(\theta)-f_i(\alpha))\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}+\varepsilon\mathcal{R}_{111}$$

$$= P.V.\fint \frac{-\kappa_i f_i(\alpha)\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}+\varepsilon\mathcal{R}_{111},$$

where $\mathcal{R}_{111}$ is regular and bounded. Hence, to prove $\mathcal{F}_{i,11}\in L^2$, we only need to estimate the rest term $\frac{1}{1+2\varepsilon f_i(\theta)+\varepsilon^2(f_i(\theta)^2+f_i'(\theta)^2)}\frac{1}{\varepsilon}P.V.\fint\frac{\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}(1+\varepsilon f_i(\alpha))(\kappa_i+\varepsilon g_i(\alpha))d\alpha$. By the odd symmetry and (3.1), we have

$$\frac{1}{1+2\varepsilon f_i(\theta)+\varepsilon^2(f_i(\theta)^2+f_i'(\theta)^2)}\frac{1}{\varepsilon}P.V.\fint\frac{\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}(1+\varepsilon f_i(\alpha))(\kappa_i+\varepsilon g_i(\alpha))d\alpha$$

$$= \frac{1}{1+2\varepsilon f_i(\theta)+\varepsilon^2(f_i(\theta)^2+f_i'(\theta)^2)}P.V.\fint\frac{\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}(\kappa_i f_i(\alpha)+g_i(\alpha)+\varepsilon f_i(\alpha)g_i(\alpha))d\alpha$$

$$= P.V.\fint\frac{(\kappa_i f_i(\alpha)+g_i(\alpha))\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}d\alpha+\varepsilon\mathcal{R}_{112}.$$

Using the expansion $f_i(\alpha)=f_i(\theta)+O(|\sin\left(\frac{\theta-\alpha}{2}\right)|)$ and $g_i(\alpha)=g_i(\theta)+O(|\sin\left(\frac{\theta-\alpha}{2}\right)|)$, then we find

$$P.V.\fint\frac{(\kappa_i f_i(\alpha)+g_i(\alpha))\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}d\alpha=\fint\frac{\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}O(|\sin\left(\frac{\theta-\alpha}{2}\right)|)d\alpha=O(1),$$

https://doi.org/10.4153/S0008414X22000190 Published online by Cambridge University Press

where we have used the fact $\left|\frac{\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}O\left(\sin\left(\frac{\theta-\alpha}{2}\right)\right)\right| \le C$. Therefore, it holds that $P.V.\fint \frac{(\kappa_i f_i(\alpha)+g_i(\alpha))\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}d\alpha$ belongs to L^∞ and hence belongs to L^2. Moreover, $\mathcal{R}_{112}$ is regular and bounded. We conclude that $\mathcal{F}_{i,11} \in L^\infty$. Furthermore, it holds

(3.3)

$$\begin{aligned}\mathcal{F}_{i,11} &= \frac{1}{1+2\varepsilon f_i(\theta)+\varepsilon^2(f_i(\theta)^2+f_i'(\theta)^2)}\frac{1}{\varepsilon}P.V.\fint \frac{\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}(1+\varepsilon f_i(\alpha))(\kappa_i+\varepsilon g_i(\alpha))d\alpha\\
&+\frac{1}{\varepsilon}\fint \mathcal{K}_R \sin(\theta-\alpha)(1+\varepsilon f_i(\alpha))(\kappa_i+\varepsilon g_i(\alpha))d\alpha\\
&= P.V.\fint \frac{g_i(\alpha))\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}d\alpha+\varepsilon\mathcal{R}_{11},\end{aligned}$$

where $\mathcal{R}_{11} = \mathcal{R}_{111} + \mathcal{R}_{112}$ is regular and bounded.

Next, we prove that $\partial^k\mathcal{F}_{i,11} \in L^2$. To simplify notation, we rewrite $\mathcal{F}_{i,11}$ as follows by changing the variable α to $\theta-\alpha$:

$$\mathcal{F}_{i,11} := \frac{1}{\varepsilon}P.V.\fint \frac{(1+\varepsilon f_i(\theta-\alpha))\sin(\alpha)}{A(\theta,\theta-\alpha)+\varepsilon B(f_i,\theta,\theta-\alpha)}(\kappa_i+\varepsilon g_i(\theta-\alpha))d\alpha.$$

Taking kth derivatives of $\mathcal{F}_{i,11}$, we see that the most singular term is

$$\begin{aligned}&P.V.\fint \frac{\partial^k f_i(\theta-\alpha)\sin(\alpha)}{A(\theta,\theta-\alpha)+\varepsilon B(f_i,\theta,\theta-\alpha)}(\kappa_i+\varepsilon g_i(\theta-\alpha))d\alpha\\
&+P.V.\fint \frac{(1+\varepsilon f_i(\theta-\alpha))\sin(\alpha)}{A(\theta,\theta-\alpha)+\varepsilon B(f_i,\theta,\theta-\alpha)}\partial^k g_i(\theta-\alpha)d\alpha\\
&-P.V.\fint \frac{(1+\varepsilon f_i(\theta-\alpha))(\kappa_i+\varepsilon g_i(\theta-\alpha))\sin(\alpha)}{(A(\theta,\theta-\alpha)+\varepsilon B(f_i,\theta,\theta-\alpha))^2}\Bigg[4(\partial^k f_i(\theta)+\partial^k f_i(\theta-\alpha))\sin^2\left(\frac{\theta-\alpha}{2}\right)\\
&+2\varepsilon(f_i(\theta)-f_i(\theta-\alpha))(\partial^k f_i(\theta)-\partial^k f_i(\theta-\alpha))\\
&+4\varepsilon(\partial^k f_i(\theta)f_i(\theta-\alpha)+f_i(\theta)\partial^k f_i(\theta-\alpha))\sin^2\left(\frac{\theta-\alpha}{2}\right)\Bigg]d\alpha\\
&=: I_1+I_2+I_3.\end{aligned}$$

We first deal with I_1. By the splitting of the kernel (3.2), we derive

$$\begin{aligned}I_1 &= P.V.\fint \frac{\partial^k f_i(\theta-\alpha)\sin(\alpha)}{A(\theta,\theta-\alpha)+\varepsilon B(f_i,\theta,\theta-\alpha)}(\kappa_i+\varepsilon g_i(\theta-\alpha))d\alpha\\
&= P.V.\fint \frac{\partial^k f_i(\alpha)\sin(\theta-\alpha)}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}(\kappa_i+\varepsilon g_i(\alpha))d\alpha\end{aligned}$$

https://doi.org/10.4153/S0008414X22000190 Published online by Cambridge University Press

$$= \frac{1}{1+2\varepsilon f_i(\theta)+\varepsilon^2(f_i(\theta)^2+f_i'(\theta)^2)} P.V.\!\!\fint \frac{\partial^k f_i(\alpha)(\kappa_i+\varepsilon g_i(\alpha))\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}$$

$$+ P.V.\!\!\fint \mathcal{K}_R \sin(\theta-\alpha)\partial^k f_i(\alpha)(\kappa_i+\varepsilon g_i(\alpha))d\alpha.$$

Noting that $|\mathcal{K}_R\sin(\theta-\alpha)|\le C\varepsilon$, we have

$$\|P.V.\!\!\fint \mathcal{K}_R \sin(\theta-\alpha)\partial^k f_i(\alpha)(\kappa_i+\varepsilon g_i(\alpha))d\alpha\|_{L^2} \le C\|f_i\|_{H^k}(1+\|g\|_{L^2})$$

is bounded. Since $P.V.\!\fint \frac{\partial^k f_i(\alpha)(\kappa_i+\varepsilon g_i(\alpha))\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}$ is the Hilbert transformation of the function $\partial^k f_i(\alpha)(\kappa_i+\varepsilon g_i(\alpha))$, we have

$$\left\|P.V.\!\!\fint \frac{\partial^k f_i(\alpha)(\kappa_i+\varepsilon g_i(\alpha))\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}\right\|_{L^2} \le \|\partial^k f_i(\alpha)(\kappa_i+\varepsilon g_i(\alpha))\|_{L^2}\le C\|f\|_{H^k}(1+\|g\|_{L^2}).$$

Similarly, one can check that $\|I_2\|_{L^2}\le C(1+\|f\|_{L^2})\|g\|_{H^k}$.

To estimate I_3, we split the kernel as follows:

$$\frac{4\sin^2\left(\frac{\alpha}{2}\right)}{(A(\theta,\theta-\alpha)+\varepsilon B(f_i,\theta,\theta-\alpha))^2} = \frac{1}{4\sin^2\left(\frac{\alpha}{2}\right)}\cdot\frac{1}{(1+2\varepsilon f_i(\theta)+\varepsilon^2(f_i(\theta)^2+f_i'(\theta)^2))^2}+\tilde{\mathcal{K}}_R,$$

where $\tilde{\mathcal{K}}_R$ satisfies $|\tilde{\mathcal{K}}_R\sin\alpha|\le C$. Since convolution with the kernel $\frac{\sin\alpha}{4\sin^2\left(\frac{\alpha}{2}\right)}$ defines the Hilbert transformation, we find that

$$P.V.\!\!\fint \frac{(1+\varepsilon f_i(\theta-\alpha))(\kappa_i+\varepsilon g_i(\theta-\alpha))\sin(\alpha)}{(A(\theta,\theta-\alpha)+\varepsilon B(f_i,\theta,\theta-\alpha))^2}$$

$$\sin^2\left(\frac{\theta-\alpha}{2}\right)\Big((\partial^k f_i(\theta)+\partial^k f_i(\theta-\alpha))(\partial^k f_i(\theta)f_i(\theta-\alpha)+f_i(\theta)\partial^k f_i(\theta-\alpha))\Big)d\alpha$$

belongs to L^2 due to the L^2 boundedness of Hilbert transformation and the regularity of $\tilde{\mathcal{K}}_R$. For the remaining term in I_3

$$2\varepsilon P.V.\!\!\fint \frac{(1+\varepsilon f_i(\theta-\alpha))(\kappa_i+\varepsilon g_i(\theta-\alpha))\sin(\alpha)}{(A(\theta,\theta-\alpha)+\varepsilon B(f_i,\theta,\theta-\alpha))^2}(f_i(\theta)-f_i(\theta-\alpha))(\partial^k f_i(\theta)-\partial^k f_i(\theta-\alpha))d\alpha,$$

we decompose the kernel

$$\frac{(1+\varepsilon f_i(\theta-\alpha))(\kappa_i+\varepsilon g_i(\theta-\alpha))\sin(\alpha)}{(A(\theta,\theta-\alpha)+\varepsilon B(f_i,\theta,\theta-\alpha))^2}(f_i(\theta)-f_i(\theta-\alpha)) = \frac{(1+\varepsilon f_i(\theta))(\kappa_i+\varepsilon g_i(\theta))f_i'(\theta)}{4\sin^2\left(\frac{\alpha}{2}\right)}+\bar{\mathcal{K}}_R,$$

where $\bar{\mathcal{K}}_R$ satisfies $|\bar{\mathcal{K}}_R\sin\alpha|\le C$. Then, we deduce

$$P.V.\!\!\fint \frac{(1+\varepsilon f_i(\theta-\alpha))(\kappa_i+\varepsilon g_i(\theta-\alpha))\sin(\alpha)}{(A(\theta,\theta-\alpha)+\varepsilon B(f_i,\theta,\theta-\alpha))^2}(f_i(\theta)-f_i(\theta-\alpha))(\partial^k f_i(\theta)-\partial^k f_i(\theta-\alpha))d\alpha$$

$$= (1+\varepsilon f_i(\theta))(\kappa_i+\varepsilon g_i(\theta))f'(\theta)P.V.\!\!\fint \frac{\partial^k f_i(\theta)-\partial^k f_i(\theta-\alpha)}{4\sin^2\left(\frac{\alpha}{2}\right)}d\alpha$$

$$+ P.V.\!\!\fint \bar{\mathcal{K}}_R(\partial^k f_i(\theta)-\partial^k f_i(\theta-\alpha))d\alpha$$

$$= (1+\varepsilon f_i(\theta))(\kappa_i+\varepsilon g_i(\theta))f_i'(\theta)\left((-\Delta)^{\frac{1}{2}}(\partial^k f_i)\right)(\theta)+P.V.\!\!\fint \bar{\mathcal{K}}_R(\partial^k f_i(\theta)-\partial^k f_i(\theta-\alpha))d\alpha.$$

