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Amalgamated free product type III factors with at most one Cartan subalgebra

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Abstract

We investigate Cartan subalgebras in nontracial amalgamated free product von Neumann algebras $M_1 *_B M_2$ over an amenable von Neumann subalgebra B. First, we settle the problem of the absence of Cartan subalgebra in arbitrary free product von Neumann algebras. Namely, we show that any nonamenable free product von Neumann algebra $(M_1, \varphi_1) * (M_2, \varphi_2)$ with respect to faithful normal states has no Cartan subalgebra. This generalizes the tracial case that was established by A. Ioana [*Cartan subalgebras of amalgamated free product* II_1 *factors*, arXiv:1207.0054]. Next, we prove that any countable nonsingular ergodic equivalence relation $\mathcal{R}$ defined on a standard measure space and which splits as the free product $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ of recurrent subequivalence relations gives rise to a nonamenable factor $\mathrm{L}(\mathcal{R})$ with a unique Cartan subalgebra, up to unitary conjugacy. Finally, we prove unique Cartan decomposition for a class of group measure space factors $\mathrm{L}^\infty(X) \rtimes \Gamma$ arising from nonsingular free ergodic actions $\Gamma \curvearrowright (X, \mu)$ on standard measure spaces of amalgamated groups $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ over a finite subgroup Σ.

Contents

1. Introduction and main results

A *Cartan subalgebra* A in a von Neumann algebra M is a unital maximal abelian $*$-subalgebra $A \subset M$ such that there exists a faithful normal conditional expectation $E_A : M \to A$ and such that

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the group of normalizing unitaries of A inside M defined by $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$ generates M.

By a classical result of Feldman and Moore [FM77], any Cartan subalgebra A in a von Neumann algebra M with separable predual arises from a countable nonsingular equivalence relation $\mathcal{R}$ on a standard measure space (X, μ) and a 2-cocycle $v \in \mathrm{H}^2(\mathcal{R}, \mathbf{T})$. Namely, we have the following isomorphism of inclusions:

$$(A \subset M) \cong (\mathrm{L}^\infty(X) \subset \mathrm{L}(\mathcal{R}, v)).$$

In particular, for any nonsingular free action $\Gamma \curvearrowright (X, \mu)$ of a countable discrete group Γ on a standard measure space (X, μ), $\mathrm{L}^\infty(X)$ is a Cartan subalgebra in the group measure space von Neumann algebra $\mathrm{L}^\infty(X) \rtimes \Gamma$.

The presence of a Cartan subalgebra A in a von Neumann algebra M with separable predual is therefore an important feature which allows us to divide the classification problem for M up to $*$-isomorphism into two different questions: uniqueness of the Cartan subalgebra A inside M up to conjugacy and classification of the underlying countable nonsingular equivalence relation $\mathcal{R}$ up to orbit equivalence.

In [CFW81], Connes *et al.* showed that any amenable countable nonsingular ergodic equivalence relation is hyperfinite and thus implemented by an ergodic **Z**-action. This implies, together with [Kri76], that any two Cartan subalgebras inside an amenable factor are always conjugate by an automorphism.

The uniqueness of Cartan subalgebras up to conjugacy is no longer true in general for nonamenable factors. In [CJ82], Connes and Jones discovered the first examples of II_1 factors with at least two Cartan subalgebras which are not conjugate by an automorphism. More concrete examples were later found by Ozawa and Popa in [OP10b]. We also refer to the recent work of Speelman and Vaes [SV12] on II_1 factors with uncountably many non (stably) conjugate Cartan subalgebras.

In the past decade, Popa's *deformation/rigidity theory* [Pop06a, Pop06b] has led to a lot of progress in the classification of II_1 factors arising from probability measure preserving (pmp) actions of countable discrete groups on standard probability spaces and from countable pmp equivalence relations. We refer to the recent surveys [Io12b, Pop07, Vae10] for an overview of this topic.

We highlight below three breakthrough results regarding uniqueness of Cartan subalgebras in nonamenable II_1 factors. In his pioneering article [Pop06a], Popa showed that any *rigid* Cartan subalgebra inside group measure space II_1 factors $\mathrm{L}^\infty(X) \rtimes \mathbf{F}_n$ arising from *rigid* pmp free ergodic actions $\mathbf{F}_n \curvearrowright (X, \mu)$ of the free group $\mathbf{F}_n$ $(n \geqslant 2)$ is necessarily unitarily conjugate to $\mathrm{L}^\infty(X)$. In [OP10a], Ozawa and Popa proved that any *compact* pmp free ergodic action of the free group $\mathbf{F}_n$ $(n \geqslant 2)$ gives rise to a II_1 factor $\mathrm{L}^\infty(X) \rtimes \mathbf{F}_n$ with unique Cartan decomposition, up to unitary conjugacy. This was the first result in the literature proving the uniqueness of Cartan subalgebras in nonamenable II_1 factors. Recently, Popa and Vaes [PV11] proved that *any* pmp free ergodic action of the free group $\mathbf{F}_n$ $(n \geqslant 2)$ gives rise to a II_1 factor $\mathrm{L}^\infty(X) \rtimes \mathbf{F}_n$ with unique Cartan decomposition, up to unitary conjugacy. We refer to [CS13, CSU13, Hou10, HV13, Io12a, OP10b, PV12] for further results in this direction.

Very recently, using [PV11], Ioana [Io12a] obtained new results regarding the Cartan decomposition of *tracial* amalgamated free product von Neumann algebras $M_1 *_B M_2$. Let us highlight below two of Ioana's results [Io12a]: any nonamenable tracial free product $M_1 * M_2$ has no Cartan subalgebra and any pmp free ergodic action $\Gamma \curvearrowright (X, \mu)$ of a free product

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group $\Gamma = \Gamma_1 * \Gamma_2$ with $|\Gamma_1| \geqslant 2$ and $|\Gamma_2| \geqslant 3$ gives rise to a II_1 factor with unique Cartan decomposition, up to unitary conjugacy.

In the present paper, we use Popa's deformation/rigidity theory to investigate Cartan subalgebras in *nontracial* amalgamated free product (AFP) von Neumann algebras $M_1 *_B M_2$ over an amenable von Neumann subalgebra B. We generalize some of Ioana's recent results [Io12a] to this setting. The methods of the proofs rely on a combination of results and techniques from [HV13, Io12a, PV11].

Statement of the main results

Using his free probability theory, Voiculescu [Voi96] proved that the free group factors $\mathrm{L}(\mathbf{F}_n)$ $(n \geqslant 2)$ have no Cartan subalgebra. This exhibited the first examples of II_1 factors with no Cartan decomposition. This result was generalized later in [Jun07] to free product II_1 factors $M_1 * M_2$ of diffuse subalgebras which are embeddable into R^ω. Finally, the general case of arbitrary tracial free product von Neumann algebras was recently obtained in [Io12a] using Popa's deformation/rigidity theory.

The first examples of type III factors with no Cartan subalgebra were obtained in [Shl00] as a consequence of [Voi96]. Namely, it was shown that the unique free Araki–Woods factor of type III_λ $(0 < \lambda < 1)$ has no Cartan subalgebra. This result was vastly generalized later in [HR11], where it was proven that in fact any free Araki–Woods factor has no Cartan subalgebra.

Our first result settles the question of the absence of Cartan subalgebra in arbitrary free product von Neumann algebras.

Theorem A. *Let (M_1, φ_1) and (M_2, φ_2) be any von Neumann algebras with separable predual endowed with faithful normal states such that $\dim M_1 \geqslant 2$ and $\dim M_2 \geqslant 3$. Then, the free product von Neumann algebra $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ has no Cartan subalgebra.*

Observe that when $\dim M_1 = \dim M_2 = 2$, the free product $M = M_1 * M_2$ is hyperfinite by [Dyk93, Theorem 1.1] and so has a Cartan subalgebra. Note that the questions of factoriality, type classification, and fullness for arbitrary free product von Neumann algebras were recently settled in [Ued11]. These results are used in the proof of Theorem A.

We next investigate more generally Cartan subalgebras in nontracial AFP von Neumann algebras $M = M_1 *_B M_2$ over an amenable von Neumann subalgebra B. Even though we do not get a complete solution in that setting, our second result shows that, under fairly general assumptions, any Cartan subalgebra $A \subset M$ can be embedded into B inside M, in the sense of Popa's intertwining techniques. We refer to § 2 for more information on these intertwining techniques and the notation $A \preceq_M B$. Recall from [HV13, Definition 5.1] that an inclusion of von Neumann algebras $P \subset M$ has no *trivial corner* if for all nonzero projections $p \in P' \cap M$, we have $Pp \neq pMp$.

Theorem B. *For $i \in \{1, 2\}$, let $B \subset M_i$ be any inclusion of von Neumann algebras with separable predual and with faithful normal conditional expectation $E_i : M_i \to B$. Let $(M, E) = (M_1, E_1) *_B (M_2, E_2)$ be the corresponding amalgamated free product von Neumann algebra. Assume that B is a finite amenable von Neumann algebra.*

Assume moreover that:

- *either both M_1 and M_2 have no amenable direct summand;*
- *or B is of finite type* I*, M_1 has no amenable direct summand, and the inclusion $B \subset M_2$ has no trivial corner.*

If $A \subset M$ is a Cartan subalgebra, then $A \preceq_M B$.

A similar result was obtained for tracial AFP von Neumann algebras in [Io12a, Theorem 1.3].

The first examples of type III factors with unique Cartan decomposition were recently obtained in [HV13]. Namely, it was shown that any nonamenable nonsingular free ergodic action $\Gamma \curvearrowright (X, \mu)$ of a Gromov hyperbolic group on a standard measure space gives rise to a factor $\mathrm{L}^\infty(X) \rtimes \Gamma$ with unique Cartan decomposition, up to unitary conjugacy. This generalized the probability measure preserving case that was established in [PV12].

In order to state our next results, we need to introduce some terminology. Let $\mathcal{R}$ be a countable nonsingular equivalence relation on a standard measure space (X, μ) and denote by $\mathrm{L}(\mathcal{R})$ the von Neumann algebra of the equivalence relation $\mathcal{R}$ [FM77]. Following [Ada94, Definition 2.1], we say that $\mathcal{R}$ is *recurrent* if for all measurable subsets $\mathcal{U} \subset X$ such that $\mu(\mathcal{U}) > 0$, the set $[x]_\mathcal{R} \cap \mathcal{U}$ is infinite for almost every $x \in \mathcal{U}$. This is equivalent to saying that $\mathrm{L}(\mathcal{R})$ has no type I direct summand. We then say that a nonsingular action $\Gamma \curvearrowright (X, \mu)$ of a countable discrete group on a standard measure space is *recurrent* if the corresponding orbit equivalence relation $\mathcal{R}(\Gamma \curvearrowright X)$ is recurrent.

Our next result provides a new class of type III factors with unique Cartan decomposition, up to unitary conjugacy. These factors arise from countable nonsingular ergodic equivalence relations $\mathcal{R}$ which split as a free product $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ of arbitrary recurrent subequivalence relations. We refer to [Gab00, Definition IV.6] for the notion of *free product* of countable nonsingular equivalence relations.

Theorem C. *Let $\mathcal{R}$ be any countable nonsingular ergodic equivalence relation on a standard measure space (X, μ) which splits as a free product $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ such that the subequivalence relation $\mathcal{R}_i$ is recurrent for all $i \in \{1, 2\}$.*

Then the nonamenable factor $\mathrm{L}(\mathcal{R})$ has $\mathrm{L}^\infty(X)$ as its unique Cartan subalgebra, up to unitary conjugacy. In particular, for any nonsingular ergodic equivalence relation $\mathcal{S}$ on a standard measure space (Y, η) such that $\mathrm{L}(\mathcal{R}) \cong \mathrm{L}(\mathcal{S})$, we have $\mathcal{R} \cong \mathcal{S}$.

Observe that Theorem C generalizes [Io12a, Corollary 1.4] where the same result was obtained for countable pmp equivalence relations under additional assumptions. Note that in the case when $\mathcal{R}_1$ is *nowhere amenable*, that is, $\mathrm{L}(\mathcal{R}_1)$ has no amenable direct summand and $\mathcal{R}_2$ is recurrent, Theorem C is a consequence of Theorem B and [HV13, Theorem 2.5]. However, Theorem B does not cover the case when both $\mathcal{R}_1$ and $\mathcal{R}_2$ are amenable. So, in the setting of von Neumann algebras arising from countable nonsingular equivalence relations, Theorem C is a generalization of Theorem B in the sense that we are able to remove the nonamenability assumption on $M_1 = \mathrm{L}(\mathcal{R}_1)$.

Finally, when dealing with certain nonsingular free ergodic actions $\Gamma \curvearrowright (X, \mu)$ of amalgamated groups $\Gamma_1 *_\Sigma \Gamma_2$, we obtain new examples of group measure space type III factors with unique Cartan decomposition, up to unitary conjugacy.

Theorem D. *Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ be any amalgamated free product of countable discrete groups such that Σ is finite and Γ_i is infinite for all $i \in \{1, 2\}$. Let $\Gamma \curvearrowright (X, \mu)$ be any nonsingular free ergodic action on a standard measure space such that for all $i \in \{1, 2\}$, the restricted action $\Gamma_i \curvearrowright (X, \mu)$ is recurrent.*

Then the group measure space factor $\mathrm{L}^\infty(X) \rtimes \Gamma$ has $\mathrm{L}^\infty(X)$ as its unique Cartan subalgebra, up to unitary conjugacy.

Observe that Theorem D generalizes the probability measure preserving case that was established in [Io12a, Theorem 1.1].

In the spirit of [HV13, Corollary B], we obtain the following interesting consequence. Let $\Gamma = \Gamma_1 * \Gamma_2$ be an arbitrary free product group such that Γ_1 is amenable and infinite and $|\Gamma_2| \geqslant 2$. Then we get group measure space factors of the form $\mathrm{L}^\infty(X) \rtimes \Gamma$ with unique Cartan decomposition, having any possible type and with any possible flow of weights in the type III_0 case.

We finally mention that, unlike the probability measure preserving case [Io12a, Theorem 1.1], the assumption of recurrence of the action $\Gamma_i \curvearrowright (X, \mu)$ for all $i \in \{1, 2\}$ is necessary. Indeed, using [SV12], we exhibit in § 8 a class of nonamenable infinite measure preserving free ergodic actions $\Gamma \curvearrowright (X, \mu)$ of free product groups $\Gamma = \Gamma_1 * \Gamma_2$ such that the corresponding type II_∞ group measure space factor $\mathrm{L}^\infty(X) \rtimes \Gamma$ has uncountably many nonconjugate Cartan subalgebras.

Comments on the proofs

As we have already mentioned above, the proofs of our main results rely heavily on results and techniques from [HV13, Io12a, PV11]. Let us describe below the main three ingredients which are needed. We will mainly focus on the proof of Theorem A.

Denote by $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ an arbitrary free product of von Neumann algebras as in Theorem A. For simplicity, we may assume that M is a factor. In the case when both M_1 and M_2 are amenable, M is already known to have no Cartan subalgebra by [HR11, Theorem 5.5]. So we may assume that M_1 is not amenable. Using [Dyk93, Ued11], we may further assume that M_1 has no amenable direct summand and $M_2 \neq \mathbf{C}$. By contradiction, assume that $A \subset M$ is a Cartan subalgebra.

We first use Connes and Takesaki's *noncommutative flow of weights* [Con73, CT77, Tak03] in order to work inside the semifinite von Neumann algebra $\mathrm{c}(M)$ which is the *continuous core* of M. We obtain a canonical decomposition of $\mathrm{c}(M)$ as the semifinite amalgamated free product von Neumann algebra $\mathrm{c}(M) = \mathrm{c}(M_1) *_{\mathrm{L}(\mathbf{R})} \mathrm{c}(M_2)$. Moreover, $\mathrm{c}(A) \subset \mathrm{c}(M)$ is a Cartan subalgebra.

Next, we use Popa's intertwining techniques in the setting of nontracial von Neumann algebras that were developed in [HV13, § 2]. Since A is diffuse, we show that necessarily $\mathrm{c}(A) \npreceq_{\mathrm{c}(M)} \mathrm{L}(\mathbf{R})$ (see Proposition 2.10).

Finally, we extend Ioana's techniques from [Io12a, §§ 3, 4] to *semifinite* AFP von Neumann algebras (see Theorems 3.4 and 4.1). The major difference, though, between our approach and Ioana's approach is that we cannot use the spectral gap techniques from [Io12a, § 5]. The main reason why Ioana's approach cannot work here is that $\mathrm{c}(M)$ is not full in general, even though M is a full factor. Instead, we strengthen [Io12a, Theorem 4.1] in the following way. We show that the presence of the Cartan subalgebra $\mathrm{c}(A) \subset \mathrm{c}(M)$ which satisfies $\mathrm{c}(A) \npreceq_{\mathrm{c}(M)} \mathrm{L}(\mathbf{R})$ forces *both* $\mathrm{c}(M_1)$ and $\mathrm{c}(M_2)$ to have an amenable direct summand. Therefore, both M_1 and M_2 have an amenable direct summand as well. Since we have assumed that M_1 has no amenable direct summand, this is a contradiction.

2. Preliminaries

Since we want the paper to be as self-contained as possible, we recall in this section all the necessary background that will be needed for the proofs of the main results.

2.1 Intertwining techniques

All the von Neumann algebras that we consider in this paper are always assumed to be σ-finite. Let M be a von Neumann algebra. We say that a von Neumann subalgebra $P \subset 1_P M 1_P$ is with *expectation* if there exists a faithful normal conditional expectation $E_P : 1_P M 1_P \to P$.

Whenever $\mathcal{V} \subset M$ is a linear subspace, we denote by $\mathrm{Ball}(\mathcal{V})$ the *unit ball* of $\mathcal{V}$ with respect to the uniform norm $\|\cdot\|_\infty$. We will sometimes say that a von Neumann algebra (M, τ) is *tracial* if M is endowed with a faithful normal tracial state τ.

