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On Conway's conjecture for integer sets

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Let $A = \{a_i\}$ be a finite set of integers and let p, m denote the orders of $A + A = \{a_i+a_j\}$ and $A - A = \{a_i-a_j\}$ respectively. J.H. Conway conjectured that $p \leq m$ always and that $p = m$ only if A is symmetric about 0. This conjecture has since been disproved; here we make several other observations on the values of p and m.

Let $A = \{a_i\}$ be a finite set of integers and let p, m denote the orders of $A + A = \{a_i+a_j\}$ and $A - A = \{a_i-a_j\}$ respectively. In [1] Conway asks for a proof that $p \leq m$ always and $p = m$ only if A is symmetric about 0.

In [2] Marica shows that not only is symmetry about any point sufficient to give equality but also that there exist nonsymmetric sets with $p = m$ and even $p > m$. Stein, [4], has gone even further and proved that the ratio m/p can be made as large or as small as we please. In [3] Spohn produces other counterexamples and makes various conjectures based on the observation that if A consists of the $n + 1$ integers $a_0 < a_1 < \ldots < a_n$, then the values of p and m depend only on the set $\{d_i\}$ where $d_i = a_i - a_{i-1}$.

Conjecture 1 of [3] says that for nonsymmetric A, $p < m$ for $n < 4$, and this may be checked by case-by-case evaluation. For $n = 1$, the result is trivial since a two-element set is necessarily symmetric. For $n = 2$, if $A = \{0, a, b\}$, then $A + A = \{0, a, b, 2a, a+b, 2b\}$ and $A - A = \{0, \pm a, \pm b, \pm(b-a)\}$, so $p = 6$, $m = 7$ unless $2a = b$ in which

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case the set A is symmetric. For $n = 3$, if $A = \{0, a, b, c\}$, then $A + A = \{0, a, b, c, 2a, a+b, a+c, 2b, b+c, 2c\}$ and $A - A = \{0, \pm a, \pm b, \pm c, \pm(b-a), \pm(c-b), \pm(c-a)\}$. If no two elements of $A + A$ are equal, then $p = 10$ and $m = 13$. Two elements of $A + A$ may be equal in any of the following five ways:

(1) $2a = b$,

(2) $2a = c$,

(3) $a + b = c$,

(4) $2b = c$,

(5) $2b = a + c$.

Any of (1), (2), (4) or (5) implies that $p = 9$ and $m = 11$. (3) implies that the set A is symmetric. No three elements of $A + A$ can be equal, but two of the above equations can be true simultaneously. There are three possibilities for this:

(1) and (4) together imply that $p = 8$, $m = 9$; so do (2) and (5). The only other possible combination is (1) with (5) and these imply that the set A is symmetric.

Conjecture 4 of [3], which states that $p \le m$ if $d_i \le 3$ for every i, is in fact false, for the set

$$\{0, 1, 4, 6, 9, 10, 13, 14, 16, 18, 19, 22, 23\},$$

with differences 1, 3, 2, 3, 1, 3, 1, 2, 2, 1, 3, 1, has $p = 44$, $m = 43$. However, we do have the following result.

THEOREM. *If $d_i \le 2$ for all i, then $p \le m$.*

Proof. We can take A to be $\{0, d_1, d_1+d_2, \ldots, d_1 + \ldots + d_n\}$. If $k = d_1 + \ldots + d_n$ then the integers in $A + A$ lie between 0 and $2k$ and those in $A - A$ between $-k$ and k. Hence if $m = 2k + 1$ we necessarily have $p \le m$, and to prove $m = 2k + 1$ it is sufficient to prove that every integer from 1 to k can be expressed as a sum of successive d_i's.

Case 1. $d_1 = 1$. We prove by induction on i that either $i = d_1 + \ldots + d_s$ for some s or $i = d_2 + \ldots + d_s$ for some s. This

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is certainly true for 1, so suppose it true for $i-1$. If $i-1 = d_2 + \dots + d_s$ then $i = d_1 + d_2 + \dots + d_s$. If $i-1 = d_1 + \dots + d_s$ and $d_{s+1} = 1$ then $i = d_1 + \dots + d_s + d_{s+1}$. If $i-1 = d_1 + \dots + d_s$ and $d_{s+1} = 2$ then $i = d_2 + \dots + d_s + d_{s+1}$.

Case 2. $d_n = 1$. As above, every i has an expression of the form $d_s + d_{s+1} + \dots + d_n$ or $d_s + d_{s+1} + \dots + d_{n-1}$.

Case 3. $d_1 = d_n = 2$. If $d_i = 2$ for every i, then A is symmetric and Marica's result shows that $p \le m$, so we may suppose that $d_1 = d_2 = \dots = d_a = 2$, $d_{a+1} = 1$ and $d_{n-b} = 1$, $d_{n-b+1} = \dots = d_{n-1} = d_n = 2$. As in the previous cases, every integer up to the larger of $d_1 + d_2 + \dots + d_{n-b}$ and $d_{a+1} + d_{a+2} + \dots + d_n$ can be expressed as a sum of successive d_i's but thereafter only alternate integers can be so expressed. Thus we have that $m = 2k + 1 - 2\min\{a, b\}$. But then $A = \{0, 2, 4, \dots, 2a, 2a+1, \dots, k-2b-1, k-2b, \dots, k-2, k\}$ and $A + A = \{0, 2, 4, \dots, 2a, 2a+1, \dots, 2k-2b-1, 2k-2b, \dots, 2k-2, 2k\}$ so that $p \le 2k + 1 - (a+b) \le m$, as required.

This argument can be extended to show that $p \le m$ if $d_i = 1$ or n for all i, provided that the first and last times that 1 occurs as a difference, it occurs in a block of at least $n-1$ consecutive differences, each of which equals 1.

Another possible way of salvaging Conway's conjecture would be to replace $A + A$ and $A - A$ by the deleted sum and difference $A \oplus A = \{a_i + a_j \mid i \ne j\}$ and $A \ominus A = \{a_i - a_j \mid i \ne j\}$. Since $|A \oplus A| \le p - 2$ whereas $|A \ominus A| = m - 1$ we have strict inequality in the symmetric case. However if

$$A = \{0, 1, 2, 4, 5, 9, 12, 13, 17, 20, 21, 25, 28, 30, 32, 33\}$$

we have $|A \oplus A| = 61$ and $|A \ominus A| = 60$, so this does not work either.

Note added in proof (6 April 1973). Conway has informed us that his conjecture was merely that $p \le m$ and that he is not responsible for the patently false rider that $p = m$ only if A is symmetric about 0.

References

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