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CENTRALIZING AUTOMORPHISMS OF PRIME RINGS

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ABSTRACT. Let R be a prime ring and T be a nontrivial automorphism of R. If $xx^T - x^Tx$ is in the center of the ring for every x in R, then R is a commutative integral domain.

An additive mapping L of a ring R to itself is called *centralizing* if $x(xL)-(xL)x$ is in the center of R for every x in R. In [4] Posner showed that a prime ring must be commutative if it has a nontrivial centralizing derivation (see [1] for another proof). In this note the analogous result for a centralizing automorphism is proved.

THEOREM. *If R is a prime ring with a nontrivial centralizing automorphism, then R is a commutative integral domain.*

This generalizes the results of Divinsky [2] and Luh [3]. Divinsky showed that a simple ring is commutative if it has a nontrivial automorphism T such that $xx^T = x^Tx$ for all x in the ring and Luh extended this result to prime rings.

Let $[x, y]=xy-yx$ and note that $[x, yz]=y[x, z]+[x, y]z$. Assume that R is a prime ring and let Z be the center of R. The next two lemmas will be used in the proof of the theorem.

LEMMA 1. [3] *Let T be a nontrivial automorphism of R. If $[x, x^T]=0$ for all x in R, then R is commutative.*

Proof. Linearizing $[x, x^T]=0$ gives $[x, y^T]=[x^T, y]$ and thus $[x, (xy)^T]=[x^T, xy]$. But $[x, (xy)^T]=x^T[x, y^T]$ and $[x^T, xy]=x[x^T, y]=x[x, y^T]$. Thus $(x-x^T)[x, y^T]=0$ and since T is an automorphism $(x-x^T)[x, z]=0$ for all x and z in R. Since $y[x, z]=[x, yz]-[x, y]z$, $(x-x^T)R[x, z]=0$. If $x\neq x^T$, then x is in the center since R is prime. Since T is nontrivial, there must be at least one x such that $x\neq x^T$. Suppose y is not in the center of R. Then $x+y$ is not in the center and $y^T=y$, $(x+y)^T=x+y$. But then $x=x^T$ which is a contradiction. Hence R is commutative.

LEMMA 2. *If $xy=0$ and x is a nonzero element in Z, then $y=0$.*

Proof. If $xy=0$, then $zxy=xzy=0$ for all z in R. Since R is prime, and $x\neq 0$, y must be 0.

Proof of the theorem. Let T be a nontrivial automorphism of R such that $[x, x^T]$ is in Z for all x in R. The proof will consist of showing that $[x, x^T]=0$ for

[1] A portion of the research for this paper was done while the author was a Faculty Research Participant at the Argonne National Laboratory.

all x in R and then using Lemma 1 to conclude that R is commutative. Linearization of $[x, x^T]$ in Z gives

(1) $$[x, y^T]+[y, x^T] \text{ is in } Z \text{ for all } x \text{ and } y \text{ in } R,$$

and thus

(2) $$[x, [x, y^T]+[y, x^T]] = 0 \text{ for all } x \text{ and } y \text{ in } R.$$

Now R is a prime ring so R is either of characteristic two or $2x=0$ implies $x=0$ for x in R.

Suppose R is not of characteristic two and let $y=x^2$ in (2). Then $0=[x, x, (x^2)^T]+[x^2, x^T]]=[x, 2x^T[x, x^T]]+[x, 2x[x, x^T]]=2[x, x^T]^2$. Hence $[x, x^T]^2=0$. By Lemma 2 $[x, x^T]=0$ for all x in R and thus R is commutative.