By Fourier transformation and the Hardy inequality, we obtain

$$\|(-\Delta)^{\frac{1}{2}}(\partial^k f_i)\|_{L^2} \le \|\nabla \partial^k f_i\|_{L^2} \le \|f_i\|_{H^{k+1}}$$

and

$$\left\| \fint \bar{\mathcal{K}}_R(\partial^k f_i(\theta) - \partial^k f_i(\theta-\alpha))d\alpha \right\|_{L^2} \le \|\nabla \partial^k f_i\|_{L^2} \le \|f_i\|_{H^{k+1}}.$$

Consequently, we have $\partial^k \mathcal{F}_{i,11} \in L^2$ and hence $\mathcal{F}_{i,11} \in H^k$.

Now, we turn to the second term

$$\mathcal{F}_{i,12} := \frac{1}{1+\varepsilon f_i(\theta)} P.V.\fint \frac{f_i'(\theta)(1+\varepsilon f_i(\alpha))(1-\cos(\theta-\alpha))}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}(\kappa_i + \varepsilon g_i(\alpha))d\alpha.$$

Since $|1-\cos(\theta-\alpha)| = \sin^2\left(\frac{\theta-\alpha}{2}\right)$, the kernel of this term is actually regular and bounded. Therefore, it is easy to see that $\mathcal{F}_{i,12} \in H^k$. Moreover, by (3.1), we find

$$\begin{aligned} \mathcal{F}_{i,12} &= \frac{1}{1+\varepsilon f_i(\theta)} P.V.\fint \frac{f_i'(\theta)(1+\varepsilon f_i(\alpha))(1-\cos(\theta-\alpha))}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}(\kappa_i + \varepsilon g_i(\alpha))d\alpha \\ &= P.V.\fint \frac{\kappa_i f_i'(\theta)(1-\cos(\theta-\alpha))}{A(\theta,\alpha)}d\alpha + \varepsilon \mathcal{R}_{12} \end{aligned}$$

$$= \frac{\kappa_i}{2} f_i'(\theta) + \varepsilon \mathcal{R}_{12}, \tag{3.4}$$

where $\mathcal{R}_{12}$ is smooth and we have used the identity $1-\cos(\theta-\alpha) = 2\sin^2\left(\frac{\theta-\alpha}{2}\right) = \frac{A(\theta,\alpha)}{2}$.

For $\mathcal{F}_{i,13}$, taking kth derivatives of $\mathcal{F}_{i,13}$, we see that the most singular terms are

$$\begin{aligned} &\frac{\varepsilon \partial^{k+1} f_i(\theta)}{1+\varepsilon f_i(\theta)} P.V.\fint \frac{f_i(\theta)-f_i(\alpha)}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}(\kappa_i+\varepsilon g_i(\alpha))d\alpha \\ &\quad + \frac{\varepsilon f_i'(\theta)}{1+\varepsilon f_i(\theta)} P.V.\fint \frac{\partial^k f_i(\theta)-\partial^k f_i(\theta-\alpha)}{A(\theta,\theta-\alpha)+\varepsilon B(f_i,\theta,\theta-\alpha)}(\kappa_i+\varepsilon g_i(\theta-\alpha))d\alpha \\ &\quad + \frac{\varepsilon^2 f_i'(\theta)}{1+\varepsilon f_i(\theta)} P.V.\fint \frac{(f_i(\theta)-f_i(\theta-\alpha))\partial^k g_i(\theta-\alpha)}{A(\theta,\theta-\alpha)+\varepsilon B(f_i,\theta,\theta-\alpha)}d\alpha \\ &\quad - \frac{\varepsilon f_i'(\theta)}{1+\varepsilon f_i(\theta)} P.V.\fint \frac{(f_i(\theta)-f_i(\theta-\alpha))(\kappa_i+\varepsilon g_i(\theta-\alpha))}{(A(\theta,\theta-\alpha)+\varepsilon B(f_i,\theta,\theta-\alpha))^2}\Big[4(\partial^k f_i(\theta)+\partial^k f_i(\theta-\alpha))\sin^2\left(\tfrac{\theta-\alpha}{2}\right) \\ &\qquad + 2\varepsilon(f_i(\theta)-f_i(\theta-\alpha))(\partial^k f_i(\theta)-\partial^k f_i(\theta-\alpha)) \\ &\qquad + 4\varepsilon(\partial^k f_i(\theta) f_i(\theta-\alpha) + f_i(\theta)\partial^k f_i(\theta-\alpha))\sin^2\left(\tfrac{\theta-\alpha}{2}\right)\Big]d\alpha \\ &=: J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Since $P.V.\fint \frac{f_i(\theta)-f_i(\alpha)}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}(\kappa_i+\varepsilon g_i(\alpha))d\alpha \in L^2$, by Taylor's expansion of $f_i(\alpha)$ at $\alpha=\theta$, we know that the first term J_1 is bounded in L^2. To deal with J_2, we split the kernel

$$\frac{\kappa_i+\varepsilon g_i(\theta-\alpha)}{A(\theta,\theta-\alpha)+\varepsilon B(f_i,\theta,\theta-\alpha)} = \frac{\kappa_i+\varepsilon g_i(\theta)}{4\sin^2\left(\frac{\alpha}{2}\right)} + \hat{\mathcal{K}}_R,$$

https://doi.org/10.4153/S0008414X22000190 Published online by Cambridge University Press

where $|\hat{\mathcal{K}}_R \sin\frac{\alpha}{2}| \le C$. Therefore, we conclude

$$\begin{aligned}
&\frac{\varepsilon f_i'(\theta)}{1+\varepsilon f_i(\theta)} P.V.\fint \frac{\partial^k f_i(\theta)-\partial^k f_i(\theta-\alpha)}{A(\theta,\theta-\alpha)+\varepsilon B(f_i,\theta,\theta-\alpha)}(\kappa_i+\varepsilon g_i(\theta-\alpha))d\alpha\\
&=\frac{\varepsilon f_i'(\theta)(\kappa_i+\varepsilon g_i(\theta))}{1+\varepsilon f_i(\theta)} P.V.\fint \frac{\partial^k f_i(\theta)-\partial^k f_i(\theta-\alpha)}{4\sin^2\left(\frac{\alpha}{2}\right)}d\alpha\\
&\quad+\frac{\varepsilon f_i'(\theta)}{1+\varepsilon f_i(\theta)} P.V.\fint \hat{\mathcal{K}}_R(\partial^k f_i(\theta)-\partial^k f_i(\theta-\alpha))d\alpha\\
&=\frac{\varepsilon f_i'(\theta)(\kappa_i+\varepsilon g_i(\theta))}{1+\varepsilon f_i(\theta)}(-\Delta)^{\frac{1}{2}}(\partial^k f_i)(\theta)\\
&\quad+\frac{\varepsilon f_i'(\theta)}{1+\varepsilon f(\theta)} P.V.\fint \hat{\mathcal{K}}_R(\partial^k f_i(\theta)-\partial^k f_i(\theta-\alpha))d\alpha.
\end{aligned}$$

By Fourier transformation and Hardy inequality, we obtain

$$\|(-\Delta)^{\frac{1}{2}}(\partial^k f_i)\|_{L^2} \le \|\nabla\partial^k f_i\|_{L^2} \le \|f_i\|_{H^{k+1}}$$

and

$$\left\|\fint \hat{\mathcal{K}}_R(\partial^k f_i(\theta)-\partial^k f_i(\theta-\alpha))d\alpha\right\|_{L^2} \le \|\nabla\partial^k f_i\|_{L^2} \le \|f_i\|_{H^{k+1}}.$$

We can show that the remaining terms J_3 and J_4 are bounded in L^2 similarly. Moreover, it can be seen that

(3.5)

$$\mathcal{F}_{i,13} = \frac{1}{1+\varepsilon f_i(\theta)} P.V.\fint \frac{\varepsilon f_i'(\theta)(f_i(\theta)-f_i(\alpha))}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}(\kappa_i+\varepsilon g_i(\alpha))d\alpha = \varepsilon\mathcal{R}_{13},$$

where $\mathcal{R}_{13}$ is regular.

Since $H(x,y)$ is smooth in Ω, the terms $\mathcal{F}_{i,14}$, $\mathcal{F}_{i,15}$, and $\mathcal{F}_{i,16}$ are apparently smooth and belong to H^k. Furthermore, we have

(3.6)

$$\begin{aligned}
&\mathcal{F}_{i,14}+\mathcal{F}_{i,15}+\mathcal{F}_{i,16}\\
&=\textstyle\sum_{j\neq i}\fint \frac{\kappa_j(x_i-x_j)\cdot(-\sin\theta,\cos\theta)}{|x_{i,\varepsilon}-x_{j,\varepsilon}|^2}d\alpha-\sum_{j=1}^m 2\pi\fint \kappa_j\nabla H(x_i,x_j)\cdot(-\sin\theta,\cos\theta)d\alpha+\varepsilon\mathcal{R}_{14}\\
&=\textstyle\sum_{j\neq i}2\pi\fint \kappa_j\nabla G(x_i,x_j)\cdot(-\sin\theta,\cos\theta)d\alpha\\
&\quad-2\pi\fint \kappa_i\nabla H(x_i,x_i)\cdot(-\sin\theta,\cos\theta)d\alpha+\varepsilon\mathcal{R}_{14},
\end{aligned}$$

where $\mathcal{R}_{14}$ is bounded and smooth.

