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VOLATILITY DETERMINATION IN AN AMBIT PROCESS SETTING

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VOLATILITY DETERMINATION IN AN AMBIT PROCESS SETTING

BY OLE E. BARNDORFF-NIELSEN AND SVEND ERIK GRAVERSEN

Abstract

The limit behaviour in probability of realised quadratic variation is discussed under a relatively simple ambit process setting. The relation of this to the underlying volatility/intermittency field is in focus, especially as concerns the question of no volatility/intermittency memory.

Keywords: Ambit field; realised quadratic variation; volatility memorylessness

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1. Introduction

Dynamic stochastic phenomena frequently involve a significant element of randomness beyond the most basic types of stochastic innovation. Additional variations of this kind, for instance in the form of latent stochastic variance changing over time, are often referred to as volatility or intermittency, and they are of key importance, particularly in finance and turbulence.

In many cases the volatility is expressed in stochastic modelling by a multiplicative term specified as a positive process σ. Thus, for example, we consider stochastic processes symbolically written as

$$Y_t = \int_{A_t} g(t-s)\sigma_s \mathrm{d}W_s, \tag{1}$$

where A_t is a t-dependent interval of $\mathbb{R}$, g is a deterministic function, and W is a Brownian motion that is independent of the process σ. The question of what can be learned about σ from observations of the process is then often of central interest, and the main tool to study that is (realised) multipower variations, in particular (realised) quadratic variation; see [2], [3], [4], [5], [6], [7], [10], and the references therein.

There are two main types of (1). In the case when g is constant and $A_t = [0, t]$ we are in the framework of Brownian semimartingales, while if g is nontrivial and A_t is of the form $[t-c, t]$ for some $c \in (0, \infty]$ we have a Brownian semistationary process, as defined in [3]. Note that in the latter case if the process σ is stationary then Y is in fact a strictly stationary process on $\mathbb{R}$. These two types are substantially different. In particular, Brownian semistationary processes are generally not semimartingales, and this, in particular, implies major differences between the theory of multipower variations for the two types; see [6]. To exemplify, in the Brownian semimartingale case the realised quadratic variation over $[0, t]$ will converge in probability to $\sigma^{2+}_{[0,t]}$, where, for $a < b$, $\sigma^{2+}_{(a,b]} = \int_a^b \sigma_s^2 \mathrm{d}s$. On the other hand, for Brownian semistationary processes, where a normalisation of the realised quadratic variation is generally required, it may, for instance, happen that the convergence is to $\lambda\sigma^{2+}_{[0,t]} + (1-\lambda)\sigma^{2+}_{[-1,t-1]}$ for

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some constant $\lambda \in (0, 1)$ (cf. [3] and [6]). When the limit is in fact $\sigma^{2+}_{[0,t]}$, we speak of a volatility memoryless process.

Brownian semistationary processes constitute the subfamily of Brownian-based ambit processes for which the spatial component is trivial. The general form (except for a drift term that will not concern us here) is based on an ambit field Y, i.e. a stochastic field in space–time

$$Y(x,t) = \int_{A_t(x)} g(x,\xi,t-s)\sigma_s(\xi)W(\mathrm{d}\xi\mathrm{d}s),$$

where $A_t(x)$ is some subset of $\mathcal{X} \times (-\infty, t]$ for some spatial region $\mathcal{X}$, g is deterministic, σ is a positive stochastic field, and W is two-dimensional white noise. Then an ambit process X is a process of the form $X = Y(\tau)$, where τ denotes a smooth curve in $\mathcal{X} \times \mathbb{R}$.

The purpose of the present paper is to explore the question of volatility memorylessness for a simple tempo-spatial setting and to draw some conclusions with respect to further related research questions.

In Section 2 we present our main conclusions, while the proofs are given in Section 4. In Section 3 we summarise and provide a brief outlook.

2. Results

We restrict the discussion to the case in which $\mathcal{X} = \mathbb{R}$ and to ambit fields of the form

$$Y(x,t) = \int_{A_t(x)} g(x-\xi,t-s)\sigma_s(\xi)W(\mathrm{d}\xi\,\mathrm{d}s), \tag{2}$$

where $A_t(x) = A + (x, t)$ for some $A \in \mathcal{B}_\mathrm{b}(\mathbb{R}^2)$, the bounded Borel sets in $\mathbb{R}^2$, g is a Lebesgue locally square integrable function on $\mathbb{R}^2$, and $(\sigma_s(\xi))_{(\xi,s)\in\mathbb{R}^2}$ is a real-valued continuous random field independent of W with $(\xi, s) \mapsto \mathrm{E}[\sigma_s^2(\xi)]$ locally bounded. Here we are primarily interested in the case where $A = \{(\xi, s) \in \mathbb{R}^2 \mid -M \le s \le 0,\ c_1(s) \le \xi \le c_2(s)\}$ for some $M \in \mathbb{R}_+$, and smooth functions $c_1\colon [-M, 0] \to \mathbb{R}_-$ and $c_2\colon [-M, 0] \to \mathbb{R}_+$ such that c_1 is increasing and c_2 is decreasing. Note that A is a closed set and that if $c_1(0) = c_2(0)$ then $(0, 0)$ is the unique top point, i.e. the only point in A for which $s = 0$.

To specify the meaning of (2), let λ_2 denote the Lebesgue measure on $\mathbb{R}^2$ and, for $f \in L^2(\lambda_2)$, let $W_f = \int_{\mathbb{R}^2} f(\xi, s)W(\mathrm{d}\xi\mathrm{d}s)$, the integral being a Wiener integral. The field $(W_f)_{f\in L^2(\lambda_2)}$ is then isonormal, that is, a centred Gaussian process with covariance function $(f, h) \mapsto \int_{\mathbb{R}^2} fh\,\mathrm{d}\lambda_2$. As $(f, \omega) \mapsto W_f(\omega)$ can be assumed measurable, we can, for every $B \in \mathcal{B}_\mathrm{b}(\mathbb{R}^2)$, consider the variable W_{T_B}, where $T_B\colon \Omega \to L^2(\lambda_2)$; $\omega \mapsto \sigma_\cdot(\cdot)(\omega)g\,\mathbf{1}_B$. Consequently, $(W_{T_B})_{B\in\mathcal{B}_\mathrm{b}(\mathbb{R}^2)}$ is a well-defined square integrable centred process whose distribution given $\sigma = \psi$ equals the distribution of $(\int_B g\psi\,\mathrm{d}W)_{B\in\mathcal{B}_\mathrm{b}(\mathbb{R}^2)}$; in other words, (W_{T_B}) is a centred Gaussian process with covariance function $(B_1, B_2) \mapsto \int_{B_1\cap B_2} g^2\psi^2\,\mathrm{d}\lambda_2$. This last observation both justifies and motivates the writing used in (2).