Now suppose that R is of characteristic two. Then $[x^2, x^T]=2x[x, x^T]=0$ and $[(x^T)^2, x]=2x^T[x, x^T]=0$. Let $y=x^T$ in (1), then $[x, x^{TT}]+[x^T, x^T]=[x, x^{TT}]$ is in Z. Using the Jacobi identity (2) can be rewritten as

(3) $$[x, [y^T, x]]+[x^T, [x, y]] = 0.$$

Letting $y=x^3x^T$ in (3) gives

(4) $$[x, [(x^3x^T)^T, x]]+[x^T, [x, x^3x^T]] = 0.$$

Now $[x, [(x^3x^T)^T, x]]=[x, (x^3x^T)^Tx+x(x^3x^T)^T]=[x^2, (x^3x^T)^T]$. But expanding the last commutator gives

$$\begin{aligned} x[x, (x^3x^T)^T]+[x, (x^3x^T)^T]x &= x(x^T)^3[x, x^{TT}]+x[x, (x^T)^3]x^{TT}+(x^T)^3[x, x^{TT}]x+[x, (x^T)^3]x^{TT}x \\ &= [x, (x^T)^3][x, x^{TT}]+x(x^T)^2[x, x^T]x^{TT}+(x^T)^2[x, x^T]x^{TT}x \end{aligned}$$

since

$$[x, (x^T)^2] = 0.$$

Hence

$$\begin{aligned} [x, [(x^3x^T)^T, x]] &= [x, (x^T)^3][x, x^{TT}]+(x^T)^2[x, x^T][x, x^{TT}] \\ &= 2[x, (x^T)^3][x, x^{TT}] = 0. \end{aligned}$$

Thus (4) reduces to

(5) $$[x^T, [x, x^3x^T]] = 0.$$

But then $0=[x^T, x^3[x, x^T]]=[x^T, x^3][x, x^T]$ and using $[x^T, x^2]=0$ results in

(6) $$x^2[x, x^T]^2 = 0 \text{ for all } x \text{ in } R.$$

By Lemma 2, if $[x, x^T]\neq 0$, then $x^2=0$. So assume $x^2=0$, then $(x^T)^2=0$ and $(x^{TT})^2=0$. Now $(x^Tx)(xx^T)=0$ and $[x, x^T]=xx^T+x^Tx=z$ for some z in Z. Therefore $(xx^T+z)(xx^T)=0$ and thus $(xx^T)^2=z(xx^T)$. If $(xx^T)^2=0$, then $z(xx^T)=0$ and so either $z=0=[x, x^T]$ or $xx^T=0$. But if $xx^T=0$, then $[x, x^T]x=(x^Tx)x=0$ and hence $[x, x^T]=0$ or $x=0$. So from now on, assume that $x^2=0$ and $(xx^T)^2\neq 0$.

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Now (6) with xx^T replacing x implies that $[xx^T, (xx^T)^T]=0$. Expanding gives $x[x^T, x^Tx^{TT}]+[x, x^Tx^{TT}]x^T=0$. If this equation is left multiplied by x, then $x[x, x^Tx^{TT}]x^T=0$ and so $xx^T[x, x^{TT}]x^T+x[x, x^T]x^{TT}x^T=0$. But $xx^T[x, x^{TT}]x^T$ $=x(x^T)^2[x, x^{TT}]=0$. Thus $x[x, x^T]x^{TT}x^T=[x, x^T]xx^{TT}x^T=0$. If $[x, x^T]\neq 0$, then $xx^{TT}x^T=0$.

Thus $[x, x^{TT}]x^T=x^{TT}xx^T$, and so $x^{TT}[x, x^{TT}]x^T=(x^{TT})^2xx^T=0$. Hence if $[x, x^{TT}]\neq 0$, then $x^{TT}x^T=0$. But this forces $x^Tx=0$ and so $x=0$ or $[x, x^T]=0$. Suppose then that $[x, x^{TT}]=0$. Letting $y=xx^T$ in (2) results in $[x, [x^T, xx^T]+[x, (xx^T)^T]]=0$. Thus $[x, x^T[x, x^T]+x^T[x, x^{TT}]+[x, x^T]x^{TT}]=0$. But then $[x, x^T]^2+2[x, x^T][x, x^{TT}]=[x, x^T]^2=0$. Therefore $[x, x^T]=0$ for all x in R and by Lemma 1, R is commutative.

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