By (3.3)–(3.6), we conclude

(3.7)

$$\begin{aligned}
\mathcal{F}_{i,1}(\varepsilon,\mathbf{x},\boldsymbol{f},\boldsymbol{g}) = P.V.\fint \frac{\sin(\theta-\alpha)g_i(\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}d\alpha+\frac{\kappa_i}{2}f_i'(\theta)+\textstyle\sum_{j\neq i}2\pi\kappa_j\nabla G(x_i,x_j)\cdot(-\sin\theta,\cos\theta)\\
-2\pi\kappa_i\nabla H(x_i,x_i)\cdot(-\sin\theta,\cos\theta)+\varepsilon\mathcal{R}_1,
\end{aligned}$$

https://doi.org/10.4153/S0008414X22000190 Published online by Cambridge University Press

where $\mathcal{R}_1 := \mathcal{R}_{11} + \mathcal{R}_{12} + \mathcal{R}_{13} + \mathcal{R}_{14}$ is regular. Hence, we can define

(3.8)
$$\mathcal{F}_{i,1}(0,\mathbf{x},\boldsymbol{f},\boldsymbol{g}) := P.V.\fint \frac{\sin(\theta-\alpha)g_i(\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}d\alpha + \frac{\kappa_i}{2}f_i'(\theta) + \sum_{j\neq i} 2\pi\kappa_j \nabla G(x_i,x_j)\cdot(-\sin\theta,\cos\theta) \\ -2\pi\kappa_i\nabla H(x_i,x_i)\cdot(-\sin\theta,\cos\theta).$$

Next, we prove the continuity of $\mathcal{F}_{i,1}$. By (3.7) and the definition of $\mathcal{F}_{i,1}(0,\mathbf{x},\boldsymbol{f},\boldsymbol{g})$, one can easily check that $\mathcal{F}_{i,1}$ is continuous with respect to ε at $\varepsilon = 0$. Thus, we only need to prove that $\mathcal{F}_{i,1}$ is continuous with respect to ε for $\varepsilon \neq 0$. However, it is easy to see that the continuity of $\mathcal{F}_{i,1}$ with respect to ε is a consequence of its continuity with respect to $\boldsymbol{f}$ and $\boldsymbol{g}$ when $\varepsilon \neq 0$, on which we will focus below.

We only prove the continuity of $\mathcal{F}_{i,11}$ with respect to f_i and g_i, and the continuity of other terms in $\mathcal{F}_{i,1}$ can be proved by a similar or even easier way. We will use the following notations: for a general function h, we denote $\Delta h = h(\theta) - h(\alpha)$, $\quad h = h(\theta)$, $\quad \tilde{h} = h(\alpha)$, and

$$D(h) = \varepsilon^2(\Delta h)^2 + 4(1+\varepsilon h)(1+\varepsilon\tilde{h})\sin^2\left(\frac{\theta-\alpha}{2}\right).$$

To show the continuity of $\mathcal{F}_{i,11}$ with respect to f_i, let $(f_1,g),(f_2,g)\in X_i^k$. Then, we can calculate the difference

$$\mathcal{F}_{i,11}(\varepsilon,f_2,g) - \mathcal{F}_{i,11}(\varepsilon,f_1,g) = P.V.\fint \frac{(f_2(\alpha)-f_1(\alpha))\sin(\theta-\alpha)}{D(f_2)}(\kappa+\varepsilon g(\alpha))d\alpha \\ +\frac{1}{\varepsilon}P.V.\fint(1+\varepsilon f_1(\alpha))(\kappa+\varepsilon g(\alpha))\sin(\theta-\alpha)\left(\frac{1}{D(f_2)}-\frac{1}{D(f_1)}\right)d\alpha =: K_1+K_2.$$

For the first term K_1, since $\frac{1}{D(f_2)}$ has the same singularity as $\frac{1}{\sin^2\left(\frac{\theta-\alpha}{2}\right)}$, it is easy to prove $\|K_1\|_{H^k} \le C\|f_1-f_2\|_{H^{k+1}}$ by the technique we have used before. For the second term K_2, since

$$\frac{1}{D(f_2)}-\frac{1}{D(f_1)} \\ = \frac{\varepsilon^2((\Delta f_1)^2-(\Delta f_2)^2)+4\varepsilon\big((f_1-f_2)(1+\varepsilon\tilde{f}_2)+(\tilde{f}_1-\tilde{f}_2)(1+\varepsilon f_1)\big)\sin^2(\frac{x-y}{2})}{D(f_1)D(f_2)} \\ = \varepsilon\frac{\varepsilon(\Delta f_1+\Delta f_2)(\Delta f_1-\Delta f_2)+4\big((f_1-f_2)(1+\varepsilon\tilde{f}_2)+(\tilde{f}_1-\tilde{f}_2)(1+\varepsilon f_1)\big)\sin^2(\frac{x-y}{2})}{D(f_1)D(f_2)},$$

it holds that the singularity of $\frac{1}{D(f_2)}-\frac{1}{D(f_1)}$ is also of the order $O\left(\frac{1}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)}\right)$, the same as the kernel in $\mathcal{F}_{i,11}$ itself. Therefore, using argument similar to the above, we can prove that $\|K_2\|_{H^k} \le C\|f_2-f_1\|_{H^{k+1}}$, which shows the continuity of $\mathcal{F}_{i,11}$ with respect to f_i. Notice that $\mathcal{F}_{i,11}$ is linear with respect to g_i. Then, the continuity of $\mathcal{F}_{i,11}$ with respect to g can be obtained by argument similar to the proof of boundedness of $\mathcal{F}_{i,11}$ in H^k.

We have shown that the conclusion of Proposition 3.1 holds true for $\mathcal{F}_{i,1}$. The fact that $\mathcal{F}_{i,2}$ is well defined and continuous can be verified in a similar way. Attention should be paid to the fact that the projection operator $I-P_0$ eliminates all constant terms in $\tilde{\mathcal{F}}_{i,2}$, which also removes singularity in $\mathcal{F}_{i,2}$. By (3.1), we obtain

$$\begin{aligned}\tilde{\mathcal{F}}_{i,2}(\varepsilon,\mathbf{x}_\varepsilon,\boldsymbol{f},\boldsymbol{g}) = \kappa_i^2 f_i(\theta) - \kappa_i^2 P.V.\int \tfrac{f_i(\theta)-f_i(\alpha)}{4\sin^2(\frac{\theta-\alpha}{2})}d\alpha - \kappa_i \tfrac{g_i(\theta)}{2} \\ - \textstyle\sum_{j\neq i} 2\pi\kappa_i\kappa_j \nabla^\perp G(x_i,x_j)\cdot(-\sin\theta,\cos\theta) \\ +2\pi\kappa_i^2\nabla^\perp H(x_i,x_i)\cdot(-\sin\theta,\cos\theta) + \varepsilon\mathcal{R}_2,\end{aligned} \tag{3.9}$$

where $\mathcal{R}_2$ is smooth. Thus, we define

$$\begin{aligned}\tilde{\mathcal{F}}_{i,2}(0,\mathbf{x},\boldsymbol{f},\boldsymbol{g})(\theta) = \kappa_i^2 f_i(\theta) - \kappa_i^2 P.V.\int \tfrac{f_i(\theta)-f_i(\alpha)}{4\sin^2(\frac{\theta-\alpha}{2})}d\alpha - \kappa_i \tfrac{g_i(\theta)}{2} \\ - \textstyle\sum_{j\neq i} 2\pi\kappa_i\kappa_j \nabla^\perp G(x_i,x_j)\cdot(-\sin\theta,\cos\theta) \\ +2\pi\kappa_i^2\nabla^\perp H(x_i,x_i)\cdot(-\sin\theta,\cos\theta).\end{aligned} \tag{3.10}$$

■

Our next proposition concerns the C^1 regularity.

Proposition 3.2 *The Gateaux derivatives $\partial_{(\boldsymbol{f},\boldsymbol{g})}\mathcal{F}_{i,1}$ and $\partial_{(\boldsymbol{f},\boldsymbol{g})}\mathcal{F}_{i,2}$ exist and are continuous.*

Proof We first prove that the derivative of $\mathcal{F}_{i,11}$ with respect to f_i exists and is as follows:

$$\partial_{f_i}\mathcal{F}_{i,11}h = Fh, \qquad \forall\, h\in X^{k+1}, \tag{3.11}$$

where Fh is given by

(3.12)

$$\begin{aligned}Fh := P.V.\int \frac{h(\alpha)\sin(\theta-\alpha)}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}(\kappa_i+\varepsilon g_i(\alpha))d\alpha \\ - P.V.\int \frac{(1+\varepsilon f_i(\alpha))(\kappa_i+\varepsilon g_i(\alpha))\sin(\theta-\alpha)}{(A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha))^2}\left[4(h(\theta)+h(\alpha))\sin^2\left(\frac{\theta-\alpha}{2}\right)\right. \\ \left. +2\varepsilon(f_i(\theta)-f_i(\alpha))(h(\theta)-h(\alpha)) + 4\varepsilon\,(h(\theta)f_i(\alpha)+h(\alpha)f_i(\theta))\sin^2\left(\frac{\theta-\alpha}{2}\right)\right]d\alpha.\end{aligned}$$

To prove (3.11), one needs to verify

$$\lim_{t\to 0}\left\|\frac{\mathcal{F}_{i,11}(\varepsilon,f_i+th,g_i)-\mathcal{F}_{i,11}(\varepsilon,f_i,g_i)}{t} - Fh\right\|_{H^k} = 0. \tag{3.13}$$

Using the notations given in the proof of Proposition 3.1, we deduce

$$\begin{aligned}&\frac{\mathcal{F}_{i,11}(\varepsilon,f_i+th,g_i)-\mathcal{F}_{i,11}(\varepsilon,f_i,g_i)}{t} - Fh \\ &= \frac{1}{t\varepsilon}\int (1+\varepsilon f_i(\alpha))(\kappa_i+\varepsilon g_i(\alpha))\sin(\theta-\alpha) \\ &\quad\times\left(\frac{1}{D(f_i+th)} - \frac{1}{D(f_i)} + t\frac{2\varepsilon^2\Delta f_i\Delta h + 4\varepsilon((1+\varepsilon\tilde{f}_i)h + (1+\varepsilon f_i(\alpha))\tilde{h})\sin^2(\frac{x-y}{2})}{D(f_i)^2}\right) \\ &\quad+\int h(\alpha)(\kappa_i+\varepsilon g_i(\alpha))\sin(\theta-\alpha)\left(\frac{1}{D(f_i+th)}-\frac{1}{D(f_i)}\right)dy \\ &=: F_1+F_2.\end{aligned}$$

https://doi.org/10.4153/S0008414X22000190 Published online by Cambridge University Press

By the mean value theorem, we find

$$\frac{1}{D(f_i+th)}-\frac{1}{D(f_i)}=O\left(\frac{t\varepsilon}{4\sin^2\left(\frac{x-y}{2}\right)}\right),$$

$$\frac{1}{D(f_i+th)}-\frac{1}{D(f_i)}+t\frac{2\varepsilon^2\Delta f_i\Delta h+4\big(\varepsilon\tilde{R}h+\varepsilon\tilde{h}R\big)\sin^2(\frac{x-y}{2})}{D(f_i)^2}=O\left(\frac{t^2\varepsilon^2}{4\sin^2\left(\frac{x-y}{2}\right)}\right),$$

which means that the kernels in F_1 and F_2 are of the same order as the kernel in $\mathcal{F}_{i,11}$. Therefore, by argument similar to Proposition 3.1, we have

$$\|F_1\|_{H^k}+\|F_2\|_{H^k}\le Ct\|h\|_{X^{k+1}}.$$

Letting $t\to 0$, we obtain (3.13) and hence obtain the existence of Gateaux derivative of $\mathcal{F}_{i,11}$. To prove the continuity of $\partial_{f_i}\mathcal{F}_{i,11}(\varepsilon,f_i,g_i)h$, one just needs to verify by definition. Since there is no other new idea than the proof of continuity for $\mathcal{F}_{i,11}$, we omit it therefore. The existence and continuity of Gateaux derivatives of other terms in $\mathcal{F}_{i,1}$ and $\mathcal{F}_{i,2}$ can be obtained via similar argument, which we leave out here. Noting that $\mathcal{F}_{i,1}$ is linearly dependent on g and $\mathcal{F}_{i,2}$ is quadratically dependent on g, it is much easier to compute their Gateaux derivatives with respect to g, so we leave them to our reader. For readers' convenience, we also write down the derivatives of $\mathcal{F}_{i,1}$ and $\mathcal{F}_{i,2}$ in the following form directly without proof here.