In [Pop06a, Pop06b], Popa discovered the following powerful method to unitarily conjugate subalgebras of a finite von Neumann algebra. Let M be a finite von Neumann algebra and $A \subset 1_A M 1_A$, $B \subset 1_B M 1_B$ von Neumann subalgebras. By [Pop06b, Corollary 2.3] and [Pop06a, Theorem A.1], the following statements are equivalent:

- there exist projections $p \in A$ and $q \in B$, a nonzero partial isometry $v \in pMq$, and a unital normal $*$-homomorphism $\varphi : pAp \to qBq$ such that $av = v\varphi(a)$ for all $a \in A$;
- there exist $n \geqslant 1$, a possibly nonunital normal $*$-homomorphism $\pi : A \to \mathbf{M}_n(B)$, and a nonzero partial isometry $v \in \mathbf{M}_{1,n}(1_A M 1_B)$ such that $av = v\pi(a)$ for all $a \in A$;
- there is no net of unitaries (w_k) in $\mathcal{U}(A)$ such that $E_B(x^* w_k y) \to 0$ $*$-strongly for all $x, y \in 1_A M 1_B$.

If one of the previous equivalent conditions is satisfied, we say that A *embeds into* B *inside* M and write $A \preceq_M B$.

We will need the following generalization of Popa's Intertwining Theorem, which was proven in [HV13, Theorems 2.3, 2.5]. A further generalization can also be found in [Ued13, Proposition 3.1].

Theorem 2.1. *Let M be any von Neumann algebra. Let $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be von Neumann subalgebras such that B is finite and with expectation $E_B : 1_B M 1_B \to B$. The following are equivalent:*

(1) *there exist $n \geqslant 1$, a possibly nonunital normal $*$-homomorphism $\pi : A \to \mathbf{M}_n(B)$ and a nonzero partial isometry $v \in \mathbf{M}_{1,n}(1_A M 1_B)$ such that $av = v\pi(a)$ for all $a \in A$;*

(2) *there is no net of unitaries (w_k) in $\mathcal{U}(A)$ such that $E_B(x^* w_k y) \to 0$ $*$-strongly for all $x, y \in 1_A M 1_B$.*

Moreover, when M is a factor and $A, B \subset M$ are both Cartan subalgebras, the previous conditions are equivalent to the following:

(3) *there exists a unitary $u \in \mathcal{U}(A)$ such that $uAu^* = B$.*

Definition 2.2. Let M be any von Neumann algebra. Let $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be von Neumann subalgebras such that B is finite and with expectation. We say that A *embeds into* B *inside* M and denote $A \preceq_M B$ if one of the equivalent conditions of Theorem 2.1 is satisfied.

Observe that when 1_A and 1_B are finite projections in M, then $1_A \vee 1_B$ is finite, and $A \preceq_M B$ in the sense of Definition 2.2 if and only if $A \preceq_{(1_A \vee 1_B) M (1_A \vee 1_B)} B$ holds in the usual sense for finite von Neumann algebras.

In case of semifinite von Neumann algebras, we recall the following useful intertwining result (see [HR11, Lemma 2.2]). When $(\mathcal{B}, \mathrm{Tr})$ is a semifinite von Neumann algebra endowed with a semifinite faithful normal trace, we will denote by $\mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$ the set of all nonzero finite trace projections of $\mathcal{B}$. We will denote by $\|\cdot\|_{2,\mathrm{Tr}}$ the L^2-norm associated with the trace Tr.

Lemma 2.3. *Let $(\mathcal{M}, \mathrm{Tr})$ be a semifinite von Neumann algebra endowed with a semifinite faithful normal trace. Let $\mathcal{B} \subset \mathcal{M}$ be a von Neumann subalgebra such that $\mathrm{Tr}\,|\mathcal{B}$ is semifinite. Denote by $E_{\mathcal{B}} : \mathcal{M} \to \mathcal{B}$ the unique trace-preserving faithful normal conditional expectation.*

Let $p \in \mathrm{Proj_f}(\mathcal{M})$ and $\mathcal{A} \subset p\mathcal{M}p$ be any von Neumann subalgebra. The following conditions are equivalent:

(1) *for every $q \in \mathrm{Proj_f}(\mathcal{B})$, we have $\mathcal{A} \not\preceq_{\mathcal{M}} q\mathcal{B}q$;*

(2) *there exists an increasing sequence of projections $q_n \in \mathrm{Proj_f}(\mathcal{B})$ such that $q_n \to 1$ strongly and $\mathcal{A} \not\preceq_{\mathcal{M}} q_n\mathcal{B}q_n$ for all $n \in \mathbf{N}$;*

(3) *there exists a net of unitaries $w_k \in \mathcal{U}(\mathcal{A})$ such that $\lim_k \|E_{\mathcal{B}}(x^* w_k y)\|_{2,\mathrm{Tr}} = 0$ for all $x, y \in p\mathcal{M}$.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Let $\mathcal{F} \subset \mathrm{Ball}(p\mathcal{M})$ be a finite subset and $\varepsilon > 0$. We need to show that there exists $w \in \mathcal{U}(\mathcal{A})$ such that $\|E_{\mathcal{B}}(x^* w y)\|_{2,\mathrm{Tr}} < \varepsilon$ for all $x, y \in \mathcal{F}$. Since the projection p has finite trace, there exists $n \in \mathbf{N}$ large enough such that

$$\|q_n x^* p - x^* p\|_{2,\mathrm{Tr}} + \|p y q_n - p y\|_{2,\mathrm{Tr}} < \frac{\varepsilon}{2}, \quad \forall x, y \in \mathcal{F}.$$

Put $q = q_n$. Since $\mathcal{A} \not\preceq_{\mathcal{M}} q\mathcal{B}q$, there exists a net $w_k \in \mathcal{U}(\mathcal{A})$ such that $\lim_k \|E_{q\mathcal{B}q}(a^* w_k b)\|_{2,\mathrm{Tr}} = 0$ for all $a, b \in p\mathcal{M}q$. Applying this to $a = pxq$ and $b = pyq$, if we take $w = w_k$ for k large enough, we get $\|E_{\mathcal{B}}(qx^*p\, w\, pyq)\|_{2,\mathrm{Tr}} = \|E_{q\mathcal{B}q}(qx^*p\, w\, pyq)\|_{2,\mathrm{Tr}} < \varepsilon/2$. Therefore, $\|E_{\mathcal{B}}(x^* w y)\|_{2,\mathrm{Tr}} < \varepsilon$.

(3) $\Rightarrow$ (1) Let $q \in \mathrm{Proj_f}(\mathcal{B})$ and put $e = p \vee q$. Let $\lambda = \mathrm{Tr}(e) < \infty$ and denote by $\|\cdot\|_2$ the L^2-norm with respect to the normalized trace on $e\mathcal{M}e$. For all $x, y \in p\mathcal{M}q$, we have

$$\lim_k \|E_{q\mathcal{B}q}(x^* w_k y)\|_2 = \lambda^{-1/2} \lim_k \|E_{q\mathcal{B}q}(x^* w_k y)\|_{2,\mathrm{Tr}} = 0.$$

This means exactly that $\mathcal{A} \not\preceq_{e\mathcal{M}e} q\mathcal{B}q$ in the usual sense for tracial von Neumann algebras and so $\mathcal{A} \not\preceq_{\mathcal{M}} q\mathcal{B}q$. □

Let Γ be any countable discrete group and $\mathcal{S}$ any nonempty collection of subgroups of Γ. Following [BO08, Definition 15.1.1], we say that a subset $\mathcal{F} \subset \Gamma$ is *small relative to* $\mathcal{S}$ if there exist $n \geqslant 1$, $\Sigma_1, \dots, \Sigma_n \in \mathcal{S}$, and $g_1, h_1, \dots, g_n, h_n \in \Gamma$ such that $\mathcal{F} \subset \bigcup_{i=1}^n g_i \Sigma_i h_i$.

We will need the following generalization of [Vae13, Proposition 2.6] and [HV13, Lemma 2.7].

Proposition 2.4. *Let $(\mathcal{B}, \mathrm{Tr})$ be a semifinite von Neumann algebra endowed with a semifinite faithful normal trace. Let $\Gamma \curvearrowright (\mathcal{B}, \mathrm{Tr})$ be a trace-preserving action of a countable discrete group Γ on $(\mathcal{B}, \mathrm{Tr})$ and denote by $\mathcal{M} = \mathcal{B} \rtimes \Gamma$ the corresponding semifinite crossed product von Neumann algebra. Let $p \in \mathrm{Proj_f}(\mathcal{M})$ and $\mathcal{A} \subset p\mathcal{M}p$ be any von Neumann subalgebra. Denote $\mathcal{P} = \mathcal{N}_{p\mathcal{M}p}(\mathcal{A})''$.*

For every subset $\mathcal{F} \subset \Gamma$ which is small relative to $\mathcal{S}$, denote by $P_{\mathcal{F}}$ the orthogonal projection from $\mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$ onto the closed linear span of $\{x u_g : x \in \mathcal{B} \cap \mathrm{L}^2(\mathcal{B}, \mathrm{Tr}), g \in \mathcal{F}\}$.

(1) *The set $\mathcal{J} = \{e \in \mathcal{A}' \cap p\mathcal{M}p : \mathcal{A}e \not\preceq_{\mathcal{M}} q(\mathcal{B} \rtimes \Sigma)q, \forall \Sigma \in \mathcal{S}, \forall q \in \mathrm{Proj_f}(\mathcal{B})\}$ is directed and attains its maximum in a projection z which belongs to $\mathcal{Z}(\mathcal{P})$.*

(2) *There exists a net (w_k) in $\mathcal{U}(\mathcal{A}z)$ such that $\lim_k \|P_{\mathcal{F}}(w_k)\|_{2,\mathrm{Tr}} = 0$ for every subset $\mathcal{F} \subset \Gamma$ which is small relative to $\mathcal{S}$.*

(3) *For every $\varepsilon > 0$, there exists a subset $\mathcal{F} \subset \Gamma$ which is small relative to $\mathcal{S}$ such that $\|a - P_{\mathcal{F}}(a)\|_{2,\mathrm{Tr}} < \varepsilon$ for all $a \in \mathcal{A}(p - z)$.*

Proof. (1) In order to show that the set $\mathcal{J}$ is directed and attains its maximum, it suffices to prove that whenever $(e_i)_{i \in I}$ is a family of projections in $\mathcal{A}' \cap p\mathcal{M}p$ and $e = \bigvee_{i \in I} e_i$, if $e \notin \mathcal{J}$, then there exists $i \in I$ such that $e_i \notin \mathcal{J}$. If $e \notin \mathcal{J}$, there exist $\Sigma \in \mathcal{S}$ and $q \in \mathrm{Proj_f}(\mathcal{B})$ such that $\mathcal{A}e \preceq_{\mathcal{M}} q(\mathcal{B} \rtimes \Sigma)q$. Let $n \geqslant 1$, a nonzero partial isometry $v \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes e\mathcal{M}q$ and a normal

$*$-homomorphism $\varphi : \mathcal{A}e \to \mathbf{M}_n(q(\mathcal{B} \rtimes \Sigma)q)$ such that $av = v\varphi(a)$ for all $a \in \mathcal{A}e$. By definition, we have $ev = v$. Choose $i \in I$ such that $e_i v \neq 0$ and denote by $w \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes e_i \mathcal{M} q$ the polar part of $e_i v$. Since $aw = w\varphi(a)$ for all $a \in \mathcal{A}e$, it follows that $\mathcal{A}e_i \preceq_{\mathcal{M}} q(\mathcal{B} \rtimes \Sigma)q$. Hence, $e_i \notin \mathcal{J}$.

Denote by z the maximum of the set $\mathcal{J}$. It is easy to see that $uzu^* \in \mathcal{J}$ whenever $u \in \mathcal{N}_{p\mathcal{M}p}(\mathcal{A})$; hence $uzu^* = z$. Therefore, $z \in \mathcal{Z}(\mathcal{P})$.

(2) We have that $\mathcal{A}z \npreceq_{\mathcal{M}} q(\mathcal{B} \rtimes \Sigma)q$ for all $\Sigma \in \mathcal{S}$ and all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$. Let $\varepsilon > 0$ and $\mathcal{F} \subset \Gamma$ be a subset which is small relative to $\mathcal{S}$. We show that we can find $w \in \mathcal{U}(\mathcal{A}z)$ such that $\|P_{\mathcal{F}}(w)\|_{2,\mathrm{Tr}} < \varepsilon$.

Let $\mathcal{F} \subset \bigcup_{i=1}^n g_i \Sigma_i h_i$ with $\Sigma_1, \ldots, \Sigma_n \in \mathcal{S}$ and $g_1, h_1, \ldots, g_n, h_n \in \Gamma$. Consider the semifinite von Neumann algebra $\mathbf{M}_n(\mathcal{M})$ together with the diagonal subalgebra $\mathcal{Q} = \bigoplus_{i=1}^n \mathcal{B} \rtimes \Sigma_i$. Observe that the canonical trace on $\mathbf{M}_n(\mathcal{M})$ is still semifinite on $\mathcal{Q}$. Moreover, consider the trace-preserving $*$-embedding $\rho : \mathcal{M} \to \mathbf{M}_n(\mathcal{M}) : x \mapsto x \oplus \cdots \oplus x$.

Since $\mathcal{A}z \npreceq_{\mathcal{M}} q(\mathcal{B} \rtimes \Sigma_i)q$ for all $i \in \{1, \ldots, n\}$ and all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$, we get that $\rho(\mathcal{A}z) \npreceq_{\mathbf{M}_n(\mathcal{M})} \rho(q)\mathcal{Q}\rho(q)$ for all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$ by the first criterion in Lemma 2.3. Then, by the second criterion in Lemma 2.3, there exists a net $w_k \in \mathcal{U}(\mathcal{A}z)$ such that

$$\lim_k \|E_{\mathcal{B} \rtimes \Sigma_i}(x w_k y)\|_{2,\mathrm{Tr}} = 0, \quad \forall x, y \in \mathcal{M}, \forall i \in \{1, \ldots, n\}.$$

Recall that $P_{g\Sigma h}(x) = u_g E_{\mathcal{B} \rtimes \Sigma}(u_g^* x u_h^*) u_h$ for all $x \in \mathcal{M} \cap \mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$. Applying what we have just proved to $x = u_{g_i}^*$ and $y = u_{h_i}^*$, we get that $\lim_k \|P_{g_i \Sigma_i h_i}(w_k)\|_{2,\mathrm{Tr}} = 0$ for all $i \in \{1, \ldots, n\}$. Therefore, $\lim_k \|P_{\mathcal{F}}(w_k)\|_{2,\mathrm{Tr}} = 0$.

(3) By construction, for any projection $e \leqslant p - z$, there exist $\Sigma \in \mathcal{S}$ and $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$ such that $\mathcal{A}e \preceq_{\mathcal{M}} q(\mathcal{B} \rtimes \Sigma)q$. Let $\varepsilon > 0$. Choose $\ell \geqslant 1$ and $e_1, \ldots, e_\ell \in \mathcal{A}' \cap p\mathcal{M}p$ pairwise orthogonal projections such that:

- for every $i \in \{1, \ldots, \ell\}$, $e_i \leqslant p - z$ and $e = e_1 + \cdots + e_\ell$ satisfies $\|(p - z) - e\|_{2,\mathrm{Tr}} \leqslant \varepsilon/3$;
- for every $i \in \{1, \ldots, \ell\}$, there exist $n_i \geqslant 1$, $\Sigma_i \in \mathcal{S}$, a projection $q_i \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$, a nonzero partial isometry $v_i \in \mathbf{M}_{1,n_i}(\mathbf{C}) \otimes e_i \mathcal{M} q_i$, and a normal $*$-homomorphism $\varphi_i : \mathcal{A} \to \mathbf{M}_{n_i}(q_i(\mathcal{B} \rtimes \Sigma_i)q_i)$ such that $v_i v_i^* = e_i$ and $a v_i = v_i \varphi_i(a)$ for all $a \in \mathcal{A}$.

Put $n = n_1 + \cdots + n_\ell$, $q = \bigvee_{i=1}^\ell q_i$ and define $\varphi : \mathcal{A} \to \bigoplus_{i=1}^\ell q_i(\mathcal{B} \rtimes \Sigma_i)q_i \subset \mathbf{M}_n(q\mathcal{M}q)$ by putting together the φ_i diagonally. Similarly, define the partial isometry $v \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes e\mathcal{M}q$ such that $vv^* = e$ and $av = v\varphi(a)$ for all $a \in \mathcal{A}$.

Using the Kaplansky density theorem, choose $v_0 \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes q(\mathcal{B} \rtimes_{\mathrm{alg}} \Gamma)q$ such that $\|v_0\|_\infty \leqslant 1$ and $\|v - v_0\|_{2,\mathrm{Tr}} < \varepsilon/3$. Define $\mathcal{G} \subset \Gamma$ as the finite subset such that v_0 belongs to the linear span of $\{e_{1i} \otimes e x u_g q : x \in \mathcal{B}, g \in \mathcal{G}, 1 \leqslant i \leqslant \ell\}$. Put $\mathcal{F} = \bigcup_{i=1}^\ell \bigcup_{g,h \in \mathcal{G}} g \Sigma_i h^{-1}$.

Let $a \in \mathrm{Ball}(\mathcal{A}(p - z))$ and write $a = a(p - z - e) + ae$. Observe that $\|a(p - z - e)\|_{2,\mathrm{Tr}} \leqslant \|a\|_\infty \|p - z - e\|_{2,\mathrm{Tr}} < \varepsilon/3$. Since $ae = v\varphi(a)v^*$, it follows that ae lies at a distance less than $2\varepsilon/3$ from $v_0\varphi(a)v_0^*$. Observe that by construction $P_{\mathcal{F}}(v_0\varphi(a)v_0^*) = v_0\varphi(a)v_0^*$. Therefore, a lies at a distance less than ε from the range of $P_{\mathcal{F}}$. □

2.2 Amalgamated free product von Neumann algebras

For $i \in \{1, 2\}$, let $B \subset M_i$ be an inclusion of von Neumann algebras with expectation $E_i : M_i \to B$. Recall that the *amalgamated free product* $(M, E) = (M_1, E_1) *_B (M_2, E_2)$ is the von Neumann algebra M generated by M_1 and M_2 where the faithful normal conditional expectation $E : M \to B$ satisfies the freeness condition:

$$E(x_1 \cdots x_n) = 0 \quad \text{whenever } x_j \in M_{i_j} \ominus B \text{ and } i_j \neq i_{j+1}.$$

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Here and in what follows, we denote by $M_i \ominus B$ the kernel of the conditional expectation $E_i : M_i \to B$. We refer to [Ued99, VDN92, Voi85] for more details on the construction of amalgamated free products in the framework of von Neumann algebras.