For a given smooth curve $\tau = (\tau_1, \tau_2)\colon \mathbb{R} \to \mathbb{R}^2$, consider the process $X_\theta = Y(\tau(\theta))$, $\theta \ge 0$. The realised quadratic variation of X and its normalised version are, for $\delta > 0$ and $t > 0$, given by

$$[X_\delta]_t = \sum_{k=1}^{[t/\delta]} (X_{k\delta} - X_{(k-1)\delta})^2 \quad \text{and} \quad \overline{[X_\delta]}_t = \frac{\delta}{c(\delta)}[X_\delta]_t,$$

where $c(\delta)$ is a positive constant depending only on δ, whose specific form will be given below. We are interested in the asymptotic behaviour of $\overline{[X_\delta]}_t$ for $\delta \to 0$. So far we can only

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satisfactorily handle the case of τ being a straight line or, more generally, a piecewise straight line. Therefore, for ease of notation we will from now on assume that $\theta \mapsto \tau(\theta)$ is a straight line and, thus, $\Delta\tau(\delta) = \delta\Delta\tau$, where, in obvious notation,

$$\Delta\tau(\delta) = (\Delta\tau_1(\delta), \Delta\tau_2(\delta)) = (\tau_1(t+\delta) - \tau_1(t), \tau_2(t+\delta) - \tau_2(t)) \quad \text{for } t, \delta \geq 0.$$

We now introduce a probability measure π_δ which is determined by the kernel function g. The behaviour of π_δ as $\delta \to 0$ is of key importance for the probabilistic limit properties of $\overline{[X_\delta]}$.

Set

$$\psi_\delta(u, v) = \begin{cases} (g(\Delta\tau_1(\delta) + u, \Delta\tau_2(\delta) + v) - g(u, v))^2 & \text{for } (u, v) \in (-A) \cap (-A - \Delta\tau(\delta)), \\ g^2(u, v) & \text{for } (u, v) \in (-A) \setminus (-A - \Delta\tau(\delta)), \\ g^2(\Delta\tau_1(\delta) + u, \Delta\tau_2(\delta) + v) & \text{for } (u, v) \in (-A - \Delta\tau(\delta)) \setminus (-A). \end{cases}$$

Observe that

$$\psi_\delta(u, v) = 0 \quad \text{if } (u, v) \notin (-A) \cup (-A - \Delta\tau(\delta))$$

and that, under smoothness conditions, $\psi_\delta(u, v)$ for $(u, v) \in (-A) \cap (-A - \Delta\tau(\delta))$ will typically be of order δ^2. Now define

$$\pi_\delta(\mathrm{d}u\mathrm{d}v) = \frac{\psi_\delta(u, v)}{c(\delta)} \lambda_2(\mathrm{d}u\, \mathrm{d}v) \quad \text{for } \delta > 0, \tag{3}$$

where $c(\delta) = \int_{\mathbb{R}^2} \psi_\delta(u, v) \lambda_2(\mathrm{d}u\, \mathrm{d}v)$. Here it is tacitly assumed that $c(\delta) > 0$, as will be the case under the assumptions of the theorem stated below. Then, by construction, π_δ is a probability measure and, clearly, all weak limit points of π_δ for $\delta \to 0$ will be probability measures concentrated on $-A$. Simple calculations together with the continuity assumption on σ then imply that in the case where the limit $\pi_\delta \xrightarrow{\mathrm{w}} \pi_0$ exists as $\delta \to 0$,

$$\mathrm{E}[\overline{[X_\delta]}_t \mid \sigma] \to \int_{\mathbb{R}^2} \int_0^t \sigma^2_{\tau_2(s)-v}(\tau_1(s) - u)\, \mathrm{d}s \pi_0(\mathrm{d}u\, \mathrm{d}v) \quad \text{as } \delta \to 0.$$

We are particularly interested in conditions on A and g ensuring that the limit π_0 exists and is concentrated on $\partial(-A) = -\partial A$. In this case we further have $\lim_{\delta\to 0} \operatorname{var}(\overline{[X_\delta]}_t \mid \sigma) = 0$, as established by Lemma 2 in Section 4.

Under these conditions, we will thus have the key result that, as $\delta \to 0$,

$$\overline{[X_\delta]}_t \xrightarrow{\mathrm{P}} \int_{\mathbb{R}^2} \int_0^t \sigma^2_{\tau_2(s)+v}(\tau_1(s) + u)\, \mathrm{d}s \pi(\mathrm{d}u\, \mathrm{d}v). \tag{4}$$

Here π denotes the image measure of π_0 under the transformation $(u, v) \mapsto (-u, -v)$. Observe that π is concentrated on ∂A.

We can now state the main result of this paper. For proofs and further details, see Section 4. Recall that, for any bounded closed convex subset $C \subseteq \mathbb{R}^2$, containing 0 as an interior point, the function

$$T(x) = \inf\{t > 0 \mid x \in tC\}, \qquad x \in \mathbb{R}^2, \tag{5}$$

is called the gauge function of C. See [8, Chapter 5] for details and properties of T.

Theorem 1. *Let τ be a straight line, and let A be a bounded, closed, convex set with nonempty interior A° and piecewise C^∞ boundary. Then there exists a probability measure π concentrated on the boundary ∂A of A such that (4) holds provided the following condition is satisfied for some $-\frac{1}{2} < \alpha < \frac{1}{2}$.*

(i) *$g = \varphi h_\alpha$, where φ is Lipschitz continuous and not identically 0 on the part of $-\partial A$ nonparallel to τ, and*

$$h_\alpha(-x) = \begin{cases} (1 - T(x - x_0))^\alpha, & x \in A, \\ 0, & x \notin A, \end{cases}$$

where T is the gauge function of $A - x_0$ for some $x_0 \in A^\circ$.

Remark. The properties of T ensure the existence of positive constants C_1 and C_2 depending only on A and x_0 such that

$$C_1 d(x, \partial A) \le 1 - T(x - x_0) \le C_2 d(x, \partial A), \qquad x \in A,$$

where $d(x, \partial A)$ is the Euclidean distance between x and ∂A. The lower condition $-\frac{1}{2} < \alpha$ therefore comes from the requirement that g should be locally integrable, whereas the upper bound reflects the fact that in order for π to be concentrated on ∂A, the function g cannot tend too fast to 0 as its argument tends to the boundary. In the proof we need $\delta^2/c(\delta) \to 0$ for $\delta \downarrow 0$, implying that $2\alpha + 1 < 2$.