Recall the definitions $\tilde{g}_{i,\varepsilon}(t)=\kappa_i+\varepsilon g_i(t)$, $R_i(t)=1+\varepsilon f_i(t)$. For any $h_1\in X^{k+1}$ and $h_2\in X^k$, we have

(3.14)
$$\begin{aligned}
\partial_{f_i}\mathcal{F}_{i,1}(\varepsilon,\mathbf{x},\boldsymbol{f},\boldsymbol{g})h_1 =\ & P.V.\!\!\fint \frac{h_1(\alpha)\sin(\theta-\alpha)}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}\tilde{g}_{i,\varepsilon}(\alpha)d\alpha\\
&-P.V.\!\!\fint \frac{R_i(\alpha)\sin(\theta-\alpha)}{(A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha))^2}\tilde{g}_{i,\varepsilon}(\alpha)\Big[4(h_1(\theta)+h_1(\alpha))\sin^2\left(\tfrac{\theta-\alpha}{2}\right)\\
&\quad+2\varepsilon(f_i(\theta)-f_i(\alpha))(h_1(\theta)-h_1(\alpha))+4\varepsilon\left(h_1(\theta)f_i(\alpha)+h_1(\alpha)f_i(\theta)\right)\sin^2\left(\tfrac{\theta-\alpha}{2}\right)\Big]d\alpha\\
&-\frac{\varepsilon h_1(\theta)}{R_i(\theta)^2}P.V.\!\!\fint \frac{f_i'(\theta)R_i(\alpha)(1-\cos(\theta-\alpha))}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}\tilde{g}_{i,\varepsilon}(\alpha)d\alpha\\
&+\frac{1}{R_i(\theta)}P.V.\!\!\fint \frac{(h_1'(\theta)R_i(\alpha)+\varepsilon f_i'(\theta)h_1(\alpha))(1-\cos(\theta-\alpha))}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}\tilde{g}_{i,\varepsilon}(\alpha)d\alpha\\
&-\frac{1}{R_i(\theta)}P.V.\!\!\fint \frac{f_i'(\theta)R_i(\alpha)\tilde{g}_{i,\varepsilon}(\alpha)(1-\cos(\theta-\alpha))}{(A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha))^2}\Big[4(h_1(\theta)+h_1(\alpha))\sin^2\left(\tfrac{\theta-\alpha}{2}\right)\\
&\quad+2\varepsilon(f_i(\theta)-f_i(\alpha))(h_1(\theta)-h_1(\alpha))+4\varepsilon\left(h_1(\theta)f_i(\alpha)+h_1(\alpha)f_i(\theta)\right)\sin^2\left(\tfrac{\theta-\alpha}{2}\right)\Big]d\alpha\\
&-\frac{\varepsilon h_1(\theta)}{R_i(\theta)^2}P.V.\!\!\fint \frac{\varepsilon f_i'(\theta)(f_i(\theta)-f_i(\alpha))}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}\tilde{g}_{i,\varepsilon}(\alpha)d\alpha\\
&+\frac{1}{R_i(\theta)}P.V.\!\!\fint \frac{\varepsilon(h_1'(\theta)(f_i(\theta)-f_i(\alpha))+f_i'(\theta)(h_1(\theta)-h_1(\alpha)))}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}\tilde{g}_{i,\varepsilon}(\alpha)d\alpha\\
&-\frac{1}{R_i(\theta)}P.V.\!\!\fint \frac{\varepsilon f_i'(\theta)(f_i(\theta)-f_i(\alpha))\tilde{g}_{i,\varepsilon}(\alpha)}{(A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha))^2}\Big[4(h_1(\theta)+h_1(\alpha))\sin^2\left(\tfrac{\theta-\alpha}{2}\right)\\
&\quad+2\varepsilon(f_i(\theta)-f_i(\alpha))(h_1(\theta)-h_1(\alpha))+4\varepsilon\left(h_1(\theta)f_i(\alpha)+h_1(\alpha)f_i(\theta)\right)\sin^2\left(\tfrac{\theta-\alpha}{2}\right)\Big]d\alpha\\
&+O(\varepsilon),
\end{aligned}$$

(3.15)
$$\partial_{f_j}\mathcal{F}_{i,1}(\varepsilon,\mathbf{x},\boldsymbol{f},\boldsymbol{g})h_1=O(\varepsilon),$$

https://doi.org/10.4153/S0008414X22000190 Published online by Cambridge University Press

(3.16)

$$\partial_{g_i}\mathcal{F}_{i,1}(\varepsilon,\mathbf{x},\boldsymbol{f},\boldsymbol{g})h_2 = P.V.\fint \frac{R_i(\alpha)\sin(\theta-\alpha)}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}h_2(\alpha)d\alpha$$
$$+\frac{\varepsilon}{R_i(\theta)}P.V.\fint \frac{f_i'(\theta)R_i(\alpha)(1-\cos(\theta-\alpha))}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}h_2(\alpha)d\alpha + \frac{\varepsilon}{R_i(\theta)}P.V.\fint \frac{\varepsilon f_i'(\theta)(f_i(\theta)-f_i(\alpha))}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}h_2(\alpha)d\alpha$$
$$-\frac{2\pi\varepsilon}{R_i(\theta)}\fint \nabla H(z_i(\theta),z_i(\alpha))\cdot(R_i(\theta)(-\sin\theta,\cos\theta)+\varepsilon f_i'(\theta)(\cos\theta,\sin\theta))h_2(\alpha)d\alpha,$$

(3.17)

$$\partial_{g_j}\mathcal{F}_{i,1}(\varepsilon,\mathbf{x},\boldsymbol{f},\boldsymbol{g})h_2$$
$$= \frac{\varepsilon}{R_i(\theta)}\fint \frac{(x_i-x_j)\cdot[R_i(\theta)(-\sin\theta,\cos\theta)+\varepsilon f_i'(\theta)(\cos\theta,\sin\theta)]}{A_{ij}+\varepsilon B_{ij}(\theta,\alpha)}h_2(\alpha)d\alpha$$
$$+\frac{\varepsilon}{R_i(\theta)}\fint \frac{\varepsilon^2 f_i'(\theta)R_i(\theta)-\varepsilon^2 f_i'(\theta)R_j(\alpha)\cos(\theta-\alpha)+\varepsilon R_i(\theta)R_j(\alpha)\sin(\theta-\alpha)}{A_{ij}+\varepsilon B_{ij}(\theta,\alpha)}h_2(\alpha)d\alpha$$
$$-\frac{2\pi\varepsilon}{R_i(\theta)}\fint \nabla H(z_i(\theta),z_j(\alpha))\cdot[R_i(\theta)(-\sin\theta,\cos\theta)+\varepsilon f_i'(\theta)(\cos\theta,\sin\theta)]h_2(\alpha)d\alpha$$
$$= O(\varepsilon),$$

(3.18)

$$\partial_{f_i}\tilde{\mathcal{F}}_{i,2}(\varepsilon,\mathbf{x},\boldsymbol{f},\boldsymbol{g})h_1$$
$$= -\frac{2\tilde{g}_{i,\varepsilon}(\theta)(R_i(\theta)h_1(\theta)+\varepsilon f_i'(\theta)h_1'(\theta))}{(R_i(\theta)^2+(R_i'(\theta))^2)^2}P.V.\fint \frac{\varepsilon f_i'(\theta)R_i(\alpha)\sin(\theta-\alpha)}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}\tilde{g}_{i,\varepsilon}(\alpha)d\alpha$$
$$+\frac{\tilde{g}_{i,\varepsilon}(\theta)}{R_i(\theta)^2+(R_i'(\theta))^2}P.V.\fint \frac{(h_1'(\theta)R_i(\alpha)+\varepsilon f_i'(\theta)h_1(\alpha))\sin(\theta-\alpha)}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}\tilde{g}_{i,\varepsilon}(\alpha)d\alpha$$
$$-\frac{\tilde{g}_{i,\varepsilon}(\theta)}{R_i(\theta)^2+(R_i'(\theta))^2}P.V.\fint \frac{f_i'(\theta)R_i(\alpha)\tilde{g}_{i,\varepsilon}(\alpha)\sin(\theta-\alpha)}{(A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha))^2}\left[4(h_1(\theta)+h_1(\alpha))\sin^2\left(\frac{\theta-\alpha}{2}\right)\right.$$
$$\left.+2\varepsilon(f_i(\theta)-f_i(\alpha))(h_1(\theta)-h_1(\alpha))+4\varepsilon\,(h_1(\theta)f_i(\alpha)+h_1(\alpha)f_i(\theta))\sin^2\left(\frac{\theta-\alpha}{2}\right)\right]d\alpha$$
$$-\frac{2\tilde{g}_{i,\varepsilon}(\theta)(R_i(\theta)h_1(\theta)+\varepsilon f_i'(\theta)h_1'(\theta))}{(R_i(\theta)^2+(R_i'(\theta))^2)^2}P.V.\fint \frac{R_i(\theta)R_i(\alpha)(\cos(\theta-\alpha)-1)}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}\tilde{g}_{i,\varepsilon}(\alpha)d\alpha$$
$$+\frac{\tilde{g}_{i,\varepsilon}(\theta)}{R_i(\theta)^2+(R_i'(\theta))^2}P.V.\fint \frac{(h_1(\theta)R_i(\alpha)+R_i(\theta)h_1(\alpha))(\cos(\theta-\alpha)-1)}{A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha)}\tilde{g}_{i,\varepsilon}(\alpha)d\alpha$$
$$-\frac{\tilde{g}_{i,\varepsilon}(\theta)}{\varepsilon(R_i(\theta)^2+(R_i'(\theta))^2)}P.V.\fint \frac{R_i(\theta)R_i(\alpha)\tilde{g}_{i,\varepsilon}(\alpha)(\cos(\theta-\alpha)-1)}{(A(\theta,\alpha)+\varepsilon B(f_i,\theta,\alpha))^2}$$
$$\times\left[4(h_1(\theta)+h_1(\alpha))\sin^2\left(\frac{\theta-\alpha}{2}\right)+2\varepsilon(f_i(\theta)-f_i(\alpha))(h_1(\theta)-h_1(\alpha))\right.$$
$$\left.+4\varepsilon\,(h_1(\theta)f_i(\alpha)+h_1(\alpha)f_i(\theta))\sin^2\left(\frac{\theta-\alpha}{2}\right)\right]d\alpha$$

https://doi.org/10.4153/S0008414X22000190 Published online by Cambridge University Press

$$- \frac{2\tilde{g}_{i,\varepsilon}(\theta)(R_i(\theta)h_1(\theta) + \varepsilon f_i'(\theta)h_1'(\theta))}{(R_i(\theta)^2 + (R_i'(\theta))^2)^2} P.V.\fint \frac{R_i(\theta)(f_i(\alpha) - f_i(\theta))}{A(\theta,\alpha) + \varepsilon B(f_i,\theta,\alpha)} \tilde{g}_{i,\varepsilon}(\alpha)d\alpha$$

$$+ \frac{\tilde{g}_{i,\varepsilon}(\theta)}{R_i(\theta)^2 + (R_i'(\theta))^2} P.V.\fint \frac{\varepsilon h_1(\theta)(f_i(\alpha) - f_i(\theta)) + R_i(\theta)(h_1(\alpha) - h_1(\theta))}{A(\theta,\alpha) + \varepsilon B(f_i,\theta,\alpha)} \tilde{g}_{i,\varepsilon}(\alpha)d\alpha$$

$$- \frac{\tilde{g}_{i,\varepsilon}(\theta)}{R_i(\theta)^2 + (R_i'(\theta))^2} P.V.\fint \frac{R_i(\theta)(f_i(\alpha) - f_i(\theta))\tilde{g}_{i,\varepsilon}(\alpha)}{(A(\theta,\alpha) + \varepsilon B(f_i,\theta,\alpha))^2} \left[4(h_1(\theta) + h_1(\alpha)) \sin^2\left(\frac{\theta - \alpha}{2}\right)\right.$$

$$\left. +2\varepsilon(f_i(\theta) - f_i(\alpha))(h_1(\theta) - h_1(\alpha)) + 4\varepsilon\left(h_1(\theta)f_i(\alpha) + h_1(\alpha)f_i(\theta)\right) \sin^2\left(\frac{\theta - \alpha}{2}\right)\right] d\alpha$$

$$+ O(\varepsilon),$$

$$\partial_{f_j}\mathcal{F}_{i,2}(\varepsilon, \mathbf{x}, \boldsymbol{f}, \boldsymbol{g})h_1 = O(\varepsilon), \tag{3.19}$$