Assume that Tr is a semifinite faithful normal trace on B such that for all $i \in \{1, 2\}$, the weight $\mathrm{Tr} \circ E_i$ is a trace on M_i. Then the weight $\mathrm{Tr} \circ E$ is a trace on M by [Ued99, Theorem 2.6]. In that case, we will say that the amalgamated free product $(M, E) = (M_1, E_1) *_B (M_2, E_2)$ is *semifinite*. Whenever we consider a semifinite faithful normal trace on a semifinite amalgamated free product $(M, E) = (M_1, E_1) *_B (M_2, E_2)$, we will always assume that $\mathrm{Tr} \circ E = \mathrm{Tr}$ and $\mathrm{Tr}\,|B$ is semifinite.

The following proposition is a semifinite analogue of [IPP08, Theorem 1.1]. The proof of Theorem 2.5 is essentially contained in [CH10, Theorem 2.4].

Theorem 2.5. *Let $(\mathcal{M}, E) = (\mathcal{M}_1, E_1) *_{\mathcal{B}} (\mathcal{M}_2, E_2)$ be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace* Tr. *Let $p \in \mathrm{Proj_f}(\mathcal{M}_1)$ and $\mathcal{Q} \subset p\mathcal{M}_1 p$ be any von Neumann subalgebra. Assume that there exists a net of unitaries $w_k \in \mathcal{U}(\mathcal{Q})$ such that $\lim_k \|E_{\mathcal{B}}(x^* w_k y)\|_{2,\mathrm{Tr}} = 0$ for all $x, y \in p\mathcal{M}_1$.*

Then any $\mathcal{Q}$-$p\mathcal{M}_1 p$-subbimodule $\mathcal{H}$ of $\mathrm{L}^2(p\mathcal{M}p)$ which has finite dimension as a right $p\mathcal{M}_1 p$-bimodule must be contained in $\mathrm{L}^2(p\mathcal{M}_1 p)$. In particular, $\mathcal{N}_{p\mathcal{M}p}(\mathcal{Q})'' \subset p\mathcal{M}_1 p$.

Proof. Using [Tak02, Proposition V.2.36], we denote by $E_{\mathcal{M}_1} : \mathcal{M} \to \mathcal{M}_1$ the unique trace-preserving faithful normal conditional expectation which satisfies

$$E_{\mathcal{M}_1}(x_1 \cdots x_{2m+1}) = 0$$

whenever $m \geqslant 1$, $x_1, x_{2m+1} \in \mathcal{M}_1$, $x_{2j} \in \mathcal{M}_2 \ominus \mathcal{B}$ and $x_{2j+1} \in \mathcal{M}_1 \ominus \mathcal{B}$ for all $1 \leqslant j \leqslant m - 1$. Moreover, observe that we have $\mathrm{Tr} \circ E_{\mathcal{M}_1} = \mathrm{Tr}$. We denote by $\mathcal{M} \ominus \mathcal{M}_1$ the kernel of the conditional expectation $E_{\mathcal{M}_1} : \mathcal{M} \to \mathcal{M}_1$.

Claim. *We have that $\lim_k \|E_{\mathcal{M}_1}(x^* w_k y)\|_{2,\mathrm{Tr}} = 0$ for all $x, y \in p(\mathcal{M} \ominus \mathcal{M}_1)$.*

Proof of the Claim. Observe that using the Kaplansky density theorem, it suffices to prove the Claim for $x = p x_1 \cdots x_{2m+1}$ and $y = p y_1 \cdots y_{2n+1}$ with $m, n \geqslant 1$, $x_1, x_{2m+1}, y_1, y_{2n+1} \in \mathcal{M}_1$, $x_{2\ell+1}, y_{2\ell'+1} \in \mathcal{M}_1 \ominus \mathcal{B}$ and $x_{2\ell}, y_{2\ell'} \in \mathcal{M}_2 \ominus \mathcal{B}$ for all $1 \leqslant \ell \leqslant m - 1$ and all $1 \leqslant \ell' \leqslant n - 1$. Then, we have

$$E_{\mathcal{M}_1}(x^* w_k y) = E_{\mathcal{M}_1}(x^*_{2m+1} \cdots x^*_2\, E_{\mathcal{B}}(x^*_1 w_k y_1)\, y_2 \cdots y_{2n+1}).$$

Hence, $\lim_k \|E_{\mathcal{M}_1}(x^* w_k y)\|_{2,\mathrm{Tr}} = 0$. □

In particular, we get $\lim_k \|E_{p\mathcal{M}_1 p}(x^* w_k y)\|_{2,\mathrm{Tr}} = 0$ for all $x, y \in p\mathcal{M}p \ominus p\mathcal{M}_1 p$. Finally, applying [Vae07, Lemma D.3], we are done. □

Moreover, we will need the following technical results.

Proposition 2.6. *Let $(\mathcal{M}, E) = (\mathcal{M}_1, E_1) *_{\mathcal{B}} (\mathcal{M}_2, E_2)$ be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace* Tr. *Assume the following:*

- *for all $i \in \{1, 2\}$ and all nonzero projections $z \in \mathcal{Z}(\mathcal{B})$, $\mathcal{B}z \neq z\mathcal{M}_i z$;*
- *for all $p \in \mathrm{Proj_f}(\mathcal{M})$ and all $q \in \mathrm{Proj_f}(\mathcal{B})$, we have $p\mathcal{M}p \npreceq_{\mathcal{M}} q\mathcal{B}q$.*

Then, for all $i \in \{1, 2\}$, all $e \in \mathrm{Proj_f}(\mathcal{M})$, and all $f \in \mathrm{Proj_f}(\mathcal{M}_i)$, we have $e\mathcal{M}e \npreceq_{\mathcal{M}} f\mathcal{M}_i f$.

Proof. By contradiction, assume that there exist $i \in \{1, 2\}$, $e \in \mathrm{Proj_f}(\mathcal{M})$, and $f \in \mathrm{Proj_f}(\mathcal{M}_i)$, a nonzero partial isometry $v \in e\mathcal{M}f$, and a unital normal $*$-homomorphism $\varphi : e\mathcal{M}e \to f\mathcal{M}_i f$ such that $xv = v\varphi(x)$ for all $x \in e\mathcal{M}e$. We may assume without loss of generality that $i = 1$.

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Moreover, as in [Vae08, Remark 3.8], we may assume that the support projection of $E_{\mathcal{M}_1}(v^*v)$ in $\mathcal{M}_1$ equals f.

Let $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$ be arbitrary. By assumption, we have $e\mathcal{M}e \not\preceq_{\mathcal{M}} q\mathcal{B}q$. Next, we claim that $\varphi(e\mathcal{M}e) \not\preceq_{\mathcal{M}_1} q\mathcal{B}q$. Indeed, otherwise there would exist $n \geqslant 1$, a nonzero partial isometry $w \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes f\mathcal{M}_1 q$ and a normal $*$-homomorphism $\psi : \varphi(e\mathcal{M}e) \to \mathbf{M}_n(q\mathcal{B}q)$ such that $\varphi(x)w = w\psi(\varphi(x))$ for all $x \in e\mathcal{M}e$. Hence, we get $xvw = vw(\psi \circ \varphi)(x)$ for all $x \in e\mathcal{M}e$. We have $E_{\mathbf{M}_n(\mathcal{M}_1)}(w^*v^*vw) = w^* E_{\mathcal{M}_1}(v^*v) w \neq 0$, since the support projection of $E_{\mathcal{M}_1}(v^*v)$ is f and $fw = w$. By taking the polar part of vw, this would imply that $e\mathcal{M}e \preceq_{\mathcal{M}} q\mathcal{B}q$, a contradiction.

By Lemma 2.3 and Theorem 2.5, we get $\varphi(e\mathcal{M}e)' \cap f\mathcal{M}f \subset f\mathcal{M}_1 f$; hence $v^*v \in f\mathcal{M}_1 f$. Thus, we may assume that $v^*v = f$. We get $f\mathcal{M}f = v^*\mathcal{M}v \subset f\mathcal{M}_1 f \subset f\mathcal{M}f$, so $f\mathcal{M}_1 f = f\mathcal{M}f$. The proof of [HV13, Theorem 5.7] shows that there exists a nonzero projection $z \in \mathcal{Z}(\mathcal{B})$ such that $z\mathcal{M}_2 z = \mathcal{B}z$, contradicting the assumptions. □

Proposition 2.7. *Let $(\mathcal{M}, E) = (\mathcal{M}_1, E_1) *_{\mathcal{B}} (\mathcal{M}_2, E_2)$ be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace* Tr. *Let $p \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{M})$ and $\mathcal{A} \subset p\mathcal{M}p$ be any von Neumann subalgebra. Assume that there exist $i \in \{1, 2\}$ and $p_i \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{M}_i)$ such that $\mathcal{A} \preceq_{\mathcal{M}} p_i\mathcal{M}_i p_i$.*

Then either there exists $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$ such that $\mathcal{A} \preceq_{\mathcal{M}} q\mathcal{B}q$ or $\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})'' \preceq_{\mathcal{M}} p_i\mathcal{M}_i p_i$.

Proof. We assume that for all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$, we have $\mathcal{A} \not\preceq_{\mathcal{M}} q\mathcal{B}q$ and show that necessarily $\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})'' \preceq_{\mathcal{M}} p_i\mathcal{M}_i p_i$.

Since $\mathcal{A} \preceq_{\mathcal{M}} p_i\mathcal{M}_i p_i$, there exist $n \geqslant 1$, a nonzero partial isometry $v \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes p\mathcal{M}p_i$, and a possibly nonunital normal $*$-homomorphism $\varphi : \mathcal{A} \to \mathbf{M}_n(p_i\mathcal{M}_i p_i)$ such that $av = v\varphi(a)$ for all $a \in \mathcal{A}$. Since we also have $\mathcal{A} \not\preceq_{\mathcal{M}} q\mathcal{B}q$ for all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$, a reasoning entirely analogous to the one of the proof of Proposition 2.6 allows us to further assume that $\varphi(\mathcal{A}) \not\preceq_{\mathbf{M}_n(\mathcal{M}_i)} \mathbf{M}_n(q\mathcal{B}q)$ for all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$.

Let $u \in \mathcal{N}_{p\mathcal{M}p}(\mathcal{A})$. Then, for all $a \in \mathcal{A}$, we have

$$v^*uv\varphi(a) = vuav = v^*(uau^*)uv = \varphi(uau^*)v^*uv.$$

By Theorem 2.5 and Lemma 2.3, we get $v^*uv \in \mathbf{M}_n(p_i\mathcal{M}_i p_i)$ for all $u \in \mathcal{N}_{p\mathcal{M}p}(\mathcal{A})$; hence $v^*\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})''v \subset p_i\mathcal{M}_i p_i$. Therefore, we have $\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})'' \preceq_{\mathcal{M}} p_i\mathcal{M}_i p_i$. □

2.3 Hilbert bimodules

Let M and N be any von Neumann algebras. Recall that an M-N-*bimodule* $\mathcal{H}$ is a Hilbert space endowed with two commuting normal $*$-representations $\pi : M \to \mathbf{B}(\mathcal{H})$ and $\rho : N^{\mathrm{op}} \to \mathbf{B}(\mathcal{H})$. We then define $\pi_{\mathcal{H}} : M \otimes_{\mathrm{alg}} N^{\mathrm{op}} \to \mathbf{B}(\mathcal{H})$ by $\pi_{\mathcal{H}}(x \otimes y^{\mathrm{op}}) = \pi(x)\rho(y^{\mathrm{op}})$ for all $x \in M$ and all $y \in N$. We will simply write $x\xi y = \pi_{\mathcal{H}}(x \otimes y^{\mathrm{op}})\xi$ for all $x \in M$, all $y \in N$ and all $\xi \in \mathcal{H}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be M-N-bimodules. Following [Con94, Appendix V.B], we say that $\mathcal{K}$ is *weakly contained* in $\mathcal{H}$ and write $\mathcal{K} \subset_{\mathrm{weak}} \mathcal{H}$ if $\|\pi_{\mathcal{K}}(T)\|_\infty \leqslant \|\pi_{\mathcal{H}}(T)\|_\infty$ for all $T \in M \otimes_{\mathrm{alg}} N^{\mathrm{op}}$. We simply denote by $(N, \mathrm{L}^2(N), J, \mathfrak{P})$ the standard form of N (see e.g. [Tak03, ch. IX.1]). Then the N-N-bimodule $\mathrm{L}^2(N)$ with left and right action given by $x\xi y = xJy^*J\xi$ is the *trivial* N-N-bimodule, while the N-N-bimodule $\mathrm{L}^2(N) \otimes \mathrm{L}^2(N)$ with left and right action given by $x(\xi \otimes \eta)y = x\xi \otimes Jy^*J\eta$ is the *coarse* N-N-bimodule.

Recall that a von Neumann algebra N is *amenable* if as N-N-bimodules, we have $\mathrm{L}^2(N) \subset_{\mathrm{weak}} \mathrm{L}^2(N) \otimes \mathrm{L}^2(N)$. Equivalently, there exists a norm one projection $\Phi : \mathbf{B}(\mathrm{L}^2(N)) \to N$.

For any von Neumann algebras B, M, N, any M-B-bimodule $\mathcal{H}$, and any B-N-bimodule $\mathcal{K}$, there is a well-defined M-N-bimodule $\mathcal{H} \otimes_B \mathcal{K}$ called the Connes's fusion tensor product of $\mathcal{H}$ and $\mathcal{K}$ over B. For more details regarding this construction, we refer to [Con94, Appendix V.B] and [Ana95, § 1].

We will be using the following well-known fact (see [Ana95, Lemma 1.7]). For any von Neumann algebras B, M, N such that B is amenable, any M-B-bimodule $\mathcal{H}$, and any B-N-bimodule $\mathcal{K}$, we have, as M-N-bimodules,

$$\mathcal{H} \otimes_B \mathcal{K} \subset_{\text{weak}} \mathcal{H} \otimes \mathcal{K}.$$

2.4 Relative amenability

Let M be any von Neumann algebra. Denote by $(M, \mathrm{L}^2(M), J, \mathfrak{P})$ the standard form of M. Let $P \subset 1_P M 1_P$ (respectively, $Q \subset M$) be a von Neumann subalgebra with expectation $E_P : 1_P M 1_P \to P$ (respectively, $E_Q : M \to Q$). The *basic construction* $\langle M, Q\rangle$ is the von Neumann algebra $(JQJ)' \cap \mathbf{B}(H)$. Following [OP10a, § 2.1], we say that P is *amenable relative to* Q *inside* M if there exists a norm one projection $\Phi : 1_P\langle M, Q\rangle 1_P \to P$ such that $\Phi|1_P M 1_P = E_P$.

In the case when (M, τ) is a tracial von Neumann algebra and the conditional expectation $E_P : M \to P$ (respectively, $E_Q : M \to Q$) is τ-preserving, the basic construction that we denote by $\langle M, e_Q\rangle$ coincides with the von Neumann algebra generated by M and the orthogonal projection $e_Q : \mathrm{L}^2(M, \tau) \to \mathrm{L}^2(Q, \tau|Q)$. Observe that $\langle M, e_Q\rangle$ comes with a semifinite faithful normal trace given by $\mathrm{Tr}(xe_Q y) = \tau(xy)$ for all $x, y \in M$. Then [OP10a, Theorem 2.1] shows that P is amenable relative to Q inside M if and only if there exists a net of vectors $\xi_n \in \mathrm{L}^2(\langle M, e_Q\rangle, \mathrm{Tr})$ such that $\lim_n \langle x\xi_n, \xi_n\rangle_{\mathrm{Tr}} = \tau(x)$ for all $x \in 1_P M 1_P$ and $\lim_n \|y\xi_n - \xi_n y\|_{2,\mathrm{Tr}} = 0$ for all $y \in P$.

2.5 Noncommutative flow of weights

Let (M, φ) be a von Neumann algebra together with a faithful normal state. Denote by M^φ the centralizer of φ and by $M \rtimes_\varphi \mathbf{R}$ the *continuous core* of M; that is, the crossed product of M with the modular automorphism group $(\sigma_t^\varphi)_{t\in\mathbf{R}}$ associated with the faithful normal state φ. We have a canonical $*$-embedding $\pi_\varphi : M \to M \rtimes_\varphi \mathbf{R}$ and a canonical group of unitaries $(\lambda_\varphi(s))_{s\in\mathbf{R}}$ in $M \rtimes_\varphi \mathbf{R}$ such that

$$\pi_\varphi(\sigma_s^\varphi(x)) = \lambda_\varphi(s)\, \pi_\varphi(x)\, \lambda_\varphi(s)^* \quad \text{for all } x \in M, s \in \mathbf{R}.$$

The unitaries $(\lambda_\varphi(s))_{s\in\mathbf{R}}$ generate a copy of $\mathrm{L}(\mathbf{R})$ inside $M \rtimes_\varphi \mathbf{R}$.