Remark. It follows from the proof given in Section 4 (see Lemma 3 for the case in which $\alpha = 0$) that if the boundary of A is piecewise smooth then the limit measure π exists if $\lambda_{21}(\{x \in \partial A \mid \varphi(-x)^2 |\tau \cdot n(x)| > 0\}) > 0$ and is determined by $\mathrm{d}\pi = c\varphi(-\cdot)^2 |\tau \cdot n| \, \mathbf{1}_{\partial A} \, \mathrm{d}\lambda_{21}$, where c is a normalising constant, τ is the unit vector giving the direction of the straight line along which we move, n is the normalised outward normal to ∂A, with $\tau \cdot n$ being the Euclidean inner product, and λ_{21} denotes the one-dimensional Hausdorff measure in $\mathbb{R}^2$.

Remark. If A has a unique top point at 0 and if $g = d(\cdot, 0)^\alpha$ for some α with $-\frac{1}{2} < \alpha < 0$, then we can verify, similarly to the proof of Theorem 1, that the limit measure π exists and equals the delta measure at the top point, in which case the process X is volatility memoryless (in the sense of [3]).

Example. Suppose that the ambit set A is specified by $c_1(s) = -c_2(s) = s$, with $-M \le s \le 0$, and let g be given by $g(\xi, s) = |s|^\alpha$ for an $\alpha \in (-\frac{1}{2}, 0]$. Then, with $\tau(\theta) = (0, \theta)$, from the definition of $\psi_\delta(u, v)$, it is easily seen that the limit measure π exists and is proportional to the Lebesgue measure on the boundary of the triangle A in the case in which $\alpha = 0$, while if $\alpha < 0$ then π equals the delta measure at the top point of A.

3. Conclusion and outlook

We have discussed the probabilistic limit behaviour of (normalised) realised quadratic variation for a class of ambit processes, where the underlying volatility/intermittency field σ is continuous and where the mother ambit set A is a bounded, closed, and convex. In this setting a considerable variety of limits are possible, depending on the nature of the damping

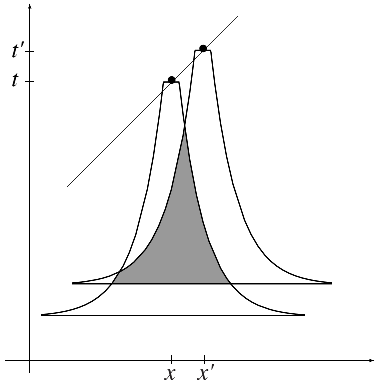

FIGURE 1: Ambit regions.

function g. All the limits are integrals of the squared volatility/intermittency field over the set A and with respect to a probability measure π on A. Under specified weak conditions, the integrals are concentrated on the boundary of A. Volatility memorylessness can be ensured if A has a single top point.

There is a range of further questions of theoretical and applied interest in this context. (i) What happens if A is not bounded, stretching to minus infinity in time, or if A is not convex? (Figure 1 shows one type of ambit set that is of interest in turbulence studies and whose shape is motivated by Taylor's frozen field hypothesis (cf. [1]).) (ii) What is the situation in the case when the curve τ is not linear? (The linearity assumption is crucial in deriving (6) below.) (iii) What happens if σ is not continuous? (iv) What is the probabilistic limit behaviour of multipower variations generally? (v) What type of central limit theorems can be established for the multipower variations? (Undoubtedly, as was the case for Brownian semimartingales (see [6]), Malliavin calculus will be a key tool.) (vi) How may such central limit theorems be used to draw inference not only on σ but also on g (cf. [6])?

4. Proofs

Maintaining the notation of Section 2, we write g_A for $g \cdot \mathbf{1}_{-A}$. Inserting this gives

$$[X_\delta]_t = \sum_{k=1}^{[t/\delta]} \biggl(\int_{\mathbb{R}^2} [g_A(\tau(k\delta) - (\xi, s)) - g_A(\tau((k-1)\delta) - (\xi, s))] \sigma_s(\xi) W(\mathrm{d}\xi \, \mathrm{d}s) \biggr)^2,$$

implying, by means of the independence between σ and W, that, for all $\delta, t > 0$,

$$\mathrm{E}[[X_\delta]_t \mid \sigma] = \sum_{k=1}^{[t/\delta]} \int_{\mathbb{R}^2} [g_A(\tau(k\delta) - (\xi, s)) - g_A(\tau((k-1)\delta) - (\xi, s))]^2 \sigma_s^2(\xi) \lambda_2(\mathrm{d}\xi \, \mathrm{d}s).$$

Writing $\Delta\tau(k\delta)$ for $\tau(k\delta) - \tau((k-1)\delta)$ and using the linear substitution

$$(u, v) = (\tau_1((k-1)\delta) - \xi, \tau_2((k-1)\delta) - s),$$

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we find that $\mathrm{E}[[X_\delta]_t \mid \sigma]$ may be written as

$$\sum_{k=1}^{[t/\delta]} \int_{\mathbb{R}^2} [g_A(\Delta\tau(k\delta) + (u, v)) - g_A(u, v)]^2 \sigma^2_{\tau_2((k-1)\delta)-v}(\tau_1((k-1)\delta) - u)\lambda_2(\mathrm{d}u\,\mathrm{d}v).$$

Thus, if $\Delta\tau(k\delta) = \Delta\tau(\delta)$, independent of k, in particular if $\theta \mapsto \tau(\theta)$ is a straight line, we have

$$\mathrm{E}[[X_\delta]_t \mid \sigma] = \int_{\mathbb{R}^2} \psi_\delta(u, v) \sum_{k=1}^{[t/\delta]} \sigma^2_{\tau_2((k-1)\delta)-v}(\tau_1((k-1)\delta) - u)\lambda_2(\mathrm{d}u\,\mathrm{d}v), \tag{6}$$

where, for $\delta > 0$,

$$\begin{aligned}
\psi_\delta(u, v) &= (g_A(\Delta\tau_1(\delta) + u, \Delta\tau_2(\delta) + v) - g_A(u, v))^2 \\
&= \begin{cases} (g(\Delta\tau_1(\delta) + u, \Delta\tau_2(\delta) + v) - g(u, v))^2 & \text{for } (u, v) \in (-A) \cap (-A - \Delta\tau(\delta)), \\ g^2(u, v) & \text{for } (u, v) \in (-A) \setminus (-A - \Delta\tau(\delta)), \\ g^2(\Delta\tau_1(\delta) + u, \Delta\tau_2(\delta) + v) & \text{for } (u, v) \in (-A - \Delta\tau(\delta)) \setminus (-A). \end{cases}
\end{aligned}$$

Observe that $\psi_\delta(u, v) = 0$ if $(u, v) \notin (-A) \cup (-A - \Delta\tau(\delta))$. Formula (6) suggests that it is natural to let $c(\delta) = \int_{\mathbb{R}^2} \psi_\delta(u, v)\lambda_2(\mathrm{d}u\,\mathrm{d}v)$, since then, assuming that $c(\delta) > 0$,

$$\mathrm{E}[\overline{[X_\delta]}_t \mid \sigma] = \int_{\mathbb{R}^2} \delta \sum_{k=1}^{[t/\delta]} \sigma^2_{\tau_2((k-1)\delta)-v}(\tau_1((k-1)\delta) - u)\pi_\delta(\mathrm{d}u\,\mathrm{d}v),$$

where π_δ denotes the probability measure defined by (3).