(3.20)

$$\partial_{g_i}\tilde{\mathcal{F}}_{i,2}(\varepsilon, \mathbf{x}, \boldsymbol{f}, \boldsymbol{g})h_2$$

$$= \frac{h_2(\theta)}{R_i(\theta)^2 + (R_i'(\theta))^2} P.V.\fint \frac{\varepsilon f_i'(\theta)R_i(\alpha)\sin(\theta - \alpha)}{A(\theta,\alpha) + \varepsilon B(f_i,\theta,\alpha)} \tilde{g}_{i,\varepsilon}(\alpha)d\alpha$$

$$+ \frac{\tilde{g}_{i,\varepsilon}(\theta)}{R_i(\theta)^2 + (R_i'(\theta))^2} P.V.\fint \frac{\varepsilon f_i'(\theta)R_i(\alpha)\sin(\theta - \alpha)}{A(\theta,\alpha) + \varepsilon B(f_i,\theta,\alpha)} h_2(\alpha)d\alpha$$

$$+ \frac{h_2(\theta)}{R(\theta)^2 + (R'(\theta))^2} P.V.\fint \frac{R_i(\theta)R_i(\alpha)(\cos(\theta - \alpha) - 1)}{A(\theta,\alpha) + \varepsilon B(f_i,\theta,\alpha)} \tilde{g}_{i,\varepsilon}(\alpha)d\alpha$$

$$+ \frac{\tilde{g}_{i,\varepsilon}(\theta)}{R_i(\theta)^2 + (R_i'(\theta))^2} P.V.\fint \frac{R_i(\theta)R_i(\alpha)(\cos(\theta - \alpha) - 1)}{A(\theta,\alpha) + \varepsilon B(f_i,\theta,\alpha)} h_2(\alpha)d\alpha$$

$$+ \frac{\varepsilon h_2(\theta)}{R_i(\theta)^2 + (R_i'(\theta))^2} P.V.\fint \frac{R_i(\theta)(f_i(\alpha) - f_i(\theta))}{A(\theta,\alpha) + \varepsilon B(f_i,\theta,\alpha)} \tilde{g}_{i,\varepsilon}(\alpha)d\alpha$$

$$+ \frac{\varepsilon \tilde{g}_{i,\varepsilon}(\theta)}{R_i(\theta)^2 + (R_i'(\theta))^2} P.V.\fint \frac{R_i(\theta)(f_i(\alpha) - f_i(\theta))}{A(\theta,\alpha) + \varepsilon B(f_i,\theta,\alpha)} h_2(\alpha)d\alpha$$

$$+ \sum_{j\neq i} \frac{\varepsilon h_2(\theta)}{R_i(\theta)^2 + (R_i'(\theta))^2} \fint \frac{(x_{i,\varepsilon} - x_{j,\varepsilon})^\perp \cdot \left[R_i(\theta)(-\sin\theta, \cos\theta) + \varepsilon f_i'(\theta)(\cos\theta, \sin\theta)\right]}{A_{ij} + \varepsilon B_{ij}(\theta,\alpha)} \tilde{g}_{j,\varepsilon}(\alpha)d\alpha$$

$$+ \sum_{j\neq i} \frac{\varepsilon h_2(\theta)}{R_i(\theta)^2 + (R_i'(\theta))^2} \fint \frac{-\varepsilon R_i^2(\theta) + \varepsilon^2 f_i'(\theta)R_j(\alpha)\sin(\theta - \alpha) + \varepsilon R_i(\theta)R_j(\alpha)\cos(\theta - \alpha)}{A_{ij} + \varepsilon B_{ij}(\theta,\alpha)} \tilde{g}_{j,\varepsilon}(\alpha)d\alpha$$

$$- \sum_{j=1}^{m} \frac{2\pi\varepsilon h_2(\theta)}{R_i(\theta)^2 + (R_i'(\theta))^2} \fint \nabla^\perp H(z_i(\theta), z_j(\alpha)) \cdot (R_i(\theta)(-\sin\theta, \cos\theta) + \varepsilon f_i'(\theta)(\cos\theta, \sin\theta))$$

$$\times \tilde{g}_{j,\varepsilon}(\alpha)d\alpha$$

$$- \frac{2\pi\varepsilon \tilde{g}_{i,\varepsilon}(\theta)}{R_i(\theta)^2 + (R_i'(\theta))^2} \fint \nabla^\perp H(z_i(\theta), z_i(\alpha)) \cdot (R_i(\theta)(-\sin\theta, \cos\theta) + \varepsilon f_i'(\theta)(\cos\theta, \sin\theta))h_2(\alpha)d\alpha,$$

https://doi.org/10.4153/S0008414X22000190 Published online by Cambridge University Press

and

(3.21)

$$
\begin{aligned}
&\partial_{g_j}\tilde{\mathcal{F}}_{i,2}(\varepsilon,\mathbf{x},\boldsymbol{f},\boldsymbol{g})h_2\\
&\quad=\frac{\varepsilon\tilde{g}_{i,\varepsilon}(\theta)}{R_i(\theta)^2+(R_i'(\theta))^2}\bigg\{\fint\frac{(x_i-x_j)^{\perp}\cdot[R_i(\theta)(-\sin\theta,\cos\theta)+\varepsilon f_i'(\theta)(\cos\theta,\sin\theta)]}{A_{ij}+\varepsilon B_{ij}(\theta,\alpha)}h_2(\alpha)d\alpha\\
&\qquad+\fint\frac{-\varepsilon R_i^2(\theta)+\varepsilon^2 f_i'(\theta)R_j(\alpha)\sin(\theta-\alpha)+\varepsilon R_i(\theta)R_j(\alpha)\cos(\theta-\alpha)}{A_{ij}+\varepsilon B_{ij}(\theta,\alpha)}h_2(\alpha)d\alpha\\
&\qquad-2\pi\fint\nabla^{\perp}H(z_i(\theta),z_j(\alpha))\cdot[R_i(\theta)(-\sin\theta,\cos\theta)+\varepsilon f_i'(\theta)(\cos\theta,\sin\theta)]h_2(\alpha)d\alpha\bigg\}\\
&\quad=O(\varepsilon).
\end{aligned}
$$

■

4 Linearization and isomorphism

In this section, we study the linearization of the functionals defined in Section 2. Denote $\mathcal{F}_i := (\mathcal{F}_{i,1}, \mathcal{F}_{i,2})$ and $\mathcal{F} := (\mathcal{F}_1, \dots, \mathcal{F}_m)$.

By (3.8) and (3.10), one can check that $(0, \mathbf{x}, 0, 0)$ is a solution to $\mathcal{F} = 0$ if and only if $\mathbf{x}$ is a critical point of $\mathcal{W}_m$. Now, we take $\mathbf{x}_0$ to be a critical point of $\mathcal{W}_m$, and hence $(0, \mathbf{x}_0, 0, 0)$ is a solution to $\mathcal{F} = 0$. We study the linearization of $\mathcal{F}$ at $(0, \mathbf{x}_0, 0, 0)$.

According to (3.14)–(3.21) at the end of the proof of Proposition 3.2, when $\varepsilon = 0$ and $\boldsymbol{f}, \boldsymbol{g} \equiv 0$, for all $i = 1, \dots, m$, the Gateaux derivatives are

$$
\begin{cases}
\partial_{f_i}\mathcal{F}_{i,1}(0,\mathbf{x},0,0)f=\frac{\kappa_i}{2}f'(\theta),\\
\partial_{f_j}\mathcal{F}_{i,1}(0,\mathbf{x},0,0)f=0,\quad j\neq i,\\
\partial_{g_i}\mathcal{F}_{i,1}(0,\mathbf{x},0,0)g=\fint\frac{g(\alpha)\sin(\theta-\alpha)}{4\sin^2(\frac{\theta-\alpha}{2})}d\alpha,\\
\partial_{g_j}\mathcal{F}_{i,1}(0,\mathbf{x},0,0)g=0,\quad j\neq i,\\
\partial_{f_i}\mathcal{F}_{i,2}(0,\mathbf{x},0,0)f=\kappa_i^2 f(\theta)-\kappa_i^2\fint\frac{f(\theta)-f(\alpha)}{4\sin^2(\frac{\theta-\alpha}{2})}d\alpha,\\
\partial_{f_j}\mathcal{F}_{i,2}(0,\mathbf{x},0,0)f=0,\quad j\neq i,\\
\partial_{g_i}\mathcal{F}_{i,2}(0,\mathbf{x},0,0)g=-\frac{\kappa_i}{2}g(\theta),\\
\partial_{g_j}\mathcal{F}_{i,2}(0,\mathbf{x},0,0)g=0\quad j\neq i.
\end{cases}
\tag{4.1}
$$

Taking $(h_1, h_2) \in X^{k+1} \times X^k$, where

(4.2)

$$
h_1(\theta)=\sum_{j=1}^{\infty}(a_j\cos(j\theta)+b_j\sin(j\theta))\quad\text{and}\quad h_2(\theta)=\sum_{j=1}^{\infty}(c_j\cos(j\theta)+d_j\sin(j\theta)),
$$

we will prove that the linearization of $\mathcal{F}_i$ at $(0, \mathbf{x}_0, 0, 0)$ has the following Fourier series form:

$$
D_{(f_i,g_i)}\mathcal{F}_i(0,\mathbf{x}_0,0,0)(h_1,h_2):=\begin{pmatrix}\partial_{f_i}\mathcal{F}_{i,1}(0,\mathbf{x}_0,0,0)h_1+\partial_{g_i}\mathcal{F}_{i,1}(0,\mathbf{x}_0,0,0)h_2\\ \partial_{f_i}\mathcal{F}_{i,2}(0,\mathbf{x}_0,0,0)h_1+\partial_{g_i}\mathcal{F}_{i,2}(0,\mathbf{x}_0,0,0)h_2\end{pmatrix}
$$

https://doi.org/10.4153/S0008414X22000190 Published online by Cambridge University Press

$$= \sum_{j=1}^{\infty} \begin{pmatrix} \hat{a}_j \sin(j\theta) + \hat{b}_j \cos(j\theta) \\ \hat{c}_j \cos(j\theta) + \hat{d}_j \sin(j\theta) \end{pmatrix}, \tag{4.3}$$

where

$$\begin{pmatrix} \hat{a}_j \\ \hat{c}_j \end{pmatrix} = M_j \begin{pmatrix} a_j \\ c_j \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \hat{b}_j \\ \hat{d}_j \end{pmatrix} = N_j \begin{pmatrix} b_j \\ d_j \end{pmatrix}$$

with M_j and N_j two 2×2 matrices given in Lemma 4.2.