We denote by $\widehat{\varphi}$ the *dual weight* on $M \rtimes_\varphi \mathbf{R}$ (see [Tak03, Definition X.1.16]), which is a semifinite faithful normal weight on $M \rtimes_\varphi \mathbf{R}$ the modular automorphism group $(\sigma_t^{\widehat{\varphi}})_{t\in\mathbf{R}}$ of which satisfies

$$\sigma_t^{\widehat{\varphi}}(\pi_\varphi(x)) = \pi_\varphi(\sigma_t^\varphi(x)) \quad \text{for all } x \in M \quad \text{and} \quad \sigma_t^{\widehat{\varphi}}(\lambda_\varphi(s)) = \lambda_\varphi(s) \quad \text{for all } s \in \mathbf{R}.$$

We denote by $(\theta_t^\varphi)_{t\in\mathbf{R}}$ the *dual action* on $M \rtimes_\varphi \mathbf{R}$, given by

$$\theta_t^\varphi(\pi_\varphi(x)) = \pi_\varphi(x) \quad \text{for all } x \in M \quad \text{and} \quad \theta_t^\varphi(\lambda_\varphi(s)) = \exp(\mathrm{i}ts)\lambda_\varphi(s) \quad \text{for all } s \in \mathbf{R}.$$

Denote by h_φ the unique nonsingular positive selfadjoint operator affiliated with $\mathrm{L}(\mathbf{R}) \subset M \rtimes_\varphi \mathbf{R}$ such that $h_\varphi^{\mathrm{i}s} = \lambda_\varphi(s)$ for all $s \in \mathbf{R}$. Then $\mathrm{Tr}_\varphi = \widehat{\varphi}(h_\varphi^{-1}\cdot)$ is a semifinite faithful normal trace on $M \rtimes_\varphi \mathbf{R}$ and the dual action θ^φ *scales* the trace Tr_φ:

$$\mathrm{Tr}_\varphi \circ \theta_t^\varphi = \exp(t)\, \mathrm{Tr}_\varphi, \quad \forall t \in \mathbf{R}.$$

Note that Tr_φ is semifinite on $\mathrm{L}(\mathbf{R}) \subset M \rtimes_\varphi \mathbf{R}$. Moreover, the canonical faithful normal conditional expectation $E_{\mathrm{L}(\mathbf{R})} : M \rtimes_\varphi \mathbf{R} \to \mathrm{L}(\mathbf{R})$ defined by $E_{\mathrm{L}(\mathbf{R})}(x\lambda_\varphi(s)) = \varphi(x)\lambda_\varphi(s)$ preserves the trace Tr_φ; that is,

$$\mathrm{Tr}_\varphi \circ E_{\mathrm{L}(\mathbf{R})} = \mathrm{Tr}_\varphi.$$

Because of Connes's Radon–Nikodym cocycle theorem (see [Tak03, Theorem VIII.3.3]), the semifinite von Neumann algebra $M \rtimes_\varphi \mathbf{R}$, together with its trace Tr_φ and trace-scaling action θ^φ, 'does not depend' on the choice of φ in the following precise sense. If ψ is another faithful normal state on M, there is a canonical surjective $*$-isomorphism $\Pi_{\psi,\varphi} : M \rtimes_\varphi \mathbf{R} \to M \rtimes_\psi \mathbf{R}$ such that $\Pi_{\psi,\varphi} \circ \pi_\varphi = \pi_\psi$, $\mathrm{Tr}_\psi \circ \Pi_{\psi,\varphi} = \mathrm{Tr}_\varphi$ and $\Pi_{\psi,\varphi} \circ \theta^\varphi = \theta^\psi \circ \Pi_{\psi,\varphi}$. Note, however, that $\Pi_{\psi,\varphi}$ does not map the subalgebra $\mathrm{L}(\mathbf{R}) \subset M \rtimes_\varphi \mathbf{R}$ onto the subalgebra $\mathrm{L}(\mathbf{R}) \subset M \rtimes_\psi \mathbf{R}$.

Altogether, we can abstractly consider the *continuous core* $(\mathrm{c}(M), \theta, \mathrm{Tr})$, where $\mathrm{c}(M)$ is a von Neumann algebra with a faithful normal semifinite trace Tr, θ is a trace-scaling action of $\mathbf{R}$ on $(\mathrm{c}(M), \mathrm{Tr})$ and $\mathrm{c}(M)$ contains a copy of M. Whenever φ is a faithful normal state on M, we get a canonical surjective $*$-isomorphism $\Pi_\varphi : M \rtimes_\varphi \mathbf{R} \to \mathrm{c}(M)$ such that

$$\Pi_\varphi \circ \theta^\varphi = \theta \circ \Pi_\varphi, \quad \mathrm{Tr}_\varphi = \mathrm{Tr} \circ \Pi_\varphi, \quad \Pi_\varphi(\pi_\varphi(x)) = x \quad \forall x \in M.$$

A more functorial construction of the continuous core, known as the *noncommutative flow of weights*, can be given (see [Con73, CT77, FT01]).

By Takesaki's duality theorem [Tak03, Theorem X.2.3], we have that $\mathrm{c}(M) \rtimes_\theta \mathbf{R} \cong M \overline{\otimes} \mathbf{B}(\mathrm{L}^2(\mathbf{R}))$. In particular, by [Ana95, Proposition 3.4], M is amenable if and only if $\mathrm{c}(M)$ is amenable.

If $P \subset 1_P M 1_P$ is a von Neumann subalgebra with expectation, we have a canonical trace-preserving inclusion $\mathrm{c}(P) \subset 1_P \mathrm{c}(M) 1_P$.

We will also frequently use the following well-known fact: if $A \subset M$ is a Cartan subalgebra, then $\mathrm{c}(A) \subset \mathrm{c}(M)$ is still a Cartan subalgebra. For a proof of this fact, see for example [HR11, Proposition 2.6].

Proposition 2.8. *Let M be any von Neumann algebra with no amenable direct summand. Then the continuous core $\mathrm{c}(M)$ has no amenable direct summand either.*

Proof. Assume that $\mathrm{c}(M)$ has an amenable direct summand. Let $z \in \mathcal{Z}(\mathrm{c}(M))$ be a nonzero projection such that $\mathrm{c}(M)z$ is amenable. Denote by $\theta : \mathbf{R} \curvearrowright \mathrm{c}(M)$ the dual action which scales the trace Tr. Put $e = \bigvee_{t \in \mathbf{R}} \theta_t(z)$. Observe that $e \in \mathcal{Z}(\mathrm{c}(M))$ and $\theta_t(e) = e$ for all $t \in \mathbf{R}$. By [Tak03, Theorem XII.6.10], we have $e \in M \cap \mathcal{Z}(\mathrm{c}(M))$; hence $e \in \mathcal{Z}(M)$. We canonically have $\mathrm{c}(M)e = \mathrm{c}(Me)$.

Since amenability is stable under direct limits, we have that $\mathrm{c}(M)e$ is amenable; hence $\mathrm{c}(Me)$ is amenable. Applying [Tak03, Theorem XII.6.10] again, we have $\mathrm{c}(Me) \rtimes_\theta \mathbf{R} \cong (Me) \overline{\otimes} \mathbf{B}(\mathrm{L}^2(\mathbf{R}))$. We get that $\mathrm{c}(Me) \rtimes_\theta \mathbf{R}$ is amenable and so is Me. Therefore, M has an amenable direct summand. □

We will frequently use the following.

Notation 2.9. Let $A \subset M$ (respectively, $B \subset M$) be a von Neumann subalgebra with expectation $E_A : M \to A$ (respectively, $E_B : M \to B$) of a given von Neumann algebra M. Moreover, assume that A and B are both tracial. Let τ_A be a faithful normal trace on A (respectively, τ_B on B) and write $\varphi_A = \tau_A \circ E_A$ (respectively, $\varphi_B = \tau_B \circ E_B$). Write $\pi_{\varphi_A} : M \to M \rtimes_{\varphi_A} \mathbf{R}$ (respectively, $\pi_{\varphi_B} : M \to M \rtimes_{\varphi_B} \mathbf{R}$) for the canonical $*$-representation of M into its continuous core associated with φ_A (respectively, φ_B).

By Connes's Radon–Nikodym cocycle theorem, there is a surjective $*$-isomorphism

$$\Pi_{\varphi_B,\varphi_A} : M \rtimes_{\varphi_A} \mathbf{R} \to M \rtimes_{\varphi_B} \mathbf{R}$$

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which intertwines the dual actions: $\theta^{\varphi_B} \circ \Pi_{\varphi_B,\varphi_A} = \Pi_{\varphi_B,\varphi_A} \circ \theta^{\varphi_A}$, and preserves the faithful normal semifinite traces, that is, $\mathrm{Tr}_{\varphi_B} \circ \Pi_{\varphi_B,\varphi_A} = \mathrm{Tr}_{\varphi_A}$. In particular, we have $\Pi_{\varphi_B,\varphi_A}(\pi_{\varphi_A}(x)) = \pi_{\varphi_B}(x)$ for all $x \in M$.

Put $\mathrm{c}(M) = M \rtimes_{\varphi_B} \mathbf{R}$, $\mathrm{c}(B) = B \rtimes_{\varphi_B} \mathbf{R}$, and $\mathrm{c}(A) = \Pi_{\varphi_B,\varphi_A}(A \rtimes_{\varphi_A} \mathbf{R})$. We simply denote by $\mathrm{Tr} = \mathrm{Tr}_{\varphi_B}$ the canonical semifinite faithful normal trace on $\mathrm{c}(M)$. Observe that Tr is still semifinite on $\mathcal{Z}(\mathrm{c}(A))$ and $\mathcal{Z}(\mathrm{c}(B))$.

Proposition 2.10. *Assume that we are in the setup of Notation 2.9. If $A \npreceq_M B$, then for all $p \in \mathrm{Proj_f}(\mathcal{Z}(\mathrm{c}(A)))$ and all $q \in \mathrm{Proj_f}(\mathcal{Z}(\mathrm{c}(B)))$, we have $\mathrm{c}(A)p \npreceq_{\mathrm{c}(M)} \mathrm{c}(B)q$.*

Proof. Let $v_k \in \mathcal{U}(A)$ be a net such that $E_B(x^* v_k y) \to 0$ $*$-strongly for all $x, y \in M$. Recall that $\mathrm{c}(M) = M \rtimes_{\varphi_B} \mathbf{R}$, $\mathrm{c}(B) = B \rtimes_{\varphi_B} \mathbf{R}$, and $\mathrm{c}(A) = \Pi_{\varphi_B,\varphi_A}(A \rtimes_{\varphi_A} \mathbf{R})$. Let $p \in \mathrm{Proj_f}(\mathcal{Z}(\mathrm{c}(A)))$ and $q \in \mathrm{Proj_f}(\mathcal{Z}(\mathrm{c}(B)))$. Observe that since p commutes with every element in $\mathrm{c}(A)$, p commutes with every element in $\Pi_{\varphi_B,\varphi_A}(\pi_{\varphi_A}(A)) = \pi_{\varphi_B}(A) \subset \mathrm{c}(A)$. Then $w_k = \Pi_{\varphi_B,\varphi_A}(\pi_{\varphi_A}(v_k))p = \pi_{\varphi_B}(v_k)p$ is a net of unitaries in $\mathcal{U}(\mathrm{c}(A)p)$.

Write $\mathrm{c}(M)_{\mathrm{alg}} = M \rtimes^{\mathrm{alg}}_{\varphi_B} \mathbf{R}$ for the algebraic crossed product; that is, the linear span of $\{\pi_{\varphi_B}(x)\lambda_{\varphi_B}(t) : x \in M, t \in \mathbf{R}\}$. Observe that $\mathrm{c}(M)_{\mathrm{alg}}$ is a dense unital $*$-subalgebra of $\mathrm{c}(M)$. We have $E_{\mathrm{c}(B)}(x^* \pi_{\varphi_B}(v_k) y) \to 0$ $*$-strongly for all $x, y \in \mathrm{c}(M)_{\mathrm{alg}}$. Since $q \in \mathrm{Proj_f}(\mathrm{c}(B))$, we have

$$\|E_{\mathrm{c}(B)q}(q\, x^* \pi_{\varphi_B}(v_k) y\, q)\|_{2,\mathrm{Tr}} = \|q E_{\mathrm{c}(B)}(x^* \pi_{\varphi_B}(v_k) y) q\|_{2,\mathrm{Tr}} \to 0, \forall x, y \in \mathrm{c}(M)_{\mathrm{alg}}.$$

Now fix $x, y \in \mathrm{Ball}(\mathrm{c}(M))$. By the Kaplansky density theorem, choose a net $(x_i)_{i \in I}$ (respectively, $(y_j)_{j \in J}$) in $\mathrm{Ball}(\mathrm{c}(M)_{\mathrm{alg}})$ such that $x_i \to px$ (respectively, $y_j \to py$) $*$-strongly. Let $\varepsilon > 0$. Since $q \in \mathrm{Proj_f}(\mathrm{c}(B))$, we can choose $(i, j) \in I \times J$ such that

$$\|(py - y_j)q\|_{2,\mathrm{Tr}} + \|q(x^* p - x_j)\|_{2,\mathrm{Tr}} < \varepsilon.$$

Therefore, by the triangle inequality, we obtain

$$\limsup_k \|E_{\mathrm{c}(B)q}(q\, x^*\, p \pi_{\varphi_B}(v_k) p\, y\, q)\|_{2,\mathrm{Tr}} \leqslant \limsup_k \|E_{\mathrm{c}(B)q}(q\, x_i^* \pi_{\varphi_B}(v_k) y_j\, q)\|_{2,\mathrm{Tr}} + \varepsilon \leqslant \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $\lim_k \|E_{\mathrm{c}(B)q}(q x^* p\, w_k\, p y q)\|_{2,\mathrm{Tr}} = 0$. This finally proves that $\mathrm{c}(A)p \npreceq_{\mathrm{c}(M)} \mathrm{c}(B)q$. □

Example 2.11. We emphasize two well-known examples that will be used extensively in this paper.

(1) Let $\Gamma \curvearrowright (X, \mu)$ be any nonsingular action on a standard measure space. Define the *Maharam extension* (see [Mah64]) $\Gamma \curvearrowright (X \times \mathbf{R}, m)$ by

$$g \cdot (x, t) = \left(gx, t + \log\left(\frac{d(\mu \circ g^{-1})}{d\mu}(x)\right)\right),$$

where $dm = d\mu \times \exp(t)\, dt$. It is easy to see that the action $\Gamma \curvearrowright X \times \mathbf{R}$ preserves the infinite measure m and, moreover, we have that

$$\mathrm{c}(\mathrm{L}^\infty(X) \rtimes \Gamma) = \mathrm{L}^\infty(X \times \mathbf{R}) \rtimes \Gamma.$$

(2) Let $(M, E) = (M_1, E_1) *_B (M_2, E_2)$ be any amalgamated free product von Neumann algebra. Fix a faithful normal state φ on B. We still denote by φ the faithful normal state $\varphi \circ E$ on M. We realize the continuous core of M as $\mathrm{c}(M) = M \rtimes_\varphi \mathbf{R}$. Likewise, if we denote by $\varphi_i = \varphi \circ E_i$ the corresponding state on M_i, we realize the continuous core of M_i as $\mathrm{c}(M_i) = M_i \rtimes_{\varphi_i} \mathbf{R}$. We denote by $\mathrm{c}(E) : \mathrm{c}(M) \to \mathrm{c}(B)$ (respectively, $\mathrm{c}(E_i) : \mathrm{c}(M_i) \to \mathrm{c}(B)$) the canonical trace-preserving

faithful normal conditional expectation. Recall from [Ued99, § 2] that $\sigma_t^\varphi(M_i) = M_i$ for all $t \in \mathbf{R}$ and all $i \in \{1, 2\}$; hence

$$(\mathrm{c}(M), \mathrm{c}(E)) = (\mathrm{c}(M_1), \mathrm{c}(E_1)) *_{\mathrm{c}(B)} (\mathrm{c}(M_2), \mathrm{c}(E_2)).$$

Moreover, $\mathrm{c}(M)$ is a semifinite amalgamated free product von Neumann algebra.

3. Intertwining subalgebras inside semifinite AFP von Neumann algebras

3.1 Malleable deformation on semifinite AFP von Neumann algebras

First, we recall the construction of the malleable deformation on amalgamated free product von Neumann algebras discovered in [IPP08, § 2].

Let $(\mathcal{M}, E) = (\mathcal{M}_1, E_1) *_\mathcal{B} (\mathcal{M}_2, E_2)$ be any semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace Tr. We will simply write $\mathcal{M} = \mathcal{M}_1 *_\mathcal{B} \mathcal{M}_2$ when no confusion is possible. Put $\widetilde{\mathcal{M}} = \mathcal{M} *_\mathcal{B} (\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{F}_2))$ and observe that $\widetilde{\mathcal{M}}$ is still a semifinite amalgamated free product von Neumann algebra. We still denote by Tr the semifinite faithful normal trace on $\widetilde{\mathcal{M}}$. Let $u_1, u_2 \in \mathcal{U}(\mathrm{L}(\mathbf{F}_2))$ be the canonical Haar unitaries generating $\mathrm{L}(\mathbf{F}_2)$. Observe that we can decompose $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_1 *_\mathcal{B} \widetilde{\mathcal{M}}_2$ with $\widetilde{\mathcal{M}}_i = \mathcal{M}_i *_\mathcal{B} (\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{Z}))$.

Consider the unique Borel function $f : \mathbf{T} \to (-\pi, \pi]$ such that $f(\exp(\mathrm{i}t)) = t$ for all $t \in (-\pi, \pi]$. Define the selfadjoint operators $h_1 = f(u_1)$ and $h_2 = f(u_2)$ so that $\exp(\mathrm{i}u_1) = h_1$ and $\exp(\mathrm{i}u_2) = h_2$. For every $t \in \mathbf{R}$, put $u_1^t = \exp(\mathrm{i}th_1)$ and $u_2^t = \exp(\mathrm{i}th_2)$. We have

$$\tau(u_1^t) = \tau(u_2^t) = \frac{\sin(\pi t)}{\pi t}, \quad \forall t \in \mathbf{R}.$$

Define the one-parameter group of trace-preserving $*$-automorphisms $\alpha_t \in \mathrm{Aut}(\widetilde{\mathcal{M}})$ by

$$\alpha_t = \mathrm{Ad}(u_1^t) *_\mathcal{B} \mathrm{Ad}(u_2^t), \quad \forall t \in \mathbf{R}.$$

Moreover, define the trace-preserving $*$-automorphism $\beta \in \mathrm{Aut}(\widetilde{\mathcal{M}})$ by

$$\beta = \mathrm{id}_\mathcal{M} *_\mathcal{B} (\mathrm{id}_\mathcal{B} \mathbin{\overline{\otimes}} \beta_0),$$

with $\beta_0(u_1) = u_1^*$ and $\beta_0(u_2) = u_2^*$. We have $\alpha_t \beta = \beta \alpha_{-t}$ for all $t \in \mathbf{R}$. Thus, (α_t, β) is a malleable deformation in the sense of Popa [Pop07].

We will be using the following notation throughout this section.

