# A MODULE IS FLAT IF AND ONLY IF ITS CHARACTER MODULE IS INJECTIVE

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The purpose of this expository note is to establish the fact mentioned in the title. While this is not difficult and requires no new ideas, it seems worth doing, as such a simple characterization does not appear explicitly in a recent treatise on the subject of flat modules [Bourbaki XXVII, Chapter 1].

Let us begin by defining the technical terms appearing in the title. Let R be an associative ring with 1, ${}_R M$ a left R-module. (All modules are understood to be unitary.) To say that ${}_R M$ is <u>flat</u> is to assert that, whenever $A_R$ and $B_R$ are right R-modules,

$$A \subset B \Rightarrow A \otimes_R M \subset B \otimes_R M ,$$

or more precisely, when $A_R$ is a submodule of $B_R$, the canonical mapping of $A \otimes_R M$ into $B \otimes_R M$ is a monomorphism.

By the <u>character module</u> of ${}_R M$ we mean the additive group

$$M^* = \mathrm{Hom}_Z (M, D) ,$$

where D is the additive group of rationals modulo 1 and Z is the ring of integers, turned into a right R-module $M^*_R$ by defining

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$$(hr)m = h(rm)$$

for all $h \in M^*$, $r \in R$ and $m \in M$.

A right R-module $N_R$ is called <u>injective</u> if, whenever $A_R$ and $B_R$ are right R-modules such that $A_R$ is a submodule of $B_R$, any $f \in \mathrm{Hom}_R(A, N)$ can be extended to some $f' \in \mathrm{Hom}_R(B, N)$. It is well-known that an additive abelian group, regarded as a Z-module, is injective if and only if it is divisible [e.g., Northcott, page 269]. In particular, $D_Z$ is injective.

We shall not define here the tensor product $A \otimes_R M$, but merely state its main property (which actually does define it in a sense). Let $G: A \times M \to C$ be any <u>bilinear</u> mapping of the pair of modules $(A_R, {}_RM)$ into an additive group C. By this we mean that $G(a, m)$ is linear in each variable and that

$$G(ar, m) = G(a, rm)$$

for all $a \in A$, $r \in R$ and $m \in M$. Then there exists a (unique) homomorphism

$$g \in \mathrm{Hom}_Z(A \otimes_R M, C)$$

such that

$$g(a \otimes m) = G(a, m)$$

for all $a \in A$ and $m \in M$.

We shall require a well-known property of character modules:

LEMMA 1. If $0 \neq m \in M$, there exists a character $h \in M^*$ such that $hm \neq 0$.

<u>Proof.</u> Since D is injective as a Z-module, it suffices to find $k \in \mathrm{Hom}_Z(mZ, D)$ such that $km \neq 0$, which k may then

be extended to the required $h$.

If $mz = 0$ for some $0 \neq z \in Z$, let $z_o$ be the smallest positive integer such that $mz_o = 0$, and define $kmz = z_o^{-1} z$ modulo 1. If $mz = 0$ implies $z = 0$, define $kmz = \frac{1}{17} z$ modulo 1.

Here is our main result.

THEOREM. ${}_R M$ is flat if and only if $M^*_R$ is injective.

Proof. Assume that ${}_R M$ is flat and that $A_R$ is a sub-module of $B_R$. Then also $A \otimes_R M$ is a subgroup of $B \otimes_R M$. Now consider any $f \in \mathrm{Hom}_R(A, M^*)$. The mapping $(a, m) \rightarrow (fa)m$ is a bilinear mapping from $(A_R, {}_R M)$ to $D$; hence there exists

$$g \in \mathrm{Hom}_Z (A \otimes_R M, D)$$

such that

$$g(a \otimes m) = (fa)m$$

for all $a \in A$ and $m \in M$. Since $D_Z$ is injective, $g$ may be extended to

$$g' \in \mathrm{Hom}_Z (B \otimes_R M, D) .$$

Now define

$$f' \in \mathrm{Hom}_R (B, M^*)$$

by

$$(f' b)m = g' (b \otimes m)$$

for all $b \in B$ and $m \in M$. For any $a \in A$ and $m \in M$ we thus have



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$$(f'a)m = g'(a \otimes m) = g(a \otimes m) = (fa)m ;$$

hence $f'$ extends $f$. Thus $M^*_R$ is injective.

Conversely, assume that $M^*_R$ is injective and that $A_R$ is a submodule of $B_R$. We wish to infer that the canonical mapping of $A \otimes_R M$ into $B \otimes_R M$ is a monomorphism, whence it will follow that ${}_R M$ is flat.