To compute M_j and N_j, we need the following identities.

Lemma 4.1 *For all $j \geq 1$ and $j \in \mathbb{N}^*$, there hold*

$$\fint \frac{\cos(j\alpha)\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)} d\alpha = \frac{1}{2}\sin(j\theta), \tag{4.4}$$

$$\fint \frac{\sin(j\alpha)\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)} d\alpha = -\frac{1}{2}\cos(j\theta), \tag{4.5}$$

$$\fint \frac{\cos(j\theta)-\cos(j\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)} d\alpha = \frac{j}{2}\cos(j\theta), \tag{4.6}$$

$$\fint \frac{\sin(j\theta)-\sin(j\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)} d\alpha = \frac{j}{2}\sin(j\theta). \tag{4.7}$$

Proof Identities (4.4) and (4.6) were proved in Lemma A.8 [24]. Indeed, (4.4) can be deduced from the identity

$$\fint \frac{\cos(j\alpha)\sin(\theta-\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)} d\alpha = \frac{1}{2}\fint \cos(j\alpha)\cot\left(\frac{\theta-\alpha}{2}\right) d\alpha = \frac{1}{2}\mathcal{H}(\cos(j\theta))(\theta),$$

where $\mathcal{H}(\cdot)$ is the Hilbert transform on torus and hence $H(\cos(j\theta)) = \sin(j\theta)$. Identity (4.6) can be obtained by computing the fractional Laplacians

$$\fint \frac{\cos(j\theta)-\cos(j\alpha)}{4\sin^2\left(\frac{\theta-\alpha}{2}\right)} d\alpha = \frac{1}{2}(-\Delta)^{\frac{1}{2}}\cos(j\theta) = \frac{j}{2}\cos(j\theta).$$

Finally, we point out that the identities (4.5) and (4.7) can be derived by calculating derivatives of (4.4) and (4.6), respectively. ∎

Now, we can prove (4.3) and find the explicit formula for M_j and N_j.

Lemma 4.2 *The derivative of $\mathcal{F}_i$ at $(0, \mathbf{x}_0, 0, 0)$ is given by (4.3) with*

$$M_j = \begin{pmatrix} -\frac{\kappa_i j}{2} & \frac{1}{2} \\ \frac{(2-j)\kappa_i^2}{2} & -\frac{\kappa_i}{2} \end{pmatrix}, \quad N_j = \begin{pmatrix} \frac{\kappa_i j}{2} & -\frac{1}{2} \\ \frac{(2-j)\kappa_i^2}{2} & -\frac{\kappa_i}{2} \end{pmatrix}, \tag{4.8}$$

for any $j \geq 1$.

Moreover, $D_{(f_i, g_i)}\mathcal{F}_i(0, \mathbf{x}_0, 0, 0)$ is an isomorphism from X_i^k to Y_i^k and $D_{(f,g)}\mathcal{F}(0, \mathbf{x}_0, 0, 0)$ is an isomorphism from $\mathcal{X}^k$ to $\mathcal{Y}^k$.

Proof Using (4.1), (4.2), and Lemma 4.1, we obtain by direct calculations

$$\partial_{f_i}\mathcal{F}_{i,1}(0,\mathbf{x}_0,0,0)h_1 = \sum_{j=1}^{\infty}\left(\frac{-\kappa_i j}{2}a_j\sin(j\theta) + \frac{\kappa_i j}{2}b_j\cos(j\theta)\right),$$

$$\partial_{g_i}\mathcal{F}_{i,1}(0,\mathbf{x}_0,0,0)h_2 = \frac{1}{2}\sum_{j=1}^{\infty}(c_j\sin(j\theta) - d_j\cos(j\theta)),$$

$$\begin{aligned}\partial_{f_i}\mathcal{F}_{i,2}(0,\mathbf{x}_0,0,0)h_1 &= \kappa_i^2\textstyle\sum_{j=1}^{\infty}(a_j\cos(j\theta) + b_j\sin(j\theta)) - \kappa_i^2\sum_{j=1}^{\infty}\left(\frac{j}{2}a_j\cos(j\theta) + \frac{j}{2}b_j\sin(j\theta)\right)\\ &= \textstyle\sum_{j=1}^{\infty}\left(\frac{\kappa_i^2(2-j)}{2}a_j\cos(j\theta) + \frac{\kappa_i^2(2-j)}{2}b_j\sin(j\theta)\right),\end{aligned}$$

and

$$\partial_{g_i}\mathcal{F}_{i,2}(0,\mathbf{x}_0,0,0)h_2 = -\frac{\kappa_i}{2}\sum_{j=1}^{\infty}(c_j\cos(j\theta) + d_j\sin(j\theta)).$$

Then, one can easily check that the derivative of $\mathcal{F}_i$ at $(0,\mathbf{x},0,0)$ is given by (4.3) with

$$M_j = \begin{pmatrix} -\frac{\kappa_i j}{2} & \frac{1}{2}\\ \frac{(2-j)\kappa_i^2}{2} & -\frac{\kappa_i}{2}\end{pmatrix},\quad N_j = \begin{pmatrix} \frac{\kappa_i j}{2} & -\frac{1}{2}\\ \frac{(2-j)\kappa_i^2}{2} & -\frac{\kappa_i}{2}\end{pmatrix}.$$

Now we are going to prove that $D_{(f_i,g_i)}\mathcal{F}_i(0,\mathbf{x}_0,0,0)$ is an isomorphism from X_i^k to Y_i^k. Recall the definition of X_i^k and Y_i^k given at the end of Section 2. From the above calculations, one has $M_1 = \begin{pmatrix}-\kappa_i/2 & 1/2\\ \kappa_i^2/2 & -\kappa_i/2\end{pmatrix}$ and $N_1 = \begin{pmatrix}\kappa_i/2 & -1/2\\ \kappa_i^2/2 & -\kappa_i/2\end{pmatrix}$, then it is obvious that $D_{(f_i,g_i)}\mathcal{F}_i(0,\mathbf{x}_0,0,0)$ maps X_i^k to Y_i^k. Hence, only the invertibility needs to be considered.

For $j \ge 2$, $\det(M_j) = -\det(N_j) = \frac{\kappa_i^2(j-1)}{2} > 0$ which implies that M_j and N_j are invertible, and their inverse are given by

$$M_j^{-1} = \begin{pmatrix}\frac{-1}{\kappa_i(j-1)} & \frac{-1}{\kappa_i^2(j-1)}\\ \frac{j-2}{j-1} & \frac{-j}{\kappa_i(j-1)}\end{pmatrix},\quad \forall\, j\ge 2, \tag{4.9}$$

and

$$N_j^{-1} = \begin{pmatrix}\frac{1}{\kappa_i(j-1)} & \frac{-1}{\kappa_i^2(j-1)}\\ \frac{2-j}{j-1} & \frac{-j}{\kappa_i(j-1)}\end{pmatrix},\quad \forall\, j\ge 2. \tag{4.10}$$

Thus, for any $(u,v)\in Y_i^k$ with

$$u = \sum_{j=1}^{\infty}p_j\sin(j\theta) + q_j\cos(j\theta)\quad\text{and}$$

$$v = -\kappa_i p_1\cos(\theta) + \kappa_i q_1\sin(\theta) + \sum_{j=2}^{\infty}r_j\cos(j\theta) + s_j\sin(j\theta),$$

https://doi.org/10.4153/S0008414X22000190 Published online by Cambridge University Press

we can write $D_{(f_i,g_i)}\mathcal{F}_i(0,\mathbf{x}_0,0,0)^{-1}(u,v)$ as

$$D_{(f_i,g_i)}\mathcal{F}_i(0,\mathbf{x}_0,0,0)^{-1}(u,v)$$
$$=\begin{pmatrix}-\frac{p_1}{\kappa_i}\cos(\theta)+\frac{q_1}{\kappa_i}\sin(\theta)\\ p_1\cos(\theta)-q_1\sin(\theta)\end{pmatrix}+\sum_{j=2}^{\infty}M_j^{-1}\begin{pmatrix}p_j\\ r_j\end{pmatrix}\cos(j\theta)+N_j^{-1}\begin{pmatrix}q_j\\ s_j\end{pmatrix}\sin(j\theta).$$

Denote

$$\begin{pmatrix}\tilde{p}_j\\ \tilde{r}_j\end{pmatrix}=M_j^{-1}\begin{pmatrix}p_j\\ r_j\end{pmatrix},\quad \begin{pmatrix}\tilde{q}_j\\ \tilde{s}_j\end{pmatrix}=N_j^{-1}\begin{pmatrix}q_j\\ s_j\end{pmatrix},\quad \forall\, j\geq 2.$$

From (4.9) and (4.10), we have the asymptotic behavior: $\tilde{p}_j=O(j^{-1}(|p_j|+|r_j|))$, $\tilde{r}_j=O(|p_j|+|r_j|)$, $\tilde{q}_j=O(j^{-1}(|q_j|+|s_j|))$ and $\tilde{s}_j=O(|q_j|+|s_j|)$ as $j\to+\infty$, which implies that $D_{(f_i,g_i)}\mathcal{F}_i(0,\mathbf{x},0,0)^{-1}(u,v)$ does belong to X_i^k.

Noticing that by (4.1), we have $\partial_{f_j}\mathcal{F}_{i,1}(0,\mathbf{x}_0,0,0)h_1$, $\partial_{g_j}\mathcal{F}_{i,1}(0,\mathbf{x}_0,0,0)h_2$, $\partial_{f_j}\mathcal{F}_{i,2}(0,\mathbf{x}_0,0,0)h_1$, and $\partial_{g_j}\mathcal{F}_{i,2}(0,\mathbf{x}_0,0,0)h_2=0$, $j\neq i$. Therefore, we find

$$D_{(\boldsymbol{f},\boldsymbol{g})}\mathcal{F}(0,\mathbf{x}_0,0,0)=\operatorname{diag}\left(D_{(f_1,g_1)}\mathcal{F}_1(0,\mathbf{x}_0,0,0),\dots,D_{(f_m,g_m)}\mathcal{F}_m(0,\mathbf{x}_0,0,0)\right),$$

and hence $D_{(\boldsymbol{f},\boldsymbol{g})}\mathcal{F}(0,\mathbf{x}_0,0,0)$ is an isomorphism from $\mathcal{X}^k$ to $\mathcal{Y}^k$.

The proof of is thus completed. ∎

5 Existence of vortex sheets

In this section, inspired by the classical Crandall–Rabinowitz theorem on bifurcation theory, we use the implicit function theorem to obtain a branch of solutions for arbitrarily fixed small ε.