Notation 3.1. Put $\mathcal{H}_0 = \mathrm{L}^2(\mathcal{B}, \mathrm{Tr})$ and $\mathcal{K}_0 = \mathrm{L}^2(\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{F}_2), \mathrm{Tr})$. For $n \geqslant 1$, define $S_n = \{(i_1, \dots, i_n) : i_1 \neq \dots \neq i_n\}$ to be the set of the two alternating sequences of length n made of 1's and 2's. For $\mathcal{I} = (i_1, \dots, i_n) \in S_n$:

- denote by $\mathcal{H}_\mathcal{I}$ the closed linear span in $\mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$ of elements $x_1 \cdots x_n$, with $x_j \in \mathcal{M}_{i_j} \ominus \mathcal{B}$ such that $\mathrm{Tr}(x_j^* x_j) < \infty$ for all $j \in \{1, \dots, n\}$; and
- denote by $\mathcal{K}_\mathcal{I}$ the closed linear span in $\mathrm{L}^2(\widetilde{\mathcal{M}}, \mathrm{Tr})$ of elements $u_{h_1} x_1 \cdots u_{h_n} x_n u_{h_{n+1}}$, with $h_j \in \mathbf{F}_2$ for all $j \in \{1, \dots, n+1\}$ and $x_j \in \mathcal{M}_{i_j} \ominus \mathcal{B}$ such that $\mathrm{Tr}(x_j^* x_j) < \infty$ for all $j \in \{1, \dots, n\}$.

We denote by $E_\mathcal{M} : \widetilde{\mathcal{M}} \to \mathcal{M}$ the unique trace-preserving faithful normal conditional expectation as well as the orthogonal projection $\mathrm{L}^2(\widetilde{\mathcal{M}}, \mathrm{Tr}) \to \mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$. We still denote by $\alpha : \mathbf{R} \to \mathcal{U}(\mathrm{L}^2(\widetilde{\mathcal{M}}, \mathrm{Tr}))$ the Koopman representation associated with the trace-preserving action $\alpha : \mathbf{R} \to \mathrm{Aut}(\widetilde{\mathcal{M}})$.

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Lemma 3.2. *Let* $m, n \geqslant 1$, $\mathcal{I} = (i_1, \dots, i_m) \in S_m$, *and* $\mathcal{J} = (j_1, \dots, j_n) \in S_n$. *Let* $x_1 \in \mathcal{M}_{i_1} \ominus \mathcal{B}, \dots, x_m \in \mathcal{M}_{i_m} \ominus \mathcal{B}$, *and* $y_1 \in \mathcal{M}_{j_1} \ominus \mathcal{B}, \dots, y_n \in \mathcal{M}_{j_n} \ominus \mathcal{B}$ *with* $\mathrm{Tr}(a^*a) < \infty$ *for* $a = x_1, \dots, x_m, y_1, \dots, y_n$. *Let* $g_1, \dots, g_{m+1}, h_1, \dots, h_{n+1} \in \mathbf{F}_2$. *Then*

$$\langle u_{g_1} x_1 \cdots u_{g_m} x_m u_{g_{m+1}}, u_{h_1} y_1 \cdots u_{h_n} y_n u_{h_{n+1}} \rangle_{\mathrm{L}^2(\widetilde{\mathcal{M}}, \mathrm{Tr})} = \begin{cases} \langle x_1 \cdots x_m, y_1 \cdots y_n \rangle_{\mathrm{L}^2(\mathcal{M}, \mathrm{Tr})} & \text{if } m = n,\ \mathcal{I} = \mathcal{J} \text{ and } g_k = h_k, \forall k \in \{1, \dots, m+1\}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof is the same as the proof of [Io12a, Lemma 3.1]. We leave it to the reader. □

Lemma 3.2 allows us, in particular, to put $\mathcal{H}_n = \bigoplus_{\mathcal{I} \in S_n} \mathcal{H}_{\mathcal{I}}$ and $\mathcal{K}_n = \bigoplus_{\mathcal{I} \in S_n} \mathcal{K}_{\mathcal{I}}$ since the $\mathcal{K}_{\mathcal{I}}$'s are pairwise orthogonal. We then have

$$\mathrm{L}^2(\mathcal{M}, \mathrm{Tr}) = \bigoplus_{n \in \mathbf{N}} \mathcal{H}_n \ \text{ and } \ \mathrm{L}^2(\widetilde{\mathcal{M}}, \mathrm{Tr}) = \bigoplus_{n \in \mathbf{N}} \mathcal{K}_n.$$

For all $\xi \in \mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$, write $\xi = \sum_{n \in \mathbf{N}} \xi_n$ with $\xi_n \in \mathcal{H}_n$ for all $n \in \mathbf{N}$. A simple calculation shows that for all $t \in \mathbf{R}$,

$$\mathrm{Tr}(\alpha_t(\xi)\xi^*) = \mathrm{Tr}(E_{\mathcal{M}}(\alpha_t(\xi))\xi^*) = \sum_{n \in \mathbf{N}} \left(\frac{\sin(\pi t)}{\pi t} \right)^{2n} \|\xi_n\|_{2,\mathrm{Tr}}^2.$$

Observe that $t \mapsto \mathrm{Tr}(\alpha_t(\xi)\xi^*)$ is decreasing on $[0, 1]$ for all $\xi \in \mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$.

3.2 A semifinite analogue of the Ioana–Peterson–Popa intertwining theorem [IPP08]

The first result of this section is an analogue of the main technical result of [IPP08] (see [IPP08, Theorem 4.3]) for semifinite amalgamated free product von Neumann algebras. A similar result also appeared in [CH10, Theorem 4.2]. For the sake of completeness, we will give the proof.

Theorem 3.3. *Let* $\mathcal{M} = \mathcal{M}_1 *_{\mathcal{B}} \mathcal{M}_2$ *be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace* Tr. *Let* $p \in \mathrm{Proj_f}(\mathcal{M})$ *and* $\mathcal{A} \subset p\mathcal{M}p$ *be any von Neumann subalgebra. Assume that there exist* $c > 0$ *and* $t \in (0, 1)$ *such that* $\mathrm{Tr}(\alpha_t(w)w^*) \geqslant c$ *for all* $w \in \mathcal{U}(\mathcal{A})$.

Then there exists $q \in \mathrm{Proj_f}(\mathcal{B})$ *such that* $\mathcal{A} \preceq_{\mathcal{M}} q\mathcal{B}q$ *or there exists* $i \in \{1, 2\}$ *and* $q_i \in \mathrm{Proj_f}(\mathcal{M}_i)$ *such that* $\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})'' \preceq_{\mathcal{M}} q_i \mathcal{M}_i q_i$.

Proof. By assumption, there exist $c > 0$ and $t \in (0, 1)$ such that $\mathrm{Tr}(\alpha_t(w)w^*) \geqslant c$ for all $w \in \mathcal{U}(\mathcal{A})$. Choose $r \in \mathbf{N}$ large enough such that $2^{-r} \leqslant t$. Then $\mathrm{Tr}(\alpha_{2^{-r}}(w)w^*) \geqslant c$ for all $w \in \mathcal{U}(\mathcal{A})$. So, we may assume that $t = 2^{-r}$. A standard functional analysis trick yields a nonzero partial isometry $v \in \alpha_t(p)\widetilde{\mathcal{M}}p$ such that $vx = \alpha_t(x)v$ for all $x \in \mathcal{A}$. Observe that $v^*v \in \mathcal{A}' \cap p\widetilde{\mathcal{M}}p$ and $vv^* \in \alpha_t(\mathcal{A}' \cap p\widetilde{\mathcal{M}}p)$.

We prove the result by contradiction. Using Proposition 2.7 and as in the proof of Proposition 2.4, we may choose a net of unitaries $w_k \in \mathcal{U}(\mathcal{A})$ such that $\lim_k \|E_{\mathcal{M}_i}(x^* w_k y)\|_{2,\mathrm{Tr}} = 0$ for all $i \in \{1, 2\}$ and all $x, y \in p\mathcal{M}$. In particular, we get $\lim_k \|E_{\mathcal{B}}(x^* w_k y)\|_{2,\mathrm{Tr}} = 0$ for all $x, y \in p\mathcal{M}$. Regarding $\widetilde{\mathcal{M}} = \mathcal{M} *_{\mathcal{B}} (\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{F}_2))$, we get $v^*v \in \mathcal{A}' \cap p\mathcal{M}p$ by Theorem 2.5. We use now Popa's malleability trick [Pop06b] and put $w = \alpha_t(v\beta(v^*)) \in \alpha_{2t}(p)\widetilde{\mathcal{M}}p$. Since $ww^* = \alpha_t(vv^*) \neq 0$, we get $w \neq 0$ and $wx = \alpha_{2t}(x)w$ for all $x \in \mathcal{A}$. Iterating this construction, we find a nonzero partial isometry $v \in \alpha_1(p)\widetilde{\mathcal{M}}p$ such that

$$vx = \alpha_1(x)v, \quad \forall x \in \mathcal{A}. \tag{1}$$

Moreover, using Proposition 2.5 again, we get $v^*v \in \mathcal{A}' \cap p\mathcal{M}p$ and $vv^* \in \alpha_1(\mathcal{A}' \cap p\mathcal{M}p)$.

Next, exactly as in the proof of [CH10, Claim 4.3], we obtain the following.

Claim. *We have* $\lim_k \|E_{\alpha_1(\mathcal{M})}(x^* w_k y)\|_{2,\mathrm{Tr}} = 0$ *for all* $x, y \in p\widetilde{\mathcal{M}}$.

Proof of the Claim. Consider $\widetilde{\mathcal{M}} = \mathcal{M} *_{\mathcal{B}} (\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{F}_2))$. By the Kaplansky density theorem, it suffices to prove the Claim for $x = pa$ and $y = pb$ with a, b in $\mathcal{B}$ or reduced words in $\widetilde{\mathcal{M}}$ with letters alternating from $\mathcal{M} \ominus \mathcal{B}$ and $\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{F}_2) \ominus \mathcal{B} \mathbin{\overline{\otimes}} \mathbf{C}1$. Write $a = ca'$ with $c = a$ if $a \in \mathcal{B}$; $c = 1$ if a begins with a letter from $\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{F}_2) \ominus \mathcal{B} \mathbin{\overline{\otimes}} \mathbf{C}1$; c equals the first letter of a otherwise. Likewise, write $b = db'$. Then we have $x^* w_k y = a^* w_k b = a'^* \, c^* w_k d \, b'$ and note that $c^* w_k d \in \mathcal{M}$. Observe that a' (respectively, b') equals 1 or is a reduced word beginning with a letter from $\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{F}_2) \ominus \mathcal{B} \mathbin{\overline{\otimes}} \mathbf{C}1$.

Denote by P the orthogonal projection from $\mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$ onto $\mathcal{H}_0 \oplus \mathcal{H}_1$. Observe that since $c^* w_k d \in \mathcal{M} \cap \mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$, we have

$$P(c^* w_k d) = E_{\mathcal{M}_1}(c^* w_k d) + E_{\mathcal{M}_2}(c^* w_k d) - E_{\mathcal{B}}(c^* w_k d).$$

Hence, $\lim_k \|P(c^* w_k d)\|_{2,\mathrm{Tr}} = 0$. Moreover, a simple calculation shows that

$$E_{\alpha_1(\mathcal{M})}(x^* w_k y) = E_{\alpha_1(\mathcal{M})}(a'^* P(c^* w_k d) b').$$

Therefore, $\lim_k \|E_{\alpha_1(\mathcal{M})}(x^* w_k y)\|_{2,\mathrm{Tr}} = 0$. This finishes the proof of the Claim. □

Finally, combining Equation (1) together with the Claim, we get

$$\|vv^*\|_{2,\mathrm{Tr}} = \|\alpha_1(w_k) vv^*\|_{2,\mathrm{Tr}} = \|E_{\alpha_1(\mathcal{M})}(\alpha_1(w_k) vv^*)\|_{2,\mathrm{Tr}} = \|E_{\alpha_1(\mathcal{M})}(v w_k v^*)\|_{2,\mathrm{Tr}} \to 0.$$

This contradicts the fact that $v \neq 0$ and finishes the proof of Theorem 3.3. □

3.3 A semifinite analogue of Ioana's intertwining theorem [Io12a]

Let $\mathcal{M} = \mathcal{M}_1 *_{\mathcal{B}} \mathcal{M}_2$ be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace Tr. Put $\widetilde{\mathcal{M}} = \mathcal{M} *_{\mathcal{B}} (\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{F}_2))$ and observe that $\widetilde{\mathcal{M}}$ is still a semifinite amalgamated free product von Neumann algebra. We still denote by Tr the semifinite faithful normal trace on $\widetilde{\mathcal{M}}$. Let $\mathcal{N} = \bigvee \{u_g \mathcal{M} u_g^* : g \in \mathbf{F}_2\} \subset \widetilde{\mathcal{M}}$. Observe that $\mathcal{N}$ can be identified with an infinite amalgamated free product von Neumann algebra, that $\mathrm{Tr}\,|\mathcal{N}$ is semifinite, and that, under this identification, the action $\mathbf{F}_2 \curvearrowright \mathcal{N}$ is given by the *free Bernoulli shift* which preserves the canonical trace Tr. Moreover, we have $\widetilde{\mathcal{M}} = \mathcal{N} \rtimes \mathbf{F}_2$.

We will denote by $E_{\mathcal{N}} : \widetilde{\mathcal{M}} \to \mathcal{N}$ the unique trace-preserving faithful normal conditional expectation as well as the orthogonal projection $E_{\mathcal{N}} : \mathrm{L}^2(\widetilde{\mathcal{M}}, \mathrm{Tr}) \to \mathrm{L}^2(\mathcal{N}, \mathrm{Tr})$.

Next, we prove the analogue of [Io12a, Theorem 3.2] for semifinite amalgamated free product von Neumann algebras.

Theorem 3.4. *Let* $\mathcal{M} = \mathcal{M}_1 *_{\mathcal{B}} \mathcal{M}_2$ *be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace* Tr. *Let* $p \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$, $\mathcal{A} \subset p\mathcal{M}p$ *be any von Neumann subalgebra and* $t \in (0,1)$. *Assume that there is no net of unitaries* $w_k \in \mathcal{U}(\mathcal{A})$ *such that*

$$\lim_k \|E_{\mathcal{N}}(x^* \alpha_t(w_k) y)\|_{2,\mathrm{Tr}} = 0, \quad \forall x, y \in p\widetilde{\mathcal{M}}.$$

Then there exists $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$ *such that* $\mathcal{A} \preceq_{\mathcal{M}} q\mathcal{B}q$ *or there exists* $i \in \{1,2\}$ *and* $q_i \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{M}_i)$ *such that* $\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})'' \preceq_{\mathcal{M}} q_i \mathcal{M}_i q_i$.

The main technical lemma that will be used to prove Theorem 3.4 is a straightforward generalization of [Io12a, Lemma 3.4]. We include a proof for the sake of completeness.

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LEMMA 3.5. *Let $t \in (0,1)$ and $g, h \in \mathbf{F}_2$. For all $n \geqslant 0$, define*

$$c_n = \sup_{x \in \mathcal{H}_n,\, \|x\|_{2,\mathrm{Tr}} \leqslant 1} \|E_{\mathcal{N}}(u_g \alpha_t(x) u_h)\|_{2,\mathrm{Tr}}.$$

Then $\lim_n c_n = 0$.

Proof. First, observe that for all $g_1 \dots, g_{n+1} \in \mathbf{F}_2$ and all $x_1, \dots, x_n \in \mathcal{M}$, we have

$$E_{\mathcal{N}}(u_{g_1} x_1 \cdots u_{g_n} x_n u_{g_{n+1}}) = \begin{cases} u_{g_1} x_1 \cdots u_{g_n} x_n u_{g_{n+1}} & \text{if } g_1 \cdots g_{n+1} = 1; \\ 0 \quad \text{otherwise.} \end{cases} \tag{2}$$

Thus for all $\mathcal{I} \in S_n$, we have $E_{\mathcal{N}}(\mathcal{K}_{\mathcal{I}}) \subset \mathcal{K}_{\mathcal{I}}$ and since $\alpha_t(\mathcal{H}_{\mathcal{I}}) \subset \mathcal{K}_{\mathcal{I}}$, we get that $E_{\mathcal{N}}(u_g \alpha_t(x) u_h) \in \mathcal{K}_{\mathcal{I}}$ for all $x \in \mathcal{H}_{\mathcal{I}}$. So defining

$$c_{\mathcal{I}} = \sup_{x \in \mathcal{H}_{\mathcal{I}},\, \|x\|_{2,\mathrm{Tr}} \leqslant 1} \|E_{\mathcal{N}}(u_g \alpha_t(x) u_h)\|_{2,\mathrm{Tr}},$$

we see that $c_n = \max_{\mathcal{I} \in S_n} c_{\mathcal{I}}$, since the subspaces $\mathcal{K}_{\mathcal{I}}$ are pairwise orthogonal.

Let us fix $\mathcal{I} = (i_1, \dots, i_n) \in S_n$ and calculate $c_{\mathcal{I}}$. Denote by a and b the canonical generators of $\mathbf{F}_2$ so that $u_1 = u_a$, $u_2 = u_b$ and put $G_1 = \langle a \rangle$ and $G_2 = \langle b \rangle$. For $g_1, h_1 \in G_{i_1}, \dots, g_n, h_n \in G_{i_n}$, define a map

$$V_{g_1,h_1,\dots,g_n,h_n}(x_1 \cdots x_n) = u_{g_1} x_1 u_{h_1}^* \cdots u_{g_n} x_n u_{h_n}^*$$

for all $x_j \in \mathcal{M}_{i_j} \ominus \mathcal{B}$ such that $\mathrm{Tr}(x_j^* x_j) < \infty$ for all $j \in \{1, \dots, n\}$. By Lemma 3.2, these maps $V_{g_1,h_1,\dots,g_n,h_n}$ extend to isometries $V_{g_1,h_1,\dots,g_n,h_n} : \mathcal{H}_{\mathcal{I}} \to \mathcal{K}_{\mathcal{I}}$ with pairwise orthogonal ranges when $(g_1, h_1, \dots, g_n, h_n)$ are pairwise distinct. Indeed, we have $V_{g_1,h_1,\dots,g_n,h_n}(\mathcal{H}_{\mathcal{I}}) \perp V_{g'_1,h'_1,\dots,g'_n,h'_n}(\mathcal{H}_{\mathcal{I}})$ unless $g_1 = g'_1, h_1^{-1} g_2 = h_1'^{-1} g'_2, \dots, h_{n-1}^{-1} g_n = h_{n-1}'^{-1} g'_n, h_n^{-1} = h_n'^{-1}$. Since, moreover, $G_1 \cap G_2 = \{e\}$, this further implies that $g_j = g'_j$ and $h_j = h'_j$ for all $j \in \{1, \dots, n\}$.