Assume from now on that $\theta \mapsto \tau(\theta)$ is a straight line and, thus, $\Delta\tau(\delta) = \delta\Delta\tau$. As already observed, for all $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that $\pi_\delta(\mathbb{R}^2 \setminus A_\varepsilon) = 0$ for all $0 < \delta < \delta_\varepsilon$, where, using the notation $d((\xi, s), B) := \inf_{(u,v)\in B} |(\xi, s) - (u, v)|$ for any $B \subseteq \mathbb{R}^2$, the set A_ε is given by $A_\varepsilon = \{(\xi, s) \in \mathbb{R}^2 \mid d((\xi, s), -A) \le \varepsilon\}$. Thus, all weak limit points of π_δ for $\delta \to 0$ will be probability measures concentrated on $-A$. Using the continuity assumption on σ, we see that in the case where the limit $\pi_\delta \xrightarrow{\mathrm{w}} \pi_0$ exists as $\delta \to 0$,

$$\mathrm{E}[\overline{[X_\delta]}_t \mid \sigma] \to \int_{\mathbb{R}^2} \int_0^t \sigma^2_{\tau_2(s)-v}(\tau_1(s) - u)\,\mathrm{d}s\,\pi_0(\mathrm{d}u\,\mathrm{d}v) \quad \text{as } \delta \to 0.$$

We are interested in conditions on A and g ensuring that the limit π_0 exists and is concentrated on $\partial(-A) = -\partial A$, implying of course that π is concentrated on ∂A. Before discussing specific conditions for this to happen we establish the following lemma.

Lemma 1. *Under the assumption that π_0 exists and is concentrated on $-\partial A$, we have*

$$\lim_{\delta\to 0} \mathrm{var}(\overline{[X_\delta]}_t \mid \sigma) = 0. \tag{7}$$

Proof. For given $\delta, t > 0$, $\mathrm{var}(\overline{[X_\delta]}_t \mid \sigma)$ equals $\delta^2/c(\delta)^2$ times

$$\sum_{k=1}^{[t/\delta]} \mathrm{var}((X_{k\delta} - X_{(k-1)\delta})^2 \mid \sigma) + 2 \sum_{1\le k<l\le[t/\delta]} \mathrm{cov}((X_{k\delta} - X_{(k-1)\delta})^2, (X_{l\delta} - X_{(l-1)\delta})^2 \mid \sigma).$$

Applying the fact that, for any centred jointly Gaussian vector (U, V), we have $\operatorname{cov}(U^2, V^2) = 2\operatorname{cov}(U, V)^2$ and $\operatorname{var}(U^2) = 2\operatorname{var}(U)^2$, we may write $\operatorname{var}(\overline{[X_\delta]}_t \mid \sigma) = I_\delta + II_\delta$, where

$$I_\delta = \frac{2\delta^2}{c(\delta)^2} \sum_{k=1}^{[t/\delta]} \mathrm{E}[(X_{k\delta} - X_{(k-1)\delta})^2 \mid \sigma]^2$$

and

$$II_\delta = \frac{4\delta^2}{c(\delta)^2} \sum_{k=1}^{[t/\delta]} \mathrm{E}[(X_{k\delta} - X_{(k-1)\delta})(X_{l\delta} - X_{(l-1)\delta}) \mid \sigma]^2.$$

Simple manipulations show that, for all $\delta > 0$,

$$I_\delta \le \frac{2\delta}{c(\delta)} \max_{1\le k\le [t/\delta]} \mathrm{E}[(X_{k\delta} - X_{(k-1)\delta})^2 \mid \sigma]\, \mathrm{E}[\overline{[X_\delta]}_t \mid \sigma]$$

and, for all $1 \le k \le [t/\delta]$,

$$\begin{aligned}
&\mathrm{E}[(X_{k\delta} - X_{(k-1)\delta})^2 \mid \sigma] \\
&\quad = \int_{\mathbb{R}^2} (g_A(\tau(k\delta) - (u, v)) - g_A(\tau((k-1)\delta) - (u, v)))^2 \sigma_v^2(u) \lambda_2(\mathrm{d}u\,\mathrm{d}v) \\
&\quad \le \max_{(u,v)\in A_{|\Delta\tau|\delta}} \sigma_v^2(u) \int_{\mathbb{R}^2} (g_A(\Delta\tau(\delta) + (u, v)) - g_A(u, v))^2 \lambda_2(\mathrm{d}u\,\mathrm{d}v) \\
&\quad = \max_{(u,v)\in A_{|\Delta\tau|\delta}} \sigma_v^2(u)\, c(\delta).
\end{aligned}$$

Thus, this shows that $\lim_{\delta\to 0} I_\delta = 0$. So it remains to be seen that $\lim_{\delta\to 0} II_\delta = 0$. For all $1 \le k < l \le [t/\delta]$, we have

$$\begin{aligned}
&\mathrm{E}[(X_{k\delta} - X_{(k-1)\delta})(X_{l\delta} - X_{(l-1)\delta}) \mid \sigma] \\
&\quad = \int_{\mathbb{R}^2} (g_A(\tau(k\delta) - (u, v)) - g_A(\tau((k-1)\delta) - (u, v))) \\
&\qquad\quad \times (g_A(\tau(l\delta) - (u, v)) - g_A(\tau((l-1)\delta) - (u, v)))\sigma_v^2(u)\lambda_2(\mathrm{d}u\,\mathrm{d}v) \\
&\quad = \int_{\mathbb{R}^2} (g_A(\Delta\tau(\delta) + (u, v)) - g_A(u, v)) \\
&\qquad\quad \times (g_A((l-k+1)\Delta\tau(\delta) + (u, v)) - g_A((l-k)\Delta\tau(\delta) + (u, v))) \\
&\qquad\quad \times \mathbf{1}_{A_{|\Delta\tau|\delta}}(u, v)\sigma^2_{\tau_2((k-1)\delta)-v}(\tau_1((k-1)\delta) - u)\lambda_2(\mathrm{d}u\,\mathrm{d}v).
\end{aligned}$$