From the previous sections, we know that $(0,\mathbf{x}_0,0,0)$ is a solution to $\mathcal{F}=0$ if and only if $\mathbf{x}_0$ is a critical point of $\mathcal{W}_m$. Moreover, $D_{(\boldsymbol{f},\boldsymbol{g})}\mathcal{F}(0,\mathbf{x}_0,0,0)$ is an isomorphism from $\mathcal{X}^k$ to $\mathcal{Y}^k$. It can be seen from Lemma 4.2 that the kernel of $D_{(\boldsymbol{f},\boldsymbol{g})}\mathcal{F}(0,\mathbf{x}_0,0,0)$ in $\left(X^{k+1}\times X^k\right)^m$ is

$$\prod_{i=1}^{m}\{(a\cos(\theta)+b\sin(\theta),\kappa_i(a\cos(\theta)+b\sin(\theta)))\mid(a,b)\in\mathbb{R}^2\}.$$

We take arbitrary nontrivial $(\boldsymbol{f}_0,\boldsymbol{g}_0)\in\mathcal{X}_0^k$ and define the following new functional:

$$\overline{\mathcal{F}}(\varepsilon,\tau,\mathbf{x},\boldsymbol{f},\boldsymbol{g}):=\mathcal{F}(\varepsilon,\mathbf{x},\boldsymbol{f}+\tau\boldsymbol{f}_0,\boldsymbol{g}+\tau\boldsymbol{g}_0).\tag{5.1}$$

To apply the implicit function theorem, we need to make sure that $\overline{\mathcal{F}}$ maps a suitable subset of $\mathcal{X}^k$ into $\mathcal{Y}^k$. This aim will be achieved by choosing $\mathbf{x}$ properly. Indeed, letting $V_1:=\{(\boldsymbol{f},\boldsymbol{g})\in\mathcal{X}^k\mid\sum_{j=1}^m(\|f_i\|_{H^{k+1}}+\|g_i\|_{H^k})<1\}\subset\mathcal{X}^k$ be the unit ball, we have the following key proposition.

Proposition 5.1 *The condition that $\overline{\mathcal{F}}$ maps $(-\varepsilon_0,\varepsilon_0)\times(-\tau_1,\tau_1)\times B_{r_0}(\mathbf{x}_0)\times V_1$ into $\mathcal{Y}^k$ is equivalent to a system of $2m$ equations of the form*

$$\nabla\mathcal{W}_m(\mathbf{x})=O_{\tau_1}(\varepsilon),\tag{5.2}$$

https://doi.org/10.4153/S0008414X22000190 Published online by Cambridge University Press

where τ_1 is any fixed small positive number and $O_{\tau_1}(\varepsilon)$ means a vector that is of the order ε up to a constant depending on τ_1.

Proof For arbitrary $i = 1, \dots, m$, we take $(\boldsymbol{f}, \boldsymbol{g}) \in V_1$ with

$$f_i(\theta) = \sum_{j=1}^{\infty}(a_j \cos(j\theta) + b_j \sin(j\theta)),$$

$$g_i(\theta) = -\kappa_i a_1 \cos(\theta) - \kappa_i b_1 \sin(\theta) + \sum_{j=2}^{\infty}(c_j \cos(j\theta) + d_j \sin(j\theta)).$$

By the definition of $\mathcal{Y}^k$, in order to make $\overline{\mathcal{F}}(\varepsilon, \tau, \mathbf{x}, \boldsymbol{f}, \boldsymbol{g}) = \mathcal{F}(\varepsilon, \mathbf{x}, \boldsymbol{f} + \tau\boldsymbol{f}_0, \boldsymbol{g} + \tau\boldsymbol{g}_0) \in \mathcal{Y}^k$, we need to ensure that the following equations hold true:

(5.3)
$$-\kappa_i \fint \mathcal{F}_{i,1}(\varepsilon, \mathbf{x}, \boldsymbol{f} + \tau\boldsymbol{f}_0, \boldsymbol{g} + \tau\boldsymbol{g}_0)\sin(\theta)d\theta = \fint \mathcal{F}_{i,2}(\varepsilon, \mathbf{x}, \boldsymbol{f} + \tau\boldsymbol{f}_0, \boldsymbol{g} + \tau\boldsymbol{g}_0)\cos(\theta)d\theta,$$
$$\kappa_i \fint \mathcal{F}_{i,1}(\varepsilon, \mathbf{x}, \boldsymbol{f} + \tau\boldsymbol{f}_0, \boldsymbol{g} + \tau\boldsymbol{g}_0)\cos(\theta)d\theta = \fint \mathcal{F}_{i,2}(\varepsilon, \mathbf{x}, \boldsymbol{f} + \tau\boldsymbol{f}_0, \boldsymbol{g} + \tau\boldsymbol{g}_0)\sin(\theta)d\theta,$$

where $i = 1, \dots, m$. By (3.7), (3.9), and calculations in Lemmas 4.1 and 4.2, we obtain

(5.4)
$$\begin{aligned}\fint \mathcal{F}_{i,1}(\varepsilon, \mathbf{x}, \boldsymbol{f} + \tau\boldsymbol{f}_0, \boldsymbol{g} + \tau\boldsymbol{g}_0)\sin(\theta)d\theta = -\kappa_i a_1 - \textstyle\sum_{j\neq i} 2\pi\kappa_j \partial_{x_{i,1}} G(x_i, x_j)\\ +2\pi\kappa_i \partial_{x_{i,1}} H(x_i, x_i) + \varepsilon \fint \mathcal{R}_1 \sin(\theta)d\theta,\end{aligned}$$

(5.5)
$$\begin{aligned}\fint \mathcal{F}_{i,1}(\varepsilon, \mathbf{x}, \boldsymbol{f} + \tau\boldsymbol{f}_0, \boldsymbol{g} + \tau\boldsymbol{g}_0)\cos(\theta)d\theta = \kappa_i b_1 + \textstyle\sum_{j\neq i} 2\pi\kappa_j \partial_{x_{i,2}} G(x_i, x_j)\\ -2\pi\kappa_i \partial_{x_{i,2}} H(x_i, x_i) + \varepsilon \fint \mathcal{R}_1 \cos(\theta)d\theta,\end{aligned}$$

(5.6)
$$\begin{aligned}\fint \mathcal{F}_{i,2}(\varepsilon, \mathbf{x}, \boldsymbol{f} + \tau\boldsymbol{f}_0, \boldsymbol{g} + \tau\boldsymbol{g}_0)\sin(\theta)d\theta = \kappa_i^2 b_1 - \textstyle\sum_{j\neq i} 2\pi\kappa_i\kappa_j \partial_{x_{i,2}} G(x_i, x_j)\\ +2\pi\kappa_i^2 \partial_{x_{i,2}} H(x_i, x_i) + \varepsilon \fint \mathcal{R}_2 \sin(\theta)d\theta,\end{aligned}$$

and

(5.7)
$$\begin{aligned}\fint \mathcal{F}_{i,2}(\varepsilon, \mathbf{x}, \boldsymbol{f} + \tau\boldsymbol{f}_0, \boldsymbol{g} + \tau\boldsymbol{g}_0)\cos(\theta)d\theta = \kappa_i^2 a_1 - \textstyle\sum_{j\neq i} 2\pi\kappa_i\kappa_j \partial_{x_{i,1}} G(x_i, x_j)\\ +2\pi\kappa_i^2 \partial_{x_{i,1}} H(x_i, x_i) + \varepsilon \fint \mathcal{R}_2 \cos(\theta)d\theta.\end{aligned}$$

Then, by the above equations (5.4)–(5.7), we conclude that (5.3) is equivalent to the following equations:

(5.8)
$$\sum_{j\neq i} \kappa_i\kappa_j \partial_{x_{i,1}} G(x_i, x_j) - \kappa_i^2 \partial_{x_{i,1}} H(x_i, x_i) = \frac{\varepsilon}{4\pi}\left(\fint \mathcal{R}_2 \cos(\theta)d\theta + \kappa_i \fint \mathcal{R}_1 \sin(\theta)d\theta\right)$$

and

(5.9)

$$\sum_{j\neq i}\kappa_i\kappa_j\partial_{x_{i,2}}G(x_i,x_j)-\kappa_i^2\partial_{x_{i,2}}H(x_i,x_i)=\frac{\varepsilon}{4\pi}\Big(\fint\mathcal{R}_2\sin(\theta)d\theta-\kappa_i\fint\mathcal{R}_1\cos(\theta)d\theta\Big).$$

Since (5.8) and (5.9) hold for all $i=1,\dots,m$, we arrive at (5.2) and complete the proof of Proposition 5.1. ∎

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 Since we have the nondegeneracy condition $\deg(\nabla\mathcal{W}_m,\mathbf{x}_0)\neq 0$, equation (5.2) is solvable near $\mathbf{x}_0$ whenever ε is small. We solve (5.2) and write the solution $\mathbf{x}_{\varepsilon,\tau}$ in the form $\mathbf{x}_{\varepsilon,\tau}=\mathbf{x}_0+\varepsilon\bar{\mathcal{R}}_{\mathbf{x}}(\varepsilon,\tau,\boldsymbol{f},\boldsymbol{g})$. Then, we know that $\bar{\mathcal{R}}_{\mathbf{x}}$ defined on $(-\varepsilon_0,\varepsilon_0)\times(-\tau_1,\tau_1)\times V_1$ is at least of C^1 due to the regularity of $\mathcal{F}$.

Now, set

$$\overline{\mathcal{F}}^*(\varepsilon,\tau,\boldsymbol{f},\boldsymbol{g}):=\overline{\mathcal{F}}(\varepsilon,\tau,\mathbf{x}_0+\varepsilon\bar{\mathcal{R}}_{\mathbf{x}}(\varepsilon,\tau,\boldsymbol{f},\boldsymbol{g}),\boldsymbol{f},\boldsymbol{g}).$$

Then, we conclude from Proposition 5.1 that $\overline{\mathcal{F}}^*$ maps $(-\varepsilon_0,\varepsilon_0)\times(-\tau_1,\tau_1)\times V_1$ into $\mathcal{Y}^k$. Moreover, $\overline{\mathcal{F}}^*$ is C^1 continuous with respect to $\boldsymbol{f}$ and $\boldsymbol{g}$. Next, we need to verify that $D_{(\boldsymbol{f},\boldsymbol{g})}\mathcal{F}^*(0,0,0,0)$ is an isomorphism from $\mathcal{X}^k$ to $\mathcal{Y}^k$. In fact, by the chain rule, we get

$$D_{(\boldsymbol{f},\boldsymbol{g})}\overline{\mathcal{F}}^*=D_{(\boldsymbol{f},\boldsymbol{g})}\overline{\mathcal{F}}+D_{\mathbf{x}}\overline{\mathcal{F}}\cdot D_{(\boldsymbol{f},\boldsymbol{g})}\left(\mathbf{x}_0+\varepsilon\bar{\mathcal{R}}_{\mathbf{x}}(\varepsilon,\tau,\boldsymbol{f},\boldsymbol{g})\right),$$

which implies

$$D_{(\boldsymbol{f},\boldsymbol{g})}\overline{\mathcal{F}}^*(0,0,0,0)=D_{(\boldsymbol{f},\boldsymbol{g})}\mathcal{F}(0,\mathbf{x}_0,0,0).$$

Therefore, $D_{(\boldsymbol{f},\boldsymbol{g})}\overline{\mathcal{F}}^*(0,0,0,0)$ is an isomorphism from $\mathcal{X}^k$ to $\mathcal{Y}^k$ by Lemma 4.2.