Suppose $\Sigma_{i=1}^n a_i \otimes m_i = 0$ in $B \otimes_R M$, where $m_i \in M$ and $a_i \in A$. We will show that it is also $0$ in $A \otimes_R M$. Indeed, if it is not, then by the lemma there exists a character $g \in (A \otimes_R M)^*$ such that

$$g(\Sigma_{i=1}^n a_i \otimes m_i) \neq 0 .$$

Define

$$f \in \mathrm{Hom}_R(A, M^*)$$

by

$$(fa)m = g(a \otimes m)$$

for all $a \in A$ and $m \in M$. Since $M^*_R$ is injective, we may extend $f$ to

$$f' \in \mathrm{Hom}_R(B, M^*) .$$

The mapping $(b, m) \to (f'b)m$ is bilinear from $(B_R, {}_R M)$ to $D$, hence there exists $g' \in (B \otimes_R M)^*$ such that

$$g'(b \otimes m) = (f'b)m$$

for all $b \in B$ and $m \in M$. Then, in $A \otimes_R M$,



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$$0 = g'(\Sigma_{i=1}^{n} a_i \otimes m_i) = \Sigma_{i=1}^{n} (f' a_i) m_i$$

$$= \Sigma_{i=1}^{n} (f a_i) m_i = g(\Sigma_{i=1}^{n} a_i \otimes m_i) \neq 0 ,$$

a contradiction, and our proof is complete.

Actually we have proved a little more than what the theorem asserts. We have in fact shown the following:

PROPOSITION 1. Let $A_R$ be a submodule of $B_R$, ${}_R M$ a given left R-module. Then the canonical mapping of $A \otimes_R M$ into $B \otimes_R M$ is a monomorphism if and only if the canonical mapping of $\mathrm{Hom}_R(B, M^*)$ into $\mathrm{Hom}_R(A, M^*)$ is an epimorphism.

The first part of the proof of the above theorem is well-known. Essentially the same kind of argument establishes the formula

$$\mathrm{Ext}\,(N, M^*) \cong (\mathrm{Tor}\,(N, M))^*$$

[see Cartan and Eilenberg, p. 120, Proposition 5.1]. The present theorem may also easily be deduced from this formula with the help of Lemma 1.

The above proof has been addressed to the general reader without any knowledge of homological algebra. For those who are familiar with the language of functors and exact sequences, let us point out the natural isomorphism

$$\mathrm{Hom}_R(A, M^*) \cong (A \otimes_R M)^*$$

of functors in A. Our theorem follows from this and from the observation that the sequence

$$0 \to A \otimes_R M \to B \otimes_R M$$

is exact if and only if the induced sequence



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$$(B \otimes_R M)^* \to (A \otimes_R M)^* \to 0$$

is exact. This observation is readily deduced from Lemma 1.

It may be of some interest to establish more generally

PROPOSITION 2. A sequence of abelian groups

$$U \xrightarrow{p} V \xrightarrow{q} W$$

is exact if and only if the induced sequence

$$W^* \xrightarrow{q^*} V^* \xrightarrow{p^*} U^*$$

is exact.

Here $q^*$ (and similarly $p^*$) is of course defined as follows: For any $h \in W^*$ and $v \in V$,

$$(q^*h)v = h(qv) .$$

Proof. Let $g \in V^*$; then $g \in \operatorname{Ker} p^*$ if and only if

$$g(\operatorname{Im} p) = g(pU) = (p^*g)U = 0 .$$

On the other hand, $g \in \operatorname{Im} q^*$ if and only if there exists an $h \in W^*$ such that $gv = hqv$, for all $v \in V$. This is the same as saying that $g(\operatorname{Ker} q) = 0$; for then $qv \to gv$ is a homomorphism of $qV$ into $D$, which can be extended to a homomorphism $h: W \to D$, by injectivity of $D$.

Thus $\operatorname{Ker} q = \operatorname{Im} p$ clearly implies $\operatorname{Im} q^* = \operatorname{Ker} p^*$. Conversely, let us assume the second equality. Then, by the above, $g(\operatorname{Ker} q) = 0$ if and only if $g(\operatorname{Im} p) = 0$, for all $g \in V^*$. Can we deduce from this that $\operatorname{Ker} q = \operatorname{Im} p$ ? Yes, in view of the following:

LEMMA 2. Let $S$ be any subgroup of $V$ and $v$ any element of $V$ not in $S$. Then there exists a character of $V$ which annihilates $S$ but not $v$.

Proof. Apply Lemma 1 to $V/S$.

Lemma 2 may be interpreted as saying that every subgroup of an abelian group is "closed" under the obvious closure operation: The closure of a subset S of the group V consists of all elements of V which are annihilated by every character of V which annihilates S.

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