Using the continuity of the σ-process and Cauchy–Schwarz's inequality, this implies the existence of a constant K such that

$$\begin{aligned}
&\mathrm{E}[(X_{k\delta} - X_{(k-1)\delta})(X_{l\delta} - X_{(l-1)\delta}) \mid \sigma]^2 \\
&\quad \le K \int_{\mathbb{R}^2} (g_A(\Delta\tau(\delta) + (u, v)) - g_A(u, v))^2 \lambda_2(\mathrm{d}u\,\mathrm{d}v) \\
&\qquad \times \int_{\mathbb{R}^2} (g_A((l-k+1)\Delta\tau(\delta) + (u, v)) - g_A((l-k)\Delta\tau(\delta) + (u, v)))^2 \\
&\qquad\quad \times \mathbf{1}_{A_{|\Delta\tau|\delta}}(u, v)\lambda_2(\mathrm{d}u\mathrm{d}v) \\
&\quad = K c(\delta) \int_{\mathbb{R}^2} (g_A(\Delta\tau(\delta) + (u, v)) - g_A(u, v))^2 \\
&\qquad\qquad \times \mathbf{1}_{A_{|\Delta\tau|\delta}}((u, v) - (l-k)\Delta\tau(\delta))\lambda_2(\mathrm{d}u\,\mathrm{d}v).
\end{aligned}$$

Thus, $\lim_{\delta\to 0} I\!I_\delta = 0$ if $\lim_{\delta\to 0} \widetilde{I\!I}_\delta = 0$, where $\widetilde{I\!I}_\delta$ denotes the expression

$$\sum_{1\le k<l\le[t/\delta]} \frac{\delta^2}{c(\delta)} \int_{\mathbb{R}^2} (g_A(\Delta\tau(\delta)+(u,v))-g_A(u,v))^2\, \mathbf{1}_{A_{|\Delta\tau|\delta}}((u,v)-(l-k)\Delta\tau(\delta))\lambda_2(\mathrm{d}u\,\mathrm{d}v).$$

Given $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that, for $0 < \delta < \delta_\varepsilon$,

$$\begin{aligned}\widetilde{I\!I}_\delta &\le \sum_{1\le k<l\le[t/\delta]} \frac{\delta^2}{c(\delta)} \int_{\mathbb{R}^2} (g_A(\Delta\tau(\delta)+(u,v))-g_A(u,v))^2 \\ &\qquad\qquad \times \mathbf{1}_{A_\varepsilon}((u,v)-(l-k)\Delta\tau(\delta))\lambda_2(\mathrm{d}u\,\mathrm{d}v) \\ &= \sum_{1\le k<l\le[t/\delta]} \delta^2 \int_{\mathbb{R}^2} \mathbf{1}_{A_\varepsilon}((u,v)-(l-k)\delta\Delta\tau)\pi_\delta(\mathrm{d}u\,\mathrm{d}v) \\ &\le \sum_{1\le k<l\le[t/\delta]} \delta^2 \int_{\mathbb{R}^2} (f_{\varepsilon,1}+f_{\varepsilon,2})((u,v)-(l-k)\delta\Delta\tau)\pi_\delta(\mathrm{d}u\,\mathrm{d}v),\end{aligned}$$

where $f_{\varepsilon,1}, f_{\varepsilon,2} \in C_b(\mathbb{R}^2)_+$ are chosen such that $\mathbf{1}_{A_\varepsilon^-} \le f_{\varepsilon,1} + f_{\varepsilon,2}$, and

$$\operatorname{supp}(f_{\varepsilon,1}) \subseteq \left\{(u,v) \in -A \mid d((u,v), -\partial A) > \tfrac{1}{2}\varepsilon\right\}$$

and

$$\operatorname{supp}(f_{\varepsilon,2}) \subseteq \{(u,v) \in \mathbb{R}^2 \mid d((u,v), -\partial A) < 2\varepsilon\}.$$

By weak convergence,

$$\limsup_{\delta\downarrow 0} \widetilde{I\!I}_\delta \le \int_{\mathbb{R}^2} \int_0^t \int_s^t (f_{\varepsilon,1}+f_{\varepsilon,2})((u,v)-(r-s)\Delta\tau)\,\mathrm{d}r\,\mathrm{d}s\pi_0(\mathrm{d}u\,\mathrm{d}v)$$

and so, since π_0 is concentrated on $-\partial A$, we find that

$$\begin{aligned}\limsup_{\delta\downarrow 0} \widetilde{I\!I}_\delta &\le \int_{\mathbb{R}^2} \int_0^t \int_s^t f_{\varepsilon,2}((u,v)-(r-s)\Delta\tau)\,\mathrm{d}r\,\mathrm{d}s\pi_0(\mathrm{d}u\,\mathrm{d}v) \\ &\le \sup_{(u,v)\in-\partial A} \int_0^t \int_s^t f_{\varepsilon,2}((u,v)-(r-s)\Delta\tau)\,\mathrm{d}r\,\mathrm{d}s \\ &\le \sup_{(u,v)\in-\partial A} \lambda_2(\{(r,s) \mid 0\le s\le r\le t, (u,v)-(r-s)\Delta\tau \in \operatorname{supp}(f_{\varepsilon,2})\}).\end{aligned}$$

But, for all $(u,v) \in -\partial A$ and all $0 \le s \le t$,

$$\lambda_1(\{r \mid 0\le s\le r\le t, (u,v)-(r-s)\Delta\tau \in \operatorname{supp}(f_{\varepsilon,2})\}) \le c_\tau\varepsilon$$

for some constant c_τ depending only on τ and A. Thus, $\limsup_{\delta\downarrow 0} \widetilde{I\!I}_\delta \le c_\tau t\varepsilon$, and since ε was arbitrary, this means that $\widetilde{I\!I}_\delta \to 0$ for $\delta \to 0$. That is, (7) holds.

Finally, we turn to the proof of Theorem 1. We will focus first on the case in which $\alpha = 0$. Here the result will be a consequence of the two lemmas stated below.

