MATRICES IN LINEAR MECHANICAL SYSTEMS

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In a recent interesting paper [1] on the stability of linear mechanical systems S. P. Diliberto discusses certain reduction theorems for symmetric and skew-symmetric Hamiltonian matrices with respect to symplectic orthogonal similarity. In what follows it is shown that a unified simple argument will handle both of these problems together and then as an example it is indicated how the argument can be also used to obtain the reduction theorem for symplectic orthogonal matrices.

If A and B are matrices $A \otimes B$ will denote their Kronecker product. Let E be the 2-square matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and I_n be the n-square identity matrix. Define $J = E \otimes I_n$ so that $J^2 = E^2 \otimes I_n = -I_2 \otimes I_n = -I_{2n}$. A real 2n-square matrix A is called <u>Hamiltonian</u> if $(JA)' = JA$; A is called <u>symplectic</u> if $A'JA = J$. If the following matrices are all real n-square then a block multiplication easily establishes

LEMMA 1.

(1) $\begin{pmatrix} R & M \\ L & -R' \end{pmatrix} \begin{pmatrix} U & V \\ -V & U \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & -W \end{pmatrix}$

where

(2) $(R + iM)(U + iV) = X + iY, \quad X = RU - MV, \quad Y = MU + RV;$

$(R' + iL)(U + iV) = W + iZ, \quad W = R'U - LV, \quad Z = LU + R'V.$

Note that A is Hamiltonian if and only if

(3) $A = \begin{pmatrix} R & M \\ L & -R' \end{pmatrix}$,

in which L and M are symmetric; for, if

(4) $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ then

$A'J' = JA$ implies that $A'_{21} = A_{21}$, $A'_{12} = A_{12}$, $A'_{11} = -A_{22}$ and so A has the form (3).

LEMMA 2. A is symplectic orthogonal if and only if

(5) $A = I_2 \otimes U + E \otimes V$

in which $U + iV$ is n-square unitary.

Proof. Since $A' = A^{-1}$ it follows that A must commute with J. Using the notation in (4) it follows immediately by a block multiplication that $A_{11} = A_{22}$, $A_{12} = -A_{21}$ and A has the form (5). Now

$$AA' = (I_2 \otimes U + E \otimes V)(I_2 \otimes U' + E' \otimes V')$$

$$= I_2 \otimes (UU' + VV') + E \otimes (VU' - UV').$$

If $AA' = I_{2n} = I_2 \otimes I_n$ then $UU' + VV' = I_n$ and $VU' - UV' = 0$. These are precisely the necessary and sufficient conditions that $U + iV$ be unitary as can be checked by multiplying out $(U + iV)(U + iV)^* = I_n$.

In the following theorem T is an orthogonal symplectic matrix and A is Hamiltonian. Diliberto's two reduction theorems are contained in

THEOREM 1. If A is:

(a) skew-symmetric then there exists a real diagonal matrix D such that

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(6) $T^{-1}AT = E \otimes D$;

    (b) <u>symmetric then there exists a real diagonal matrix</u>
<u>D such that</u>

(7) $T^{-1}AT = F \otimes D$

<u>where</u> $F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

    Proof. (a): The matrix A has the form (3) and if in
addition $A' = -A$ it follows that $R' = -R$ and $L = -M$ is
symmetric. Thus $A = I_2 \otimes R + E \otimes M$. Let $H = R + iM$
and notice that $H^* = R' - iM' = -R - iM = -H$. Thus H is
skew-hermitian and by the diagonalization theorem for skew-
hermitian matrices there is a unitary $U + iV$, U, V real,
and a real diagonal D such that $(R + iM)(U + iV) = (U + iV)(iD)$
$= -VD + iUD$. By (2) $X = -VD$, $Y = UD$. Also $(R' + iL)(U + iV)$
$= (-R - iM)(U + iV) = -(R + iM)(U + iV) = VD - iUD$ so $W = VD$
and $Z = -UD$. Let $T = I_2 \otimes U + E \otimes V$ and by Lemma 2,
T is orthogonal symplectic. Then from (1)
$$AT = \begin{pmatrix} -VD & UD \\ -UD & -VD \end{pmatrix} = T(E \otimes D),\ \text{completing the proof of (a).}$$

    (b): If A is symmetric then from (3) again it follows
immediately that $R' = R$ and $M = L$ is also symmetric.
Then

        $A = F \otimes R + E \otimes L$.

Let H be the complex symmetric matrix $R + iL$. A classical
result of I. Schur [2] states that a unitary matrix $S = U + iV$
exists such that

        $S'(R + iL)S = D$

in which D is a real diagonal matrix (<u>whose main diagonal</u>
<u>elements are the singular values of</u> $R + iL$). Now
$(S')^{-1} = \bar{S} = U - iV$ so that $(R + iM)(U + iV) = (U - iV)D = UD - iVD$.
From (2) again $X = UD$, $Y = -VD$. Also $(R' + iL)(U + iV)$
$= (R + iM)(U + iV) = UD - iVD$ so $W = UD$, $Z = -VD$.

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Once again $T = I_2 \otimes U + E \otimes V$ is orthogonal symplectic and from (1),

$$AT = \begin{pmatrix} UD & -VD \\ -VD & -UD \end{pmatrix} = T(F \otimes D)$$

completing the proof of (b).

Another result that follows by the same kind of argument is

THEOREM 2. <u>If A is symplectic and orthogonal then there exists a symplectic orthogonal T such that</u>

$$T^{-1}AT = I_2 \otimes D + E \otimes K$$

<u>in which</u> $D = \text{diag}(d_1, \ldots, d_n)$ <u>is a real diagonal matrix and</u>
$K = \text{diag}(\pm \sqrt{1 - d_1^2}, \ldots, \pm \sqrt{1 - d_n^2})$.

Proof. If A is symplectic and orthogonal then by Lemma 2 $A = I_2 \otimes R + E \otimes L$ in which $R + iL$ is unitary. By the diagonalization theorem for unitary matrices obtain a unitary matrix $U + iV$ for which $(RU - LV) + i(LU + RV)$ $= (R + iL)(U + iV) = (U + iV)(D + iK) = (UD - VK) + i(VD + UK)$, in which D and K are diagonal matrices. Let T be the orthogonal symplectic matrix $I_2 \otimes U + E \otimes V$ and compute that

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\begin{aligned}
AT &= (I_2 \otimes R + E \otimes L)(I_2 \otimes U + E \otimes V) \\
&= I_2 \otimes (RU - LV) + E \otimes (LU + RV) \\
&= I_2 \otimes (UD - VK) + E \otimes (VD + UK) \\
&= (I_2 \otimes U + E \otimes V)(I_2 \otimes D + E \otimes K) \\
&= T(I_2 \otimes D + E \otimes K).
\end{aligned}
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The fact that A is orthogonal implies that $D^2 + K^2 = I_n$ and the proof is complete.

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REFERENCES

1. S. P. Diliberto, On stability of linear mechanical systems, Office of Naval Research Technical Report, Prepared under contract NONR 222(88). University of California, Berkeley. May, (1962).

2. I. Schur, Ein Satz ueber quadratische Formen mit komplexen Koeffizienten, Amer. J. Math. 67, (1945), 472.

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