Denote the Fourier coefficients of u_1^t and u_2^t by, respectively, $\beta_1(g_1) = \tau(u_1^t u_{g_1}^*)$ for $g_1 \in G_1$ and $\beta_2(g_2) = \tau(u_2^t u_{g_2}^*)$ for $g_2 \in G_2$. We have an explicit formula for these coefficients, given by

$$\beta_i(u_i^n) = \tau(u_i^t u_i^{-n}) = \tau(u_i^{t-n}) = \frac{\sin(\pi(t-n))}{\pi(t-n)}.$$

It follows, in particular, that $\beta_i(g_i) \in \mathbf{R}$ for all $i \in \{1,2\}$ and all $g_i \in G_i$. Since u_1^t and u_2^t are unitaries, we have, moreover,

$$\sum_{g_1 \in G_1} \beta_1(g_1)^2 = \sum_{g_2 \in G_2} \beta_2(g_2)^2 = 1.$$

If $x = x_1 \cdots x_n$ with $x_j \in \mathcal{M}_{i_j} \ominus \mathcal{B}$ satisfying $\mathrm{Tr}(x_j^* x_j) < \infty$, we have

$$\begin{aligned} u_g \alpha_t(x) u_h &= u_g\, u_{i_1}^t x_1 u_{i_1}^{t*} \cdots u_{i_n}^t x_n u_{i_n}^{t*}\, u_h \\ &= \sum_{g_1,h_1 \in G_{i_1},\dots,g_n,h_n \in G_{i_n}} \beta_{i_1}(g_1)\beta_{i_1}(h_1) \cdots \beta_{i_n}(g_n)\beta_{i_n}(h_n)\, u_g\, u_{g_1} x_1 u_{h_1}^* \cdots u_{g_n} x_n u_{h_n}^*\, u_h \\ &= \sum_{g_1,h_1 \in G_{i_1},\dots,g_n,h_n \in G_{i_n}} \beta_{i_1}(g_1)\beta_{i_1}(h_1) \cdots \beta_{i_n}(g_n)\beta_{i_n}(h_n)\, u_g V_{g_1,h_1,\dots,g_n,h_n}(x) u_h, \end{aligned}$$

where the sum converges in $\|\cdot\|_{2,\mathrm{Tr}}$. Thus, for all $x \in \mathcal{H}_{\mathcal{I}}$, we get

$$u_g \alpha_t(x) u_h = \sum_{g_1,h_1 \in G_{i_1},\dots,g_n,h_n \in G_{i_n}} \beta_{i_1}(g_1)\beta_{i_1}(h_1) \cdots \beta_{i_n}(g_n)\beta_{i_n}(h_n)\, u_g V_{g_1,h_1,\dots,g_n,h_n}(x) u_h.$$

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Now, using the calculation (2), and the fact that the isometries $V_{g_1,h_1,\dots,g_n,h_n}$ have mutually orthogonal ranges, we get that for all $x \in \mathcal{H}_{\mathcal{I}}$,

$$\|E_{\mathcal{N}}(u_g\alpha_t(x)u_h)\|_{2,\mathrm{Tr}}^2 = \|x\|_{2,\mathrm{Tr}}^2 \sum_{\substack{g_1,h_1\in G_{i_1},\dots,g_n,h_n\in G_{i_n}\\ gg_1h_1^{-1}\cdots g_nh_n^{-1}h=1}} \beta_{i_1}(g_1)^2\beta_{i_1}(h_1)^2\cdots\beta_{i_n}(g_n)^2\beta_{i_n}(h_n)^2.$$

Thus we get an explicit formula for $c_{\mathcal{I}}$ given by

$$c_{\mathcal{I}} = \sum_{\substack{g_1,h_1\in G_{i_1},\dots,g_n,h_n\in G_{i_n}\\ gg_1h_1\cdots g_nh_nh=1}} \beta_{i_1}(g_1)^2\beta_{i_1}(h_1^{-1})^2\cdots\beta_{i_n}(g_n)^2\beta_{i_n}(h_n^{-1})^2. \tag{3}$$

For $i\in\{1,2\}$, define $\mu_i\in\mathrm{Prob}(\mathbf{F}_2)$ by $\mu_i(g)=\beta_i(g)^2$ if $g\in G_i$ and $\mu_i(g)=0$ otherwise. Likewise, define $\check{\mu}_i\in\mathrm{Prob}(\mathbf{F}_2)$ by $\check{\mu}_i(g)=\mu_i(g^{-1})$ for all $g\in\mathbf{F}_2$. Put $\nu_i=\mu_i*\check{\mu}_i$. Then we have

$$c_{\mathcal{I}} = (\nu_{i_1}*\cdots*\nu_{i_n})(g^{-1}h^{-1}).$$

So if we put $\mu=\nu_1*\nu_2$, we have that

$$c_{\mathcal{I}} \in \left\{\mu^{*[n/2]}(g^{-1}h^{-1}), \mu^{*[n/2]}*\nu_1(g^{-1}h^{-1}), \nu_2*\mu^{*[n/2]}(g^{-1}h^{-1}), \nu_2*\mu^{*[(n-1)/2]}*\nu_1(g^{-1}h^{-1})\right\}.$$

Then [Io12a, Lemma 2.13] implies that $\lim_k \mu^{*k}(s)=0$ for all $s\in\mathbf{F}_2$ and so $\lim_n c_n=0$. □

Proof of Theorem 3.4. Assume by contradiction that the conclusion of the theorem does not hold. Then Theorem 3.3 implies that for $t\in(0,1)$, there exists a net $w_k\in\mathcal{U}(\mathcal{A})$ such that

$$\lim_k \mathrm{Tr}(\alpha_t(w_k)w_k^*)=0.$$

We will show that for all $x,y\in p\widetilde{\mathcal{M}}$, we have $\lim_k\|E_{\mathcal{N}}(x^*\alpha_t(w_k)y)\|_{2,\mathrm{Tr}}=0$, which will contradict the assumption of Theorem 3.4. By a linearity/density argument, it is sufficient to show that for all $g,h\in\mathbf{F}_2$,

$$\lim_k\|E_{\mathcal{N}}(u_g\alpha_t(w_k)u_h)\|_{2,\mathrm{Tr}}=0. \tag{4}$$

For all k, we have $w_k\in\mathcal{A}\subset \mathrm{L}^2(\mathcal{M},\mathrm{Tr})=\bigoplus_{n\in\mathbf{N}}\mathcal{H}_n$, so that we can write $w_k=\sum_{n\in\mathbf{N}}w_{k,n}$, with $w_{k,n}\in\mathcal{H}_n$. Recall that $\mathrm{Tr}(\alpha_t(w_k)w_k^*)=\sum_{n\in\mathbf{N}}(\sin(\pi t)/\pi t)^{2n}\|w_{k,n}\|_{2,\mathrm{Tr}}^2$.

Thus the fact that $\lim_k \mathrm{Tr}(\alpha_t(w_k)w_k^*)=0$ implies that, for all $n\geqslant 0$, $\lim_k\|w_{k,n}\|_{2,\mathrm{Tr}}=0$.

Fix $g,h\in\mathbf{F}_2$ and $\varepsilon>0$. Note that for $n\geqslant 1$, $E_{\mathcal{N}}(u_g\alpha_t(w_{k,n})u_h)\in\mathcal{K}_n$, so that all these terms are pairwise orthogonal. They are also all orthogonal to $E_{\mathcal{N}}(u_g\alpha_t(w_{k,0})u_h)$, which belongs to $\mathcal{K}_0$. Thus

$$\begin{aligned}\|E_{\mathcal{N}}(u_g\alpha_t(w_k)u_h)\|_{2,\mathrm{Tr}}^2 &= \sum_{n\geqslant 0}\|E_{\mathcal{N}}(u_g\alpha_t(w_{k,n})u_h)\|_{2,\mathrm{Tr}}^2\\ &\leqslant \sum_{n\geqslant 0} c_n^2\|w_{k,n}\|_{2,\mathrm{Tr}}^2,\end{aligned}$$

where c_n is defined in Lemma 3.5. Observe that $c_n\leqslant 1$ for all $n\in\mathbf{N}$.

Lemma 3.5 implies that there exists $n_0\geqslant 0$ such that for all $n>n_0$, $c_n^2<\varepsilon/2$. Then we can find k_0 such that for all $k\geqslant k_0$, and all $n\leqslant n_0$, $\|w_{k,n}\|_{2,\mathrm{Tr}}^2<\varepsilon/2(n_0+1)$. So we get that for all $k \geqslant k_0$,

$$\|E_{\mathcal{N}}(u_g \alpha_t(w_k) u_h)\|_{2,\mathrm{Tr}}^2 \leqslant \sum_{n=0}^{n_0} \|w_{k,n}\|_{2,\mathrm{Tr}}^2 + \frac{\varepsilon}{2} \sum_{n \geqslant n_0} \|w_{k,n}\|_{2,\mathrm{Tr}}^2 \leqslant \sum_{n=0}^{n_0} \|w_{k,n}\|_{2,\mathrm{Tr}}^2 + \frac{\varepsilon}{2} \|w_k\|_{2,\mathrm{Tr}}^2 \leqslant \varepsilon.$$

This shows (4) and finishes the proof of Theorem 3.4. □

4. Relative amenability inside semifinite AFP von Neumann algebras

Let $\mathcal{M} = \mathcal{M}_1 *_{\mathcal{B}} \mathcal{M}_2$ be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace Tr. Recall that $\widetilde{\mathcal{M}} = \mathcal{M} *_{\mathcal{B}} (\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{F}_2))$, $\mathcal{N} = \bigvee \{u_g \mathcal{M} u_g^* : g \in \mathbf{F}_2\} \subset \widetilde{\mathcal{M}}$ and observe that $\widetilde{\mathcal{M}} = \mathcal{N} \rtimes \mathbf{F}_2$. We denote by $\alpha : \mathbf{R} \to \mathrm{Aut}(\widetilde{\mathcal{M}})$ the malleable deformation from § 3.1.

The main result of this section is the following strengthening of Ioana's result [Io12a, Theorem 4.1] in the framework of semifinite amalgamated free product von Neumann algebras over an *amenable* subalgebra.

Theorem 4.1. *Let $\mathcal{M} = \mathcal{M}_1 *_{\mathcal{B}} \mathcal{M}_2$ be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace $\mathrm{Tr}_{\mathcal{M}}$. Assume that $\mathcal{B}$ is amenable. Let $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$ such that $q\mathcal{M}_1 q \neq q\mathcal{B}q \neq q\mathcal{M}_2 q$ and $t \in (0,1)$ such that $\alpha_t(q\mathcal{M}q)$ is amenable relative to $q\mathcal{N}q$ inside $q\widetilde{\mathcal{M}}q$.*

Then, for all $i \in \{1,2\}$, there exists a nonzero projection $z_i \in \mathcal{Z}(\mathcal{M}_i)$ such that $\mathcal{M}_i z_i$ is amenable.

Let $\mathrm{Tr}_{\widetilde{\mathcal{M}}}$ be the semifinite faithful normal trace on $\widetilde{\mathcal{M}} = \mathcal{M} *_{\mathcal{B}} (\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{F}_2))$. Consider the basic construction $\langle q\widetilde{\mathcal{M}}q, e_{q\mathcal{N}q}\rangle$ associated with the inclusion of tracial von Neumann algebras $q\mathcal{N}q \subset q\widetilde{\mathcal{M}}q$.

We denote by $\tau = (1/\mathrm{Tr}_{\widetilde{\mathcal{M}}}(q))\, \mathrm{Tr}_{\widetilde{\mathcal{M}}}(q \cdot q)$ the faithful normal tracial state on $q\widetilde{\mathcal{M}}q$ and by $\|\cdot\|_2$ the L^2-norm on $q\widetilde{\mathcal{M}}q$ associated with τ. We then simply denote by Tr the canonical semifinite faithful normal trace on $\langle q\widetilde{\mathcal{M}}q, e_{q\mathcal{N}q}\rangle$ given by $\mathrm{Tr}(a e_{q\mathcal{N}q} b) = \tau(ab)$ for all $a, b \in q\widetilde{\mathcal{M}}q$. Observe that $q\widetilde{\mathcal{M}}q = q\mathcal{N}q \rtimes \mathbf{F}_2$. Following [Io12a, § 4], we define the $q\mathcal{M}_1 q$-$q\mathcal{M}_1 q$-bimodule

$$\mathcal{H}_1 = \bigoplus_{g \in \mathbf{F}_2} \mathrm{L}^2(q\mathcal{M}_1 q)\, u_g e_{q\mathcal{N}q} u_g^* \subset \mathrm{L}^2(\langle q\widetilde{\mathcal{M}}q, e_{q\mathcal{N}q}\rangle).$$

Denote by $\mathcal{H} = \mathrm{L}^2(\langle q\widetilde{\mathcal{M}}q, e_{q\mathcal{N}q}\rangle, \mathrm{Tr}) \ominus \mathcal{H}_1$.

Lemma 4.2. *As $q\mathcal{M}_1 q$-$q\mathcal{M}_1 q$-bimodules, we have that $\mathcal{H} \subset_{\mathrm{weak}} \mathrm{L}^2(q\mathcal{M}_1 q) \otimes \mathrm{L}^2(q\mathcal{M}_1 q)$.*

Proof. The proof goes along the same lines as [Io12a, Lemma 4.2]. First observe that since $q\widetilde{\mathcal{M}}q = q\mathcal{N}q \rtimes \mathbf{F}_2$, we have

$$\mathrm{L}^2(\langle q\mathcal{M}q, e_{q\mathcal{N}q}\rangle) \cong \bigoplus_{g,h \in \mathbf{F}_2} \mathrm{L}^2(q\mathcal{N}q)\, u_g e_{q\mathcal{N}q} u_h.$$

So it suffices to prove that for all $g, h \in \mathbf{F}_2$ such that $h \neq g^{-1}$, as $q\mathcal{M}_1 q$-$q\mathcal{M}_1 q$-bimodules, we have

$$(\mathrm{L}^2(q\mathcal{N}q) \ominus \mathrm{L}^2(q\mathcal{M}_1 q)) u_g e_{q\mathcal{N}q} u_g^* \subset_{\mathrm{weak}} \mathrm{L}^2(q\mathcal{M}_1 q) \otimes \mathrm{L}^2(q\mathcal{M}_1 q),$$
$$\mathrm{L}^2(q\mathcal{N}q)\, u_g e_{q\mathcal{N}q} u_h \subset_{\mathrm{weak}} \mathrm{L}^2(q\mathcal{M}_1 q) \otimes \mathrm{L}^2(q\mathcal{M}_1 q).$$

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Denote by $\mathrm{L}^2(q\mathcal{N}q)^g$ the $q\mathcal{M}_1q$-$q\mathcal{M}_1q$-bimodule $\mathrm{L}^2(q\mathcal{N}q)$ with left and right action given by $x \cdot \xi \cdot y = x\xi u_g y u_g^*$ for all $x, y \in q\mathcal{M}_1q$ and all $\xi \in \mathrm{L}^2(q\mathcal{N}q)$. Likewise, define the $\mathcal{M}_1$-$\mathcal{M}_1$-bimodule $\mathrm{L}^2(\mathcal{N})^g$. As $q\mathcal{M}_1q$-$q\mathcal{M}_1q$-bimodules, we have

$$\bigoplus_{g\in\mathbf{F}_2}(\mathrm{L}^2(q\mathcal{N}q)\ominus \mathrm{L}^2(q\mathcal{M}_1q))u_g e_{q\mathcal{N}q}u_g^* \cong \bigoplus_{i=1}^{\infty}(\mathrm{L}^2(q\mathcal{N}q)\ominus \mathrm{L}^2(q\mathcal{M}_1q)),$$

$$\bigoplus_{g,h\in\mathbf{F}_2, h\neq g^{-1}} \mathrm{L}^2(q\mathcal{N}q)\, u_g e_{q\mathcal{N}q} u_h \cong \bigoplus_{i=1}^{\infty}\bigoplus_{g\in\mathbf{F}_2\setminus\{e\}} \mathrm{L}^2(q\mathcal{N}q)^g.$$

Put $\mathcal{P} = (\bigcup_{h\in\mathbf{F}_2\setminus\{e\}} u_h\mathcal{M}u_h^* \cup \mathcal{M}_2)''$ and $\mathcal{P}_g = (\bigcup_{h\in\mathbf{F}_2\setminus\{e,g\}} u_h\mathcal{M}u_h^* \cup \mathcal{M}_2 \cup u_g\mathcal{M}_2u_g^*)''$ for all $g\in\mathbf{F}_2$. Then we have

$$\mathcal{N} \cong \mathcal{M}_1 *_{\mathcal{B}} \mathcal{P} \cong \mathcal{M}_1 *_{\mathcal{B}} u_g\mathcal{M}_1u_g^* *_{\mathcal{B}} \mathcal{P}_g, \quad \forall g\in\mathbf{F}_2\setminus\{e\}.$$

Using [Ued99, § 2], there are $\mathcal{B}$-$\mathcal{B}$-bimodules $\mathcal{L}$ and $\mathcal{L}_g$ for $g\in\mathbf{F}_2\setminus\{e\}$, such that as $\mathcal{M}_1$-$\mathcal{M}_1$-bimodules, we have

$$\mathrm{L}^2(\mathcal{N})\ominus \mathrm{L}(\mathcal{M}_1) \cong \mathrm{L}^2(\mathcal{M}_1)\otimes_{\mathcal{B}}\mathcal{L}\otimes_{\mathcal{B}}\mathrm{L}^2(\mathcal{M}_1),$$
$$\mathrm{L}^2(\mathcal{N})^g \cong \mathrm{L}^2(\mathcal{M}_1)\otimes_{\mathcal{B}}\mathcal{L}_g\otimes_{\mathcal{B}}\mathrm{L}^2(\mathcal{M}_1).$$