In the following let τ be a given vector in $\mathbb{R}^2$, and let C be a bounded, closed, convex subset of $\mathbb{R}^2$ containing 0 as an interior point and having piecewise C^∞ boundary. In particular, $\lambda_{21}(\partial C) < \infty$, where λ_{21} is the one-dimensional Hausdorff measure in $\mathbb{R}^2$. The gauge function (5) of C is a convex subadditive function satisfying (i) $T(\lambda x) = \lambda T(x)$, $\lambda \geq 0$, $x \in \mathbb{R}^2$; (ii) there exist $r_1, r_2 \in (0, \infty)$ such that $r_1|x| \leq T(x) \leq r_2|x|$ for all $x \in \mathbb{R}^2$; (iii) $C = \{x \in \mathbb{R}^2 \mid T(x) \leq 1\}$ and $\partial C = \{x \in \mathbb{R}^2 \mid T(x) = 1\}$; (iv) there exist $c_1, c_2 \in (0, \infty)$ such that $c_1 d(x, \partial C)|x| \leq (1 - T(x)) \leq c_2 d(x, \partial C)$ for all $x \in \mathbb{R}^2$. Thus, $T\colon \mathbb{R}^2 \to \mathbb{R}$ is a nonnegative almost everywhere smoothly regular 1-homogeneous continuous function and so, using Equation (8.25) of [9], we have

$$\int_{\mathbb{R}^2} \varphi \, \mathrm{d}\lambda_2 = \int_0^\infty t \left\{ \int_{\partial C} \frac{\varphi(tx)}{|T'(x)|} \lambda_{21}(\mathrm{d}x) \right\} \mathrm{d}t = \int_{\partial C} \frac{1}{|T'(x)|} \left\{ \int_0^\infty t\varphi(tx) \, \mathrm{d}t \right\} \lambda_{21}(\mathrm{d}x) \tag{8}$$

for every nonnegative Borel function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$. The use of Tonelli's theorem is legitimate since $\lambda_{21}(\partial C) < \infty$. The properties of T ensure that $T'(x)$ exists and is nonzero for λ_{21}-almost all $x \in \partial C$; in fact, under the above assumptions, for all but finitely many points of ∂C. In the sequel we shall, for $x \in \partial C$, use the notation

$$n(x) = \begin{cases} \dfrac{T'(x)}{|T'(x)|} & \text{if } T'(x) \text{ exists and is nonzero,} \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Set, for $\delta > 0$, $\nu_\delta = \delta^{-1} f_\delta \, \mathrm{d}\lambda_2$, where $f_\delta(x) = (\mathbf{1}_C(x + \delta\tau) - \mathbf{1}_C(x))^2$, $x \in \mathbb{R}^2$, and observe that the ν_δs are all finite nonzero measures and, for $\delta \leq 1$, are all concentrated on a fixed compact set.

Lemma 2. *We have*

$$\nu_\delta \xrightarrow{\mathrm{w}} |\tau \cdot n| \, \mathbf{1}_{\partial C} \, \mathrm{d}\lambda_{21} \quad \textit{as } \delta \downarrow 0.$$

Proof. By the above observation, it is enough to prove that

$$\lim_{\delta \downarrow 0} \int_{\mathbb{R}^2} h \, \mathrm{d}\nu_\delta = \int_{\partial C} h(x) |\tau \cdot n(x)| \lambda_{21}(\mathrm{d}x)$$

for all Lipschitz continuous $h \in C_c(\mathbb{R}^2)_+$. Given such an h, we have, according to (8),

$$\int_{\mathbb{R}^2} h \, \mathrm{d}\nu_\delta = \frac{1}{\delta} \int_{\mathbb{R}^2} h f_\delta \, \mathrm{d}\lambda_2 = \int_{\partial C} \frac{1}{|T'(x)|} \left\{ \frac{1}{\delta} \int_0^\infty t h(tx) f_\delta(tx) \, \mathrm{d}t \right\} \lambda_{21}(\mathrm{d}x).$$

Using here the Lipschitz continuity of h and the fact that f_δ vanishes identically on $\{1 - \varepsilon < T < 1 + \varepsilon\}$ for sufficiently small δ, we see that it suffices to prove that

$$\lim_{\delta \downarrow 0} \int_{\partial C} \frac{h(x)}{|T'(x)|} \left\{ \frac{1}{\delta} \int_{1-\varepsilon}^{1+\varepsilon} f_\delta(tx) \, \mathrm{d}t \right\} \lambda_{21}(\mathrm{d}x) = \int_{\partial C} h(x) |\tau \cdot n(x)| \lambda_{21}(\mathrm{d}x) \tag{9}$$

for all $\varepsilon \in (0, 1)$. Fix $x \in \partial C$ with $n(x) \neq 0$, and consider the function $t \mapsto f_\delta(tx)$, that is, the indicator function for the set $\{t \geq 0 \mid tx \in (C - \delta\tau) \Delta C\}$. We may and will assume that $T(\delta\tau) < 1$, as this is true for sufficiently small δ. Since $tx \in C$ if and only if $t \leq 1$, we have $(1, \infty) \cap \{t \geq 0 \mid f_\delta(tx) = 1\} = (1, \infty) \cap \{t \geq 0 \mid T(tx + \delta\tau) \leq 1\}$ and, similarly, $(0, 1) \cap \{t \geq 0 \mid f_\delta(tx) = 1\} = (0, 1) \cap \{t \geq 0 \mid T(tx + \delta\tau) > 1\}$. Furthermore, since

$t \mapsto T(tx + \delta\tau)$ is convex, $\{t \geq 0 \mid T(tx + \delta\tau) \leq 1\}$ is an interval including 0. Thus, $(1, \infty) \cap \{t \geq 0 \mid f_\delta(tx) = 1\} = (1, b_\delta(x)]$ for some $b_\delta(x) \geq 1$ and $(0, 1) \cap \{t \geq 0 \mid f_\delta(tx) = 1\} = (a_\delta(x), 1)$ for some $0 < a_\delta(x) \leq 1$. Suppose that $\tau \cdot T'(x) > 0$. Since

$$T(x + \delta\tau) = T(x) + \delta\tau \cdot T'(x) + o(\delta^2) = 1 + \delta\tau \cdot T'(x) + o(\delta^2),$$

we have $T(x + \delta\tau) > 1$ and so $T(tx + \delta\tau) > 1$ for small δ and t sufficiently close to 1. Thus, $b_\delta(x) = 1$. Furthermore, since

$$\begin{aligned} T(tx + \delta\tau) &= T(tx) + \delta\tau \cdot T'(tx) + o(\delta^2) \\ &= tT(x) + t\delta\tau \cdot T'(x) + o(\delta^2) \\ &= t(1 + \delta\tau \cdot T'(x)) + o(\delta^2), \end{aligned}$$

we have

$$a_\delta(x) = \frac{1}{1 + \delta\tau \cdot T'(x)} + o(\delta^2) = 1 - \delta|\tau \cdot T'(x)| + o(\delta^2).$$

Similarly, if $\tau \cdot T'(x) < 0$, we see that $a_\delta(x) = 1$ and $b_\delta(x) = 1 + \delta|\tau \cdot T'(x)| + o(\delta^2)$, and if $\tau \cdot T'(x) = 0$, we obtain $a_\delta(x) = 1 - o(\delta^2)$ and $b_\delta(x) = 1 + o(\delta^2)$. Inserting this into (9) we obtain, by the above,

$$\lim_{\delta \downarrow 0} \int_{\mathbb{R}^2} h \, \mathrm{d}\nu_\delta = \int_{\partial C} \frac{h(x)}{|T'(x)|} |\tau \cdot T'(x)| \lambda_{21}(\mathrm{d}x) = \int_{\partial C} h(x) |\tau \cdot n(x)| \lambda_{21}(\mathrm{d}x).$$