Now, applying implicit function theorem to $\overline{\mathcal{F}}^*$ at the point $(0,0,0,0)$, we obtain that there exist $\varepsilon_0>0$ and $0<\tau_0\le\tau_1$ such that the solutions set

$$\left\{(\varepsilon,\tau,\boldsymbol{f},\boldsymbol{g})\in(-\varepsilon_0,\varepsilon_0)\times(-\tau_0,\tau_0)\times V_1\ :\ \overline{\mathcal{F}}^*(\varepsilon,\tau,\boldsymbol{f},\boldsymbol{g})=0\right\}$$

is not empty and can be parameterized by a two-dimensional surface $(\varepsilon,\tau)\in(-\varepsilon_0,\varepsilon_0)\times(-\tau_0,\tau_0)\to(\varepsilon,,\tau,\boldsymbol{f}_{\varepsilon,\tau},\boldsymbol{g}_{\varepsilon,\tau})$. So we obtain a family of nontrivial vortex sheet solutions and finish the proof of (i) in Theorem 1.1.

Since (ii) of Theorem 1.1 is obvious, to end our proof, we only need to show the convexity of the interior of Γ_i for $i=1,\dots,m$. This can be done by computing the sign of the curvature. Recall that $z_i(\theta)=x_{\varepsilon,\tau,i}+\varepsilon R_i(\theta)(\cos\theta,\sin\theta)$ with $R_i(\theta)=1+\varepsilon(f_{\varepsilon,\tau,i}(\theta)+\tau f_{0,i})$. Given $\theta\in[0,2\pi)$, the signed curvature of Γ_i at $z_i(\theta)$ is

$$\varepsilon\kappa(\theta)=\frac{R_i(\theta)^2+2R_i'(\theta)^2-R_i(\theta)R_i''(\theta)}{\left(R_i(\theta)^2+R_i'(\theta)^2\right)^{\frac{3}{2}}}=\frac{1+O(\varepsilon)}{1+O(\varepsilon)}>0,$$

for ε and τ small, which implies the convexity and thus completes the proof of Theorem 1.1. ∎

https://doi.org/10.4153/S0008414X22000190 Published online by Cambridge University Press

We point out that for fixed ε, if $\tau_1 \neq \tau_2$ with $0 < \tau_1, \tau_2 < \tau_0$, then obviously one has $\omega_{\varepsilon,\tau_1} \not\equiv \omega_{\varepsilon,\tau_2}$. Thus, we have obtained a large family of stationary solutions with vortex sheet for every $\varepsilon > 0$ small.

Acknowledgment The authors are grateful to the anonymous referees for their careful reading the paper and valuable comments that help a lot to improve the presentation of the present paper.

References

[1] T. Bartsch, A. M. Micheletti, and A. Pistoia, *The Morse property for functions of Kirchhoff–Routh path type*. Discrete Contin. Dyn. Syst. Ser. S. **12**(2019), 1867–1877.
[2] T. Bartsch and A. Pistoia, *Critical points of the N-vortex Hamiltonian in bounded planar domains and steady state solutions of the incompressible Euler equations*. SIAM J. Appl. Math. **75**(2015), 726–744.
[3] T. Bartsch, A. Pistoia, and T. Weth, *N-vortex equilibria for ideal fluids in bounded planar domains and new nodal solutions of the sinh-Poisson and the Lane–Emden–Fowler equations*. Commun. Math. Phys. **297**(2010), 653–686.
[4] G. K. Batchelor, *An introduction to fluid dynamics*. Second paperback ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1999, xviii+615 pp.
[5] G. Birkhoff, *Hydrodynamics: A study in logic, fact and similitude*. Revised ed., Princeton University Press, Princeton, NJ, 1960, xi + 184 pp.
[6] G. Birkhoff, *Helmholtz and Taylor instability*, Proceedings of Symposia in Applied Mathematics, XIII, American Mathematical Society, Providence, RI, 1962, pp. 55–76.
[7] G. Birkhoff and J. Fisher, *Do vortex sheets roll up?* Rend. Circ. Mat. Palermo (2) **8**(1959), 77–90.
[8] L. Caffarelli and A. Friedman, *Asymptotic estimates for the plasma problem*. Duke Math. J. **47**(1980), 705–742.
[9] R. E. Caflisch and O. F. Orellana, *Singular solutions and ill-posedness for the evolution of vortex sheets*. SIAM J. Math. Anal. **20**(1989), no. 2, 293–307.
[10] D. Cao, S. Peng, and S. Yan, *Planar vortex patch problem in incompressible steady flow*. Adv. Math. **270**(2015), 263–301.
[11] D. Cao, G. Wang, and W. Zhan, *Desingularization of vortices for 2D steady Euler flows via the vorticity method*. SIAM J. Math. Anal. **52**(2020), 5363–5388.
[12] D. Cao, S. Yan, and W. Yu, *Planar vortices for incompressible flow in unbounded domains with obstacles*. Preprint, 2021.
[13] A. Castro, D. Córdoba, and J. Gómez-Serrano, *Existence and regularity of rotating global solutions for the generalized surface quasi-geostrophic equations*. Duke Math. J. **165**(2016), no. 5, 935–984.
[14] A. Castro, D. Córdoba, and J. Gómez-Serrano, *Uniformly rotating analytic global patch solutions for active scalars*. Ann. PDE **2**(2016), no. 1, Article no. 1, 34 pp.
[15] A. Castro, D. Córdoba, and J. Gómez-Serrano, *Uniformly rotating smooth solutions for the incompressible 2D Euler equations*. Arch. Ration. Mech. Anal. **231**(2019), no. 2, 719–785.
[16] A. Castro, D. Córdoba, and J. Gómez-Serrano, *Global smooth solutions for the inviscid SQG equation*. Mem. Amer. Math. Soc. **266**(2020), no. 1292. v+89 pp.
[17] J. Dávila, M. Del Pino, M. Musso, and J. Wei, *Gluing methods for vortex dynamics in Euler flows*. Arch. Ration. Mech. Anal. **235**(2020), 1467–1530.
[18] F. de la Hoz, Z. Hassainia, T. Hmidi, and J. Mateu, *An analytical and numerical study of steady patches in the disc*. Anal. PDE **9**(2016), no. 7, 1609–1670.
[19] F. de la Hoz, T. Hmidi, J. Mateu, and J. Verdera, *Doubly connected V-states for the planar Euler equations*. SIAM J. Math. Anal. **48**(2016), no. 3, 1892–1928.
[20] J.-M. Delort, *Existence de nappes de tourbillon en dimension deux*. J. Amer. Math. Soc. **4**(1991), no. 3, 553–586.
[21] J. Duchon and R. Robert, *Global vortex sheet solutions of Euler equations in the plane*. J. Differential Equations 73(1988), no. 2, 215–224.
[22] L. C. Evans and S. Müller, *Hardy spaces and the two-dimensional Euler equations with nonnegative vorticity*. J. Amer. Math. Soc. 7(1994), no. 1, 199–219.
[23] J. Gómez-Serrano, J. Park, J. Shi, and Y. Yao, *Remarks on stationary and uniformly-rotating vortex sheets: rigidity results*. Commun. Math. Phys. **386**(2021), 1845–1879.

[24] J. Gómez-Serrano, J. Park, J. Shi, and Y. Yao, *Remarks on stationary and uniformly-rotating vortex sheets: flexibility results*. Phil. Trans. R. Soc. A. **380**(2022), 20210045.

[25] M. Grossi and F. Takahashi, *Nonexistence of multi-bubble solutions to some elliptic equations on convex domains*. J. Funct. Anal. **259**(2010), 904–917.

[26] Z. Hassainia and T. Hmidi, *Steady asymmetric vortex pairs for Euler equations*. Discrete Contin. Dyn. Syst. **41**(2021), no. 4, 1939–1969.

[27] T. Hmidi and J. Mateu, *Bifurcation of rotating patches from Kirchhoff vortices*. Discrete Contin. Dyn. Syst. **36**(2016), no. 10, 5401–5422.

[28] T. Hmidi and J. Mateu, *Existence of corotating and counter-rotating vortex pairs for active scalar equations*. Commun. Math. Phys. **350**(2017), 699–747.

[29] T. Hmidi, J. Mateu, and J. Verdera, *Boundary regularity of rotating vortex patches*. Arch. Ration. Mech. Anal. **209**(2013), no. 1, 171–208.

[30] R. Krasny, *A study of singularity formation in a vortex sheet by the point-vortex approximation*. J. Fluid Mech. **167**(1986), 65–93.

[31] C. C. Lin, *On the motion of vortices in two dimensions. I. Existence of the Kirchhoff–Routh function*. Proc. Natl. Acad. Sci. U. S. A. **27**(1941), 570–575.

[32] C. C. Lin, *On the motion of vortices in two dimensions. II. Some further investigations on the Kirchhoff–Routh function*. Proc. Natl. Acad. Sci. U. S. A. **27**(1941), 575–577.

[33] Y. Long, Y. Wang, and C. Zeng, *Concentrated steady vorticities of the Euler equation on 2-d domains and their linear stability*. J. Differential Equations **266**(2019), no. 10, 6661–6701.

[34] M. C. Lopes Filho, H. J. Nussenzveig Lopes, and S. Schochet, *A criterion for the equivalence of the Birkhoff–Rott and Euler descriptions of vortex sheet evolution*. Trans. Amer. Math. Soc. **359**(2007), no. 9, 4125–4142.

[35] M. C. Lopes Filho, H. J. Nussenzveig Lopes, and Z. Xin, *Existence of vortex sheets with reflection symmetry in two space dimensions*. Arch. Ration. Mech. Anal. **158**(2001), no. 3, 235–257.

[36] A. J. Majda, *Remarks on weak solutions for vortex sheets with a distinguished sign*. Indiana Univ. Math. J. **42**(1993), no. 3, 921–939.

[37] A. J. Majda and A. L. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics, 27, Cambridge University Press, Cambridge, 2002. xii+545 pp.

[38] A. M. Micheletti and A. Pistoia, *Non degeneracy of critical points of the Robin function with respect to deformations of the domain*. Potential Anal. **40**(2014), 103–116.

[39] D. W. Moore, *The spontaneous appearance of a singularity in the shape of an evolving vortex sheet*. Proc. Roy. Soc. London Ser. A. **365**(1979), no. 1720, 105–119.

[40] B. Protas and T. Sakajo, *Rotating equilibria of vortex sheets*. Phys. D **403**(2020), 132286, 9 pp.

[41] N. Rott, *Diffraction of a weak shock with vortex generation*. J. Fluid Mech. **1**(1956), 111–128.

[42] S. Schochet, *The weak vorticity formulation of the 2-D Euler equations and concentration-cancellation*. Comm. Partial Differential Equations **20**(1995), nos. 5–6, 1077–1104.

[43] C. Sulem, P.-L. Sulem, C. Bardos, and U. Frisch, *Finite time analyticity for the two- and three-dimensional Kelvin–Helmholtz instability*. Commun. Math. Phys. **80**(1981), no. 4, 485–516.

[44] S. Wu, *Mathematical analysis of vortex sheets*. Commun. Pure Appl. Math. **59**(2006), no. 8, 1065–1206.

[45] V. I. Yudovich, *Non-stationary flows of an ideal incompressible fluid*. Zh. Vych. Mat. **3**(1963), 1032–1106.

Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing, P.R. China,
School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, P.R. China
e-mail: dmcao@amt.ac.cn qinguolin18@mails.ucas.ac.cn

Department of Mathematics, Sichuan University, Chengdu, P.R. China
e-mail: zouchangjun@amss.ac.cn

https://doi.org/10.4153/S0008414X22000190 Published online by Cambridge University Press