Since $\mathcal{B}$ is amenable, we have that $\mathrm{L}^2(\mathcal{B}) \subset_{\mathrm{weak}} \mathrm{L}^2(\mathcal{B})\otimes\mathrm{L}^2(\mathcal{B})$ as $\mathcal{B}$-$\mathcal{B}$-bimodules. Using [Ana95, Lemma 1.7], we obtain that, as $q\mathcal{M}_1q$-$q\mathcal{M}_1q$-bimodules,

$$\begin{aligned}\mathrm{L}^2(q\mathcal{N}q)\ominus\mathrm{L}^2(q\mathcal{M}_1q) &\cong q(\mathrm{L}^2(\mathcal{M}_1)\otimes_{\mathcal{B}}\mathcal{L}\otimes_{\mathcal{B}}\mathrm{L}^2(\mathcal{M}_1))q\\ &\subset_{\mathrm{weak}} q(\mathrm{L}^2(\mathcal{M}_1)\otimes\mathcal{L}\otimes\mathrm{L}^2(\mathcal{M}_1))q\\ &\subset_{\mathrm{weak}} q(\mathrm{L}^2(\mathcal{M}_1)\otimes\mathrm{L}^2(\mathcal{M}_1))q.\end{aligned}$$

Since $q(\mathrm{L}^2(\mathcal{M}_1)\otimes\mathrm{L}^2(\mathcal{M}_1))q$ is isomorphic to a $q\mathcal{M}_1q$-$q\mathcal{M}_1q$-subbimodule of $\bigoplus_{i=1}^{\infty}\mathrm{L}^2(q\mathcal{M}_1q)\otimes\mathrm{L}^2(q\mathcal{M}_1q)$, we infer that, as $q\mathcal{M}_1q$-$q\mathcal{M}_1q$-bimodules,

$$\mathrm{L}^2(q\mathcal{N}q)\ominus\mathrm{L}^2(q\mathcal{M}_1q)\subset_{\mathrm{weak}}\mathrm{L}^2(q\mathcal{M}_1q)\otimes\mathrm{L}^2(q\mathcal{M}_1q).$$

Similarly, for all $g\in\mathbf{F}_2\setminus\{e\}$ we get that, as $q\mathcal{M}_1q$-$q\mathcal{M}_1q$-bimodules,

$$\mathrm{L}^2(q\mathcal{N}q)^g\subset_{\mathrm{weak}}\mathrm{L}^2(q\mathcal{M}_1q)\otimes\mathrm{L}^2(q\mathcal{M}_1q).$$

Proof of Theorem 4.1. Since $\alpha_t(q\mathcal{M}q)$ is amenable relative to $q\mathcal{N}q$ inside $q\widetilde{\mathcal{M}}q$, we find a net of vectors $\xi_n\in\mathrm{L}^2(\langle q\widetilde{\mathcal{M}}q, e_{q\mathcal{N}q}\rangle,\mathrm{Tr})$ for $n\in I$, such that:

- $\langle x\xi_n \mid \xi_n\rangle_{\mathrm{Tr}}\to\tau(x)$ for all $x\in q\widetilde{\mathcal{M}}q$; and
- $\|x\xi_n-\xi_n x\|_{2,\mathrm{Tr}}\to 0$ for all $x\in\alpha_t(q\mathcal{M}q)$.

Observe that using the proof of [OP10a, Theorem 2.1], we may assume that $\xi_n\geqslant 0$ so that $\langle x\xi_n\mid\xi_n\rangle_{\mathrm{Tr}}=\mathrm{Tr}(x\xi_n^2)=\langle\xi_n x\mid\xi_n\rangle_{\mathrm{Tr}}$ for all $x\in q\widetilde{\mathcal{M}}q$ and all $n\in I$. Since $\|\xi_n\|_{2,\mathrm{Tr}}\to 1$, we may further assume that $\|\xi_n\|_{2,\mathrm{Tr}}=1$ for all $n\in I$.

By contradiction, assume that for some $i\in\{1,2\}$, $q\mathcal{M}_iq$ has no amenable direct summand. Without loss of generality, we may assume that $i=1$. Denote by $P_{\mathcal{H}_1}:\mathrm{L}^2(\langle q\widetilde{\mathcal{M}}q, e_{q\mathcal{N}q}\rangle)\to\mathcal{H}_1$ the orthogonal projection. Observe that $P_{\mathcal{H}_1}$ is the orthogonal projection corresponding to the unique trace-preserving faithful normal conditional expectation $E_{\mathcal{Q}}:q\widetilde{\mathcal{M}}q\to\mathcal{Q}$ onto the von Neumann subalgebra $\mathcal{Q}=\bigvee\{q\mathcal{M}_1q, u_g e_{q\mathcal{N}q}u_g^*: g\in\mathbf{F}_2\}\subset q\widetilde{\mathcal{M}}q$. We claim that $\lim_n\|u_1^{t*}\xi_n u_1^t - P_{\mathcal{H}_1}(u_1^{t*}\xi_n u_1^t)\|_{2,\mathrm{Tr}}=0$. If this is not the case, let $\zeta_n=(1-P_{\mathcal{H}_1})(u_1^{t*}\xi_n u_1^t)\in\mathcal{H}$ and observe that

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$\limsup_n \|\zeta_n\|_{2,\mathrm{Tr}} > 0$. Arguing as in the proof of [Io12a, Lemma 2.3], we may further assume that $\liminf_n \|\zeta_n\|_{2,\mathrm{Tr}} > 0$.

Then $\zeta_n \in \mathcal{H}$ is a net of vectors which satisfies the following conditions:

- $\liminf_n \|\zeta_n\|_{2,\mathrm{Tr}} > 0$;
- $\limsup_n \|x\zeta_n\|_{2,\mathrm{Tr}} \leqslant \|x\|_2$ for all $x \in q\mathcal{M}_1 q$;
- $\lim_n \|y\zeta_n - \zeta_n y\|_{2,\mathrm{Tr}} = 0$ for all $y \in q\mathcal{M}_1 q$.

Since as $q\mathcal{M}_1q$-$q\mathcal{M}_1q$-bimodules, we have that $\mathcal{H} \subset_{\mathrm{weak}} \mathrm{L}^2(q\mathcal{M}_1q) \otimes \mathrm{L}^2(q\mathcal{M}_1q)$ by Lemma 4.2, it follows that $q\mathcal{M}_1q$ has an amenable direct summand by Connes's result [Con76]. This contradicts our assumption and we have shown that $\lim_n \|\xi_n - u_1^t P_{\mathcal{H}_1}(u_1^{t*}\xi_n u_1^t)u_1^{t*}\|_{2,\mathrm{Tr}} = \lim_n \|u_1^{t*}\xi_n u_1^t - P_{\mathcal{H}_1}(u_1^{t*}\xi_n u_1^t)\|_{2,\mathrm{Tr}} = 0$.

Put $\mathcal{L}_1 = u_1^t \mathcal{H}_1 u_1^{t*}$ and denote by $P_{\mathcal{L}_1} : \mathrm{L}^2(\langle q\widetilde{\mathcal{M}}q, e_{q\mathcal{N}q}\rangle) \to \mathcal{L}_1$ the orthogonal projection. Put $\eta_n = P_{\mathcal{L}_1}(\xi_n)$ and observe that $\eta_n \in \mathcal{L}_1$ and $\eta_n \geqslant 0$. Moreover, we have $\lim_n \|\xi_n - \eta_n\|_{2,\mathrm{Tr}} = 0$. So $\eta_n \in \mathcal{L}_1$ is a net of vectors which satisfy:

($*$) $\langle x\eta_n \mid \eta_n\rangle_{\mathrm{Tr}} = \langle \eta_n x \mid \eta_n\rangle_{\mathrm{Tr}} \to \tau(x)$ for all $x \in q\widetilde{\mathcal{M}}q$; and

($**$) $\|x\eta_n - \eta_n x\|_{2,\mathrm{Tr}} \to 0$ for all $x \in \alpha_t(q\mathcal{M}q)$.

We have $\eta_n = \sum_{g\in\mathbf{F}_2} u_1^t x_{n,g} u_g e_{q\mathcal{N}q} u_g^* u_1^{t*}$ with $x_{n,g} \in \mathrm{L}^2(q\mathcal{M}_1q)$. Since $\eta_n = \eta_n^*$ for all $n \in I$, we may assume that $x_{n,g} = x_{n,g}^*$ for all $n \in I$ and all $g \in \mathbf{F}_2$. Next, we claim that we may further assume that $x_{n,g} \in q\mathcal{M}_1q$ with $x_{n,g} = x_{n,g}^*$ for all $n \in I$ and all $g \in \mathbf{F}_2$.

To do so, define the set J of triples $j = (X, Y, \varepsilon)$, where $X \subset \mathrm{Ball}(q\widetilde{\mathcal{M}}q)$, $Y \subset \mathrm{Ball}(\alpha_t(q\mathcal{M}q))$ are finite subsets and $\varepsilon > 0$. We make J a directed set by putting $(X, Y, \varepsilon) \leqslant (X', Y', \varepsilon')$ if and only if $X \subset X'$, $Y \subset Y'$ and $\varepsilon' \leqslant \varepsilon$. Let $j = (X, Y, \varepsilon) \in J$. There exists $n \in I$ such that $|\langle x\eta_n \mid \eta_n\rangle_{\mathrm{Tr}} - \tau(x)| \leqslant \varepsilon/2$ and $\|y\eta_n - \eta_n y\|_{2,\mathrm{Tr}} \leqslant \varepsilon/2$ for all $x \in X$ and all $y \in Y$. Let $\upsilon \in \ell^2(\mathbf{F}_2)_+$ such that $\|\upsilon\|_{\ell^2(\mathbf{F}_2)} = 1$. For each $g \in \mathbf{F}_2$, choose $y_{j,g} \in q\mathcal{M}_1q$ such that $y_{j,g} = y_{j,g}^*$ and $\|x_{n,g} - y_{j,g}\|_2 \leqslant \upsilon(g)\,\varepsilon/4$. Put $\eta'_j = \sum_{g\in\mathbf{F}_2} u_1^t y_{j,g} u_g e_{q\mathcal{N}q} u_g^* u_1^{t*} \in \mathcal{L}_1$ and observe that $\eta'_j = \eta'^*_j$ and $\|\eta_n - \eta'_j\|_{2,\mathrm{Tr}} \leqslant \varepsilon/4$. We get $|\langle x\eta'_j \mid \eta'_j\rangle_{\mathrm{Tr}} - \tau(x)| \leqslant \varepsilon + \varepsilon^2/16$ and $\|y\eta'_j - \eta'_j y\|_{2,\mathrm{Tr}} \leqslant \varepsilon$ for all $x \in X$ and all $y \in Y$. Then the net $(\eta'_j)_{j\in J}$ clearly satisfies Conditions ($*$) and ($**$) above. This finishes the proof of the Claim.

Fix any $y \in q\mathcal{M}_2q \ominus q\mathcal{B}q$ satisfying $\|y\|_2 = 1$. Then we have

$$\langle \alpha_t(y)\eta_n \mid \eta_n\alpha_t(y)\rangle_{\mathrm{Tr}} \to 1.$$

Expanding $\alpha_t(y)$ and η_n, we obtain

$$\begin{aligned}\langle \alpha_t(y)\eta_n \mid \eta_n\alpha_t(y)\rangle_{\mathrm{Tr}} &= \sum_{g,h\in\mathbf{F}_2} \langle u_2^t y u_2^{t*}\, u_1^t x_{n,g} u_g e_{q\mathcal{N}q} u_g^* u_1^{t*} \mid u_1^t x_{n,h} u_h e_{q\mathcal{N}q} u_h^* u_1^{t*}\, u_2^t y u_2^{t*}\rangle_{\mathrm{Tr}} \\ &= \sum_{g,h\in\mathbf{F}_2} \langle u_h^* x_{n,h}^* u_1^{t*} u_2^t y u_2^{t*} u_1^t x_{n,g} u_g\, e_{q\mathcal{N}q} \mid e_{q\mathcal{N}q}\, u_h^* u_1^{t*} u_2^t y u_2^{t*} u_1^t u_g\rangle_{\mathrm{Tr}} \\ &= \sum_{g,h\in\mathbf{F}_2} \tau(E_{q\mathcal{N}q}(u_g^* u_1^{t*} u_2^t y^* u_2^{t*} u_1^t u_h)\, E_{q\mathcal{N}q}(u_h^* x_{n,h}^* u_1^{t*} u_2^t y u_2^{t*} u_1^t x_{n,g} u_g)).\end{aligned}$$

Recall from § 3.1 the definition of the Hilbert spaces $\mathcal{K}_k$ for $k \in \mathbf{N}$ and denote by $b_{n,g} = E_{q\mathcal{B}q}(x_{n,g})$. Since we have

$$E_{q\mathcal{N}q}(u_g^* u_1^{t*} u_2^t y^* u_2^{t*} u_1^t u_h) \in \mathcal{K}_1,$$
$$E_{q\mathcal{N}q}(u_h^*(x_{n,h} - b_{n,g})^* u_1^{t*} u_2^t y u_2^{t*} u_1^t b_{n,g} u_g) \text{ and } E_{q\mathcal{N}q}(u_h^* b_{n,g}^* u_1^{t*} u_2^t y u_2^{t*} u_1^t (x_{n,g} - b_{n,g}) u_g) \in \mathcal{K}_2,$$
$$E_{q\mathcal{N}q}(u_h^*(x_{n,h} - b_{n,g})^* u_1^{t*} u_2^t y u_2^{t*} u_1^t (x_{n,g} - b_{n,g}) u_g) \in \mathcal{K}_3,$$

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we get

$$\begin{aligned}\langle \alpha_t(y)\eta_n \mid \eta_n\alpha_t(y)\rangle_{\mathrm{Tr}} &= \sum_{g,h\in\mathbf{F}_2} \tau(E_{q\mathcal{N}q}(u_g^* u_1^{t*} u_2^t y^* u_2^{t*} u_1^t u_h)\, E_{q\mathcal{N}q}(u_h^* b_{n,h}^* u_1^{t*} u_2^t y u_2^{t*} u_1^t b_{n,g} u_g))\\ &= \sum_{g,h\in\mathbf{F}_2} \tau(E_{q\mathcal{N}q}(u_g^* u_1^{t*} u_2^t y^* u_2^{t*} u_1^t u_h)\, E_{q\mathcal{N}q}(u_h^* u_1^{t*} u_2^t (b_{n,h}^* y b_{n,g}) u_2^{t*} u_1^t u_g)).\end{aligned}$$

As in the proof of Theorem 3.4, for $i\in\{1,2\}$, put $G_1=\langle a\rangle$ and $G_2=\langle b\rangle$ so that $u_1=u_a$ and $u_2=u_b$. Denote by $(\beta_i(g))_{g\in G_i}$ the Fourier coefficients of u_i^t. For $g,h\in\mathbf{F}_2$, define the isometry $W_{g,h}:\mathrm{L}^2(\mathcal{M}_2)\ominus\mathrm{L}^2(\mathcal{B})\to\mathrm{L}^2(\widetilde{\mathcal{M}})$ by $W_{g,h}(x)=u_g x u_h^*$ for $x\in\mathcal{M}_2\ominus\mathcal{B}$ such that $\mathrm{Tr}_{\mathcal{M}}(x^*x)<\infty$. Thanks to Lemma 3.2, the isometries $W_{g,h}$ have pairwise orthogonal ranges when (g,h) are pairwise distinct. For all $z\in q\mathcal{M}_2 q\ominus q\mathcal{B}q$ and all $g,h\in\mathbf{F}_2$, using calculation (2), we obtain

$$\begin{aligned}E_{q\mathcal{N}q}(u_h^* u_1^{t*} u_2^t z u_2^{t*} u_1^t u_g) &= \sum_{r,r'\in G_1, s,s'\in G_2} \beta_1(r)\beta_2(s)\beta_2(s')\beta_1(r')\, E_{q\mathcal{N}q}(W_{h^{-1}r^{-1}s,\, g^{-1}r'^{-1}s'}(z))\\ &= \sum_{\substack{r,r'\in G_1, s,s'\in G_2\\ h^{-1}r^{-1}ss'^{-1}r'g=1}} \beta_1(r)\beta_2(s)\beta_2(s')\beta_1(r')\, W_{h^{-1}r^{-1}s,\, g^{-1}r'^{-1}s'}(z).\end{aligned}$$

Using the facts that $G_1\cap G_2=\{e\}$ and that the isometries $W_{g',h'}$ have pairwise orthogonal ranges when (g',h') are pairwise distinct, we get

$$\begin{aligned}&\tau(E_{q\mathcal{N}q}(u_g^* u_1^{t*} u_2^t y^* u_2^{t*} u_1^t u_h)\, E_{q\mathcal{N}q}(u_h^* u_1^{t*} u_2^t (b_{n,h}^* y b_{n,g}) u_2^{t*} u_1^t u_g))\\ &\quad= \sum_{\substack{r,r'\in G, s,s'\in G_2\\ rss'r'=hg^{-1}}} \beta_1(r^{-1})^2\beta_2(s)^2\beta_2(s'^{-1})^2\beta_1(r')^2\, \tau(y^* b_{n,h}^* y b_{n,g}).\end{aligned}$$

For $i\in\{1,2\}$, define $\mu_i\in\mathrm{Prob}(\mathbf{F}_2)$ by $\mu_i(g)=\beta_i(g)^2$ if $g\in G_i$ and $\mu_i(g)=0$ otherwise. Likewise, define $\check{\mu}_i\in\mathrm{Prob}(\mathbf{F}_2)$ by $\check{\mu}_i(g)=\mu_i(g^{-1})$ for all $g\in\mathbf{F}_2$. Put $\mu=\check{\mu}_1*\mu_2*\check{\mu}_2*\mu_1$. Since $y\in q\mathcal{M}_2q\ominus q\mathcal{B}q$ and $x_{n,g}\in q\mathcal{M}_1 q$, we obtain that

$$\begin{aligned}\tau(E_{q\mathcal{N}q}(u_g^* u_1^{t*} u_2^t y^* u_2^{t*} u_1^t u_h)\, E_{q\mathcal{N}q}(u_h^* u_1^{t*} u_2^t (b_{n,h}^* y b_{n,g}) u_2^{t*} u_1^t u_g)) &= \mu(hg^{-1})\,\tau(y^* b_{n,h}^* y b_{n,g})\\ &= \mu(hg^{-1})\,\tau(y^* x_{n,h}^* y x_{n,g}).\end{aligned}$$

Summing over all $g,h\in\mathbf{F}_2$ and using the Cauchy–Schwarz inequality, we get

$$\begin{aligned}|\langle \alpha_t(y)\eta_n \mid \eta_n\alpha_t(y)\rangle_{\mathrm{Tr}}| &= \Big|\sum_{g,h\in\mathbf{F}_2} \mu(hg^{-1})\tau(y^* x_{n,h}^* y x_{n,g})\Big|\\ &= \Big|\sum_{g,h\in\mathbf{F}_2} \mu(g)\tau(y^* x_{n,h}^* y x_{n,g^{-1}h})\Big|\\ &\leqslant \sum_{g,h\in\mathbf{F}_2} \mu(g)\|x_{n,h}y\|_2\,\|y x_{n,g^{-1}h}\|_2\\ &\leqslant \sum_{g\in\mathbf{F}_2} \mu(g)\langle \zeta_n \mid \lambda_g(\zeta_n')\rangle_{\ell^2(\mathbf{F}_2)},\end{aligned}$$

where $\zeta_n=\sum_{h\in\mathbf{F}_2}\|x_{n,h}y\|_2\,\delta_h$ and $\zeta_n'=\sum_{h\in\mathbf{F}_2}\|y x_{n,h}\|_2\,\delta_h$. Since we moreover have $u_1^{t*}\eta_n u_1^t=\sum_{g\in\mathbf{F}_2} u_g e_{q\mathcal{N}q} u_g^*\, x_{n,g}$, we get

$$\|u_1^{t*}\eta_n u_1^t y\|_{2,\mathrm{Tr}}^2 = \sum_{g\in\mathbf{F}_2}\|u_g e_{q\mathcal{N}q} u_g^*\, x_{n,g} y\|_{2,\mathrm{Tr}}^2 = \sum_{g\in\mathbf{F}_2}\|x_{n,g}y\|_2^2 = \|\zeta_n\|_{\ell^2(\mathbf{F}_2)}^2.$$

Likewise, we have $\|\zeta_n'\|_{\ell^2(\mathbf{F}_2)}=\|y u_1^{t*}\eta_n u_1^t\|_{2,\mathrm{Tr}}$.