Now let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a given Lipschitz continuous function. Set, for $\delta > 0$,

$$\varphi_\delta(x) = ((\varphi \mathbf{1}_C)(x + \delta\tau) - (\varphi \mathbf{1}_C)(x))^2, \qquad x \in \mathbb{R}^2.$$

Simple arithmetic shows that

$$\begin{aligned} \varphi_\delta(x) = (\varphi^2 f_\delta)(x) &+ (\varphi(x) - \varphi(x + \delta\tau)) \mathbf{1}_C(x + \delta\tau) \\ &\times ((\varphi(x) + \varphi(x + \delta\tau)) \mathbf{1}_C(x + \delta\tau)) - 2(\varphi \mathbf{1}_C)(x)) \end{aligned}$$

for all $x \in \mathbb{R}^2$. The Lipschitz property of φ and Lemma 2 therefore imply, using simple inequalities, that

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \int_{\mathbb{R}^2} \varphi_\delta \, \mathrm{d}\lambda_2 = \lim_{\delta \downarrow 0} \frac{1}{\delta} \int_{\mathbb{R}^2} \varphi^2 f_\delta \, \mathrm{d}\lambda_2 = \int_{\partial C} \varphi^2(x) |\tau \cdot n(x)| \lambda_{21}(\mathrm{d}x).$$

From this we may deduce the following result which establishes Theorem 1 in the case in which $\alpha = 0$.

Lemma 3. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be Lipschitz continuous such that*

$$c = \int_{\partial C} \varphi^2(x) |\tau \cdot n(x)| \lambda_{21}(\mathrm{d}x) > 0.$$

Then, maintaining the above notation,

$$\mu_\delta \xrightarrow{\mathrm{w}} c^{-1} \varphi^2 |\tau \cdot n| \mathbf{1}_{\partial C} \, \mathrm{d}\lambda_{21} \quad \text{as } \delta \downarrow 0,$$

where, for each $\delta > 0$, μ_δ is the absolutely continuous Borel probability measure on $\mathbb{R}^2$ with density proportional to φ_δ.

Next, consider an $\alpha \in (0, \frac{1}{2})$. Following the above we set, for $\delta > 0$,

$$\nu_\delta = \delta^{-(2\alpha+1)} f_\delta \, \mathrm{d}\lambda_2 \quad \text{for } f_\delta(x) = ((d_\alpha \mathbf{1}_C)(x + \delta\tau) - (d_\alpha \mathbf{1}_C)(x))^2, \ x \in \mathbb{R}^2,$$

where $d_\alpha(x) = (1 - T(x))^\alpha$. Again, the ν_δs are all nonzero measures concentrated on a fixed compact set for $\delta \le 1$. As above, we shall prove that $\lim_{\delta\downarrow} \nu_\delta$ exists in the weak sense, the limit being a finite nonzero measure concentrated on ∂C. That is, we shall prove that any weak limit point is concentrated on ∂C and that, for all Lipschitz continuous $h \in C_c(\mathbb{R}^2)_+$, the limit as $\delta \downarrow 0$ of

$$\delta^{-(2\alpha+1)} \int_{\mathbb{R}^2} h f_\delta \, \mathrm{d}\lambda_2 = \int_{\partial C} \frac{1}{|T'(x)|} \left\{ \frac{1}{\delta^{2\alpha+1}} \int_0^\infty t h(tx) f_\delta(tx) \, \mathrm{d}t \right\} \lambda_{21}(\mathrm{d}x) \tag{10}$$

exists and is positive for some h. The equality in (10) follows from (8).

Since T is Lipschitz continuous, we have, using the mean value theorem,

$$|d_\alpha(x_1) - d_\alpha(x_2)| \le \frac{\alpha}{((1 - T(x_1)) \wedge (1 - T(x_2)))^{1-\alpha}} |x_1 - x_2|, \qquad x_1, x_2 \in C^\circ.$$

Thus, for every $K \subseteq C^\circ$ compact, there exists a constant c_K such that $f_\delta(x) \le c_K \delta^2$, $x \in K$, $\delta > 0$, implying, since $2\alpha + 1 < 2$, that

$$\lim_{\delta\downarrow 0} \nu_\delta(K) = \lim_{\delta\downarrow 0} \delta^{-(2\alpha+1)} \int_K f_\delta(x) \lambda_2(\mathrm{d}x) = 0.$$

Furthermore, using the definition of ν_δ, it is trivial to see that $\lim_{\delta\downarrow 0} \nu_\delta(K) = 0$, $K \subseteq \mathbb{R}^2 \setminus C$ compact, implying all together that any weak limit point of the ν_δs for $\delta \downarrow 0$ is concentrated on ∂C. It remains to prove the existence of the limit in (10). Thus, fix an $h \in C_c(\mathbb{R}^2)_+$ which is also Lipschitz continuous. Owing to what has just been proved, the $\limsup_{\delta\downarrow 0}$ and the $\liminf_{\delta\downarrow 0}$ of the left-hand side of (10) equals, for every $\varepsilon \in (0, 1)$, the corresponding values of

$$\delta^{-(2\alpha+1)} \int_{\{1-\varepsilon \le T \le 1+\varepsilon\}} h f_\delta \, \mathrm{d}\lambda_2 = \int_{\partial C} \frac{1}{|T'(x)|} \left\{ \frac{1}{\delta^{2\alpha+1}} \int_{1-\varepsilon}^{1+\varepsilon} t h(tx) f_\delta(tx) \, \mathrm{d}t \right\} \lambda_{21}(\mathrm{d}x).$$

Hence, using the Lipschitz continuity of h, it suffices to prove that

$$\lim_{\delta\downarrow 0} \int_{\partial C} \frac{h(x)}{|T'(x)|} \left\{ \frac{1}{\delta^{2\alpha+1}} \int_{1-\varepsilon}^{1+\varepsilon} f_\delta(tx) \, \mathrm{d}t \right\} \lambda_{21}(\mathrm{d}x)$$

exists for all $0 < \varepsilon < 1$. That is, we have to investigate

$$\lim_{\delta\downarrow 0} \frac{1}{\delta^{2\alpha+1}} \int_{1-\varepsilon}^{1+\varepsilon} f_\delta(tx) \, \mathrm{d}t, \qquad x \in \partial C. \tag{11}$$

As above, we split the analysis into three parts according to whether, for given $x \in \partial C$, we have $\tau \cdot T'(x) > 0$, $\tau \cdot T'(x) < 0$, or $\tau \cdot T'(x) = 0$. Consider first $x \in \partial C$ with $a_x := \tau \cdot T'(x) > 0$. Using the Taylor expansions of T and the computations used in the proof of Lemma 2, we see, disregarding terms of size $o(\delta^2)$, which is legitimate since $2\alpha + 1 < 2$, that the limit in (11)

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corresponds to the $\lim_{\delta\downarrow 0}$ of

$$
\begin{aligned}
&\frac{1}{\delta^{2\alpha+1}}\left(\int_{1-\varepsilon}^{1-\delta a_x} f_\delta(tx)\,\mathrm{d}t + \int_{1-\delta a_x}^{1} f_\delta(tx)\,\mathrm{d}t\right)\\
&\quad = \frac{1}{\delta^{2\alpha+1}}\left(\int_{1-\varepsilon}^{1-\delta a_x} (|1-t-t\delta a_x|^\alpha - |1-t|^\alpha)^2\,\mathrm{d}t + \int_{1-\delta a_x}^{1} |1-t|^{2\alpha}\,\mathrm{d}t\right)\\
&\quad = \frac{1}{\delta^{2\alpha+1}}\left(\int_{\delta a_x}^{\varepsilon} (|t-(1-t)\delta a_x|^\alpha - |t|^\alpha)^2\,\mathrm{d}t + \int_{0}^{\delta a_x} t^{2\alpha}\,\mathrm{d}t\right)\\
&\quad = a_x^{2\alpha+1}\int_{1}^{\varepsilon/\delta a_x} (|t|^\alpha - |t-1+t\delta a_x|^\alpha)^2\,\mathrm{d}t + \frac{a_x^{2\alpha+1}}{2\alpha+1}.
\end{aligned}
$$

But this clearly increases to

$$
a_x^{2\alpha+1}\left(\int_1^\infty (|t|^\alpha - |t-1|^\alpha)^2\,\mathrm{d}t + (2\alpha+1)^{-1}\right) < \infty.
$$

Similarly, if $\tau\cdot T'(x) < 0$ then, setting $a_x = |\tau\cdot T'(x)|$, the limit in (11) corresponds to the $\lim_{\delta\downarrow 0}$ of

$$
\begin{aligned}
&\frac{1}{\delta^{2\alpha+1}}\left(\int_{1-\varepsilon}^{1} f_\delta(tx)\,\mathrm{d}t + \int_{1}^{1+\delta a_x} f_\delta(tx)\,\mathrm{d}t\right)\\
&\quad = \frac{1}{\delta^{2\alpha+1}}\left(\int_{1-\varepsilon}^{1} (|1-t-t\delta a_x|^\alpha - |1-t|^\alpha)^2\,\mathrm{d}t + \int_{1}^{1+\delta a_x} |1-t-t\delta a_x|^{2\alpha}\,\mathrm{d}t\right)\\
&\quad = \frac{1}{\delta^{2\alpha+1}}\left(\int_{0}^{\varepsilon} (|t-(1-t)\delta a_x|^\alpha - |t|^\alpha)^2\,\mathrm{d}t + \int_{0}^{\delta a_x} |t+(t+1)\delta a_x|^{2\alpha}\,\mathrm{d}t\right)\\
&\quad = a_x^{2\alpha+1}\left(\int_{0}^{\varepsilon/\delta a_x} (|t+t\delta a_x-1|^\alpha - |t|^\alpha)^2\,\mathrm{d}t + \int_0^1 |t+t\delta a_x+1|^{2\alpha}\,\mathrm{d}t\right),
\end{aligned}
$$

which converges to

$$
\begin{aligned}
&a_x^{2\alpha+1}\left(\int_0^\infty (|t|^\alpha - |t-1|^\alpha)^2\,\mathrm{d}t + \int_0^1 |t+1|^{2\alpha}\,\mathrm{d}t\right)\\
&\qquad = a_x^{2\alpha+1}\left(\int_0^\infty (|t|^\alpha - |t-1|^\alpha)^2\,\mathrm{d}t + \frac{2^{2\alpha+1}-1}{2\alpha+1}\right)\\
&\qquad < \infty.
\end{aligned}
$$

Finally, in the remaining case, $\tau\cdot T'(x) = 0$, the computations in Lemma 2 immediately show that the limit in (11) equals 0. Thus, for all $x \in \partial C$ for which $T'(x)$ exists,

$$
\lim_{\delta\downarrow 0}\frac{1}{\delta^{2\alpha+1}}\int_{1-\varepsilon}^{1+\varepsilon} f_\delta(tx)\,\mathrm{d}t = |\tau\cdot T'(x)|^{2\alpha+1}(a_1\,\mathbf{1}_{\{\tau\cdot T'(x)>0\}} + a_2\,\mathbf{1}_{\{\tau\cdot T'(x)<0\}})
$$

for suitable positive constants a_1 and a_2 independent of ε. As the above computations show, the setting is sufficiently regular for Lebesgue's dominated convergence theorem to apply, and, hence, we may deduce that $\lim_{\delta\downarrow 0}$ of the left-hand side of (10) exists and equals

$$
\int_{\partial C} h(x)|\tau\cdot T'(x)|^{2\alpha}|\tau\cdot n(x)|(a_1\,\mathbf{1}_{\{\tau\cdot T'(x)>0\}} + a_2\,\mathbf{1}_{\{\tau\cdot T'(x)<0\}})\lambda_{21}(\mathrm{d}x) \quad \text{for all } h \in C_c(\mathbb{R}^2)_+.
$$

As the limit is positive and finite for $h \equiv 1$, we have, as remarked above, proved that $\lim_{\delta\downarrow} \nu_\delta$ exists in the weak sense, the limit being a finite nonzero measure concentrated on ∂C. In fact, we know the form of the limit. From here on, to obtain an appropriate version of Lemma 3, we may proceed exactly as above, the corresponding constant c being given by

$$\int_{\partial C} \varphi(x)^2 |\tau \cdot T'(x)|^{2\alpha} |\tau \cdot n(x)| (a_1 \mathbf{1}_{\{\tau \cdot T'(x) > 0\}} + a_2 \mathbf{1}_{\{\tau \cdot T'(x) < 0\}}) \lambda_{21}(\mathrm{d}x),$$

which is nonzero if

$$c = \int_{\partial C} \varphi^2(x) |\tau \cdot n(x)| \lambda_{21}(\mathrm{d}x) > 0.$$

The proof of Theorem 1 for $-\frac{1}{2} < \alpha < 0$ proceeds along the same line of ideas and is therefore omitted.

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