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Denote by $T : \ell^2(\mathbf{F}_2) \to \ell^2(\mathbf{F}_2)$ the Markov operator defined by $T = \sum_{g\in\mathbf{F}_2} \mu(g)\lambda_g$. Since the support of μ generates $\mathbf{F}_2$ and $\mu(e) > 0$ (see the proof of [Io12a, Lemma 3.4, Claim]), Kesten's criterion for amenability [Kes59] yields $\|T\|_\infty < 1$. This gives

$$\begin{aligned}|\langle \alpha_t(y)\eta_n \mid \eta_n\alpha_t(y)\rangle_{\mathrm{Tr}}| &\leqslant \langle \zeta_n \mid T\zeta'_n\rangle_{\ell^2(\mathbf{F}_2)}\\ &\leqslant \|T\|_\infty \, \|\zeta_n\|_{\ell^2(\mathbf{F}_2)} \, \|\zeta'_n\|_{\ell^2(\mathbf{F}_2)}\\ &= \|T\|_\infty \, \|u_1^{t*}\eta_n u_1^t y\|_{2,\mathrm{Tr}} \, \|y u_1^{t*}\eta_n u_1^t\|_{2,\mathrm{Tr}}\\ &= \|T\|_\infty \, \|\eta_n u_1^t y\|_{2,\mathrm{Tr}} \, \|y u_1^{t*}\eta_n\|_{2,\mathrm{Tr}}.\end{aligned}$$

Since $\eta_n = \eta_n^*$, we obtain

$$\|\eta_n u_1^t y\|_{2,\mathrm{Tr}} \, \|y u_1^{t*}\eta_n\|_{2,\mathrm{Tr}} \to \|u_1^t y\|_2 \, \|y u_1^{t*}\|_2 = \|y\|_2^2 = 1;$$

hence $\limsup_n |\langle \alpha_t(y)\eta_n \mid \eta_n\alpha_t(y)\rangle_{\mathrm{Tr}}| \leqslant \|T\|_\infty < 1$. This, however, contradicts the fact that

$$|\langle \alpha_t(y)\eta_n \mid \eta_n\alpha_t(y)\rangle_{\mathrm{Tr}}| \to 1$$

and hence our assumption that $q\mathcal{M}_1 q$ had no amenable direct summand. Thus, for all $i \in \{1, 2\}$, $q\mathcal{M}_i q$ has an amenable direct summand and so does $\mathcal{M}_i$. This finishes the proof of Theorem 4.1. □

A combination of the proof of the above Theorem 4.1 and the one of [Io12a, Theorem 4.1] shows that 'or' can be replaced with 'and' in Ioana's result [Io12a, Theorem 4.1].

Theorem 4.3. *Let $M = M_1 *_B M_2$ be a tracial amalgamated free product von Neumann algebra. Assume that $M_1 \neq B \neq M_2$. Put $\widetilde{M} = M *_B (B \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{F}_2)) = N \rtimes \mathbf{F}_2$, where $N = \bigvee\{u_g M u_g^* : g \in \mathrm{L}(\mathbf{F}_2)\}$. Let $t \in (0, 1)$ such that $\alpha_t(M)$ is amenable relative to N.*

Then, for all $i \in \{1, 2\}$, there exists a nonzero projection $z_i \in \mathcal{Z}(M_i)$ such that $M_i z_i$ is amenable relative to B inside M.

5. Proofs of Theorems A and B

5.1 A general intermediate result

Theorems A and B will be derived from the following very general result regarding Cartan subalgebras inside semifinite amalgamated free product von Neumann algebras.

Theorem 5.1. *Let $\mathcal{M} = \mathcal{M}_1 *_\mathcal{B} \mathcal{M}_2$ be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace* Tr*. Assume that $\mathcal{B}$ is amenable, $\mathcal{M}_1$ has no amenable direct summand, and that for all nonzero projections $e \in \mathcal{B}$, we have $e\mathcal{B}e \neq e\mathcal{M}_2 e$.*

Let $p \in \mathrm{Proj_f}(\mathcal{B})$ and $\mathcal{A} \subset p\mathcal{M}p$ be any regular amenable von Neumann subalgebra. Then there exists $q \in \mathrm{Proj_f}(\mathcal{B})$ such that $\mathcal{A} \preceq_\mathcal{M} q\mathcal{B}q$.

Proof. Put $\widetilde{\mathcal{M}} = \mathcal{M} *_\mathcal{B} (\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{F}_2))$ and regard $p\widetilde{\mathcal{M}}p$ as the tracial crossed product von Neumann algebra $p\widetilde{\mathcal{M}}p = p\mathcal{N}p \rtimes \mathbf{F}_2$ with $\mathcal{N} = \bigvee\{u_g \mathcal{M} u_g : g \in \mathbf{F}_2\}$. We denote by (α_t) the malleable deformation from § 3.1. Applying the Popa–Vaes dichotomy result [PV11, Theorem 1.6] to the inclusion $\alpha_t(\mathcal{A}) \subset p\widetilde{\mathcal{M}}p$ for $t \in (0, 1)$, we get that at least one of the following holds true:

(1) either $\alpha_t(\mathcal{A}) \preceq_{p\widetilde{\mathcal{M}}p} p\mathcal{N}p$;

(2) or $\alpha_t(p\mathcal{M}p)$ is amenable relative to $p\mathcal{N}p$ inside $p\widetilde{\mathcal{M}}p$.

Since $\mathcal{M}_1$ has no amenable direct summand, case (2) cannot hold by Theorem 4.1. It remains to show that case (1) leads to the conclusion of the theorem.

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In case (1), using Lemma 2.3 and Theorem 3.4, we get that either there exists $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$ such that $\mathcal{A} \preceq_{\mathcal{M}} q\mathcal{B}q$ or there exist $i \in \{1,2\}$ and $q_i \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{M}_i)$ such that $p\mathcal{M}p \preceq_{\mathcal{M}} q_i\mathcal{M}_i q_i$. Since the latter case is impossible by Proposition 2.6, we get $\mathcal{A} \preceq_{\mathcal{M}} q\mathcal{B}q$ for some $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$. □

5.2 Proof of Theorem A

We first need to prove the following well-known result.

Lemma 5.2. *Let M be any von Neumann algebra such that $M \neq \mathbf{C}$ and φ is any faithful normal state on M. Realize the continuous core $\mathrm{c}(M) = M \rtimes_{\varphi} \mathbf{R}$. Then for every nonzero projection $p \in \mathrm{L}(\mathbf{R})$, we have $\mathrm{L}(\mathbf{R})p \neq p\mathrm{c}(M)p$.*

Proof. There are two cases to consider.

Case (1)*: assume that $M^{\varphi} \neq \mathbf{C}$.* Choose $r \in M^{\varphi}$ a projection such that $r \neq 0, 1$. Observe that $x = \varphi(1-r)\, r - \varphi(r)\,(1-r) \in M^{\varphi}$ is invertible and $\varphi(x) = 0$. Then for every nonzero projection $p \in \mathrm{L}(\mathbf{R})$, we have $xp \neq 0$ and $E_{\mathrm{L}(\mathbf{R})p}(xp) = \varphi(x)p = 0$. This proves that $\mathrm{L}(\mathbf{R})p \neq pMp$.

Case (2)*: assume that $M^{\varphi} = \mathbf{C}$.* Since $\mathcal{Z}(M) \subset \mathcal{Z}(M^{\varphi})$, it follows that M is a factor. If M is of type III, it follows from Connes's classification of type III factors [Con73] that M is necessarily of type III_1. In that case, $\mathrm{c}(M)$ is a type II_{∞} factor and thus $\mathrm{L}(\mathbf{R})p \neq p\mathrm{c}(M)p$ for every nonzero projection $p \in \mathrm{L}(\mathbf{R})$. If M is a semifinite factor with semifinite faithful normal trace Tr, there exists $b \in \mathrm{L}^1(M, \mathrm{Tr})_+$ such that $\varphi = \mathrm{Tr}(b\,\cdot)$ and $\|b\|_{1,\mathrm{Tr}} = 1$. Let $q \in M$ be a nonzero spectral projection of b. Since

$$\varphi(qx) = \mathrm{Tr}(bqx) = \mathrm{Tr}(qbx) = \mathrm{Tr}(bxq) = \varphi(xq)$$

for all $x \in M$, we get $q \in M^{\varphi}$ and so $q = 1$. This shows that $b = 1$ and $\mathrm{Tr} = \varphi$ is a finite trace on M. Hence $M = M^{\varphi} = \mathbf{C}$, which is a contradiction. □

Proof of Theorem A. By [Ued11, Theorem 4.1], we know that there exists a nonzero projection $z \in \mathcal{Z}(M)$ such that Mz is a full factor and $M(1-z)$ is a purely atomic von Neumann algebra. In particular, M is not amenable.

In the case when both M_1 and M_2 are amenable, [HR11, Theorem 5.5] implies that M has no Cartan subalgebra. It remains to consider the case when M_1 or M_2 is not amenable. Without loss of generality, we may assume that M_1 is not amenable.

By contradiction, assume that M has a Cartan subalgebra. Hence, Mz also has a Cartan subalgebra. Let $p \in \mathcal{Z}(M_1)$ be the largest nonzero projection such that M_1p has no amenable direct summand. Since $M(1-z)$ is purely atomic, we necessarily have $p \leqslant z$.

By [Ued11, Lemma 2.2], we have

$$\left(pMp, \frac{1}{\varphi(p)}\varphi(p \cdot p)\right) = \left(M_1p, \frac{1}{\varphi_1(p)}\varphi_1(\cdot p)\right) * \left(pNp, \frac{1}{\varphi(p)}\varphi(p \cdot p)\right),$$

with $N = (\mathbf{C}p \oplus M_1(1-p)) \vee M_2$. Observe that $pNp \neq \mathbf{C}p$. Indeed, let $q \in M_2$ be a projection such that $q \neq 0, 1$. Then $pqp = \varphi_2(q)p + p(q - \varphi_2(q))p \in pNp \backslash \mathbf{C}p$. Since Mz is a factor and $p \leqslant z$, it follows that pMp has a Cartan subalgebra by [Pop06b, Lemma 3.5].

From the previous discussion, it follows that we may assume that M_1 has no amenable direct summand, $M_2 \neq \mathbf{C}$, and M has a Cartan subalgebra $A \subset M$. Using Notation 2.9, denote by $\mathrm{c}(A) \subset \mathrm{c}(M)$ the Cartan subalgebra in the continuous core $\mathrm{c}(M) = \mathrm{c}(M_1) *_{\mathrm{L}(\mathbf{R})} \mathrm{c}(M_2)$. Let $q \in \mathrm{Proj}_{\mathrm{f}}(\mathrm{L}(\mathbf{R}))$. Since $\mathrm{c}(A) \subset \mathrm{c}(M)$ is maximal abelian and $\mathrm{Tr}\,|\mathrm{c}(A)$ is semifinite, [HV13, Lemma 2.1] shows that there exists a nonzero finite trace projection $p \in \mathrm{c}(A)$ and a partial isometry $v \in \mathrm{c}(M)$ such that $p = v^*v$ and $q = vv^*$. Observe that $v\mathrm{c}(A)v^* \subset q\mathrm{c}(M)q$ is still a Cartan subalgebra by [Pop06b, Lemma 3.5].

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By Lemma 2.3, Proposition 2.8, Theorem 5.1, and Lemma 5.2, there exists $q' \in \mathrm{Proj}_{\mathrm{f}}(\mathrm{L}(\mathbf{R}))$ such that $v\mathrm{c}(A)v^* \preceq_{\mathrm{c}(M)} \mathrm{L}(\mathbf{R})q'$. Then Proposition 2.10 implies that $A \preceq_M \mathbf{C}$. This contradicts the fact that A is diffuse and finishes the proof of Theorem A. □

5.3 Proof of Theorem B

Proof of Theorem B. Let $A \subset M$ be a Cartan subalgebra. Since $A, B \subset M$ are both tracial von Neumann subalgebras of M with expectation, we use Notation 2.9. Let $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{Z}(\mathrm{c}(B)))$. By [HV13, Lemma 2.1], there exists $p \in \mathrm{Proj}_{\mathrm{f}}(\mathrm{c}(A))$ and a partial isometry $v \in \mathrm{c}(M)$ such that $p = v^*v$ and $q = vv^*$. Observe that $v\mathrm{c}(A)v^* \subset q\mathrm{c}(M)q$ is still a Cartan subalgebra by [Pop06b, Lemma 3.5].

Using the assumptions, by Lemma 2.3, Proposition 2.8, [HV13, Proposition 5.5], and Theorem 5.1, there exists $q' \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{Z}(\mathrm{c}(B)))$ such that $v\mathrm{c}(A)v^* \preceq_{\mathrm{c}(M)} \mathrm{c}(B)q'$. Then Proposition 2.10 implies that $A \preceq_M B$. □

6. Proof of Theorem C

Let $\mathcal{R}$ be any countable nonsingular equivalence relation on a standard measure space (X, μ). Following [FM77], denote by m the measure on $\mathcal{R}$ given by

$$m(\mathcal{W}) = \int_X |\{y \in X : (x, y) \in \mathcal{W}\}| \, d\mu(x)$$

for all measurable subsets $\mathcal{W} \subset \mathcal{R}$. We denote by $[\mathcal{R}]$ the full group of $\mathcal{R}$, $M = \mathrm{L}(\mathcal{R})$ the von Neumann algebra of $\mathcal{R}$, and identify $\mathrm{L}^2(M) = \mathrm{L}^2(\mathcal{R}, m)$. For all $\psi \in [\mathcal{R}]$, define $u(\psi) \in \mathcal{U}(M)$ the action of which on $\mathrm{L}^2(\mathcal{R}, m)$ is given by

$$(u(\psi)\xi)(x, y) = \left(\frac{d(\mu \circ \psi^{-1})}{d\mu}(x)\right)^{1/2} \xi(\psi^{-1}(x), y).$$

We view $\mathrm{L}^\infty(\mathcal{R})$ as acting on $\mathrm{L}^2(\mathcal{R}, m)$ by multiplication operators. Note that the unitaries $u(\psi) \in \mathcal{U}(M)$ for $\psi \in [\mathcal{R}]$ normalize $\mathrm{L}^\infty(\mathcal{R})$ and that $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(\mathcal{R})$, by identifying a function $F \in \mathrm{L}^\infty(X)$, with the function on $\mathcal{R}$ given by $(x, y) \mapsto F(x)$.

Recall from [CFW81, Definition 5] that $\mathcal{R}$ is *amenable* if there exists a norm one projection $\Phi : \mathrm{L}^\infty(\mathcal{R}) \to \mathrm{L}^\infty(X)$ satisfying

$$\Phi(u(\psi)Fu(\psi)^*) = u(\psi)\Phi(F)u(\psi)^*, \quad \forall \psi \in [\mathcal{R}].$$

By [CFW81, Theorem 10], a countable nonsingular equivalence relation $\mathcal{R}$ is amenable if and only if it is hyperfinite. We will say that a countable nonsingular equivalence relation $\mathcal{R}$ is *nowhere amenable* if for every measurable subset $\mathcal{U} \subset X$ such that $\mu(\mathcal{U}) > 0$, the equivalence relation $\mathcal{R}|\mathcal{U} = \mathcal{R} \cap (\mathcal{U} \times \mathcal{U})$ is nonamenable.

Recall the following definition due to Gaboriau [Gab00, Definition IV.6].

Definition 6.1. Let $\mathcal{R}$ be a countable nonsingular equivalence relation on a standard measure space (X, μ) and $\mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{R}$ subequivalence relations. We say that $\mathcal{R}$ *splits as the free product* $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ if:

- $\mathcal{R}$ is generated by $\mathcal{R}_1$ and $\mathcal{R}_2$;
- for every $p \in \mathbf{N}_{>0}$ and almost every $2p$-tuple $(x_j)_{j \in \mathbf{Z}/2p\mathbf{Z}}$ in X such that $(x_{2i-1}, x_{2i}) \in \mathcal{R}_1$ and $(x_{2i}, x_{2i+1}) \in \mathcal{R}_2$ for all $i \in \mathbf{Z}/p\mathbf{Z}$, there exists $j \in \mathbf{Z}/2p\mathbf{Z}$ such that $x_j = x_{j+1}$.

We have the following well-known fact.

Proposition 6.2. *Let $\mathcal{R}$ be a countable nonsingular equivalence relation on a standard measure space (X,μ) and $\mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{R}$ subequivalence relations. Let $B = \mathrm{L}^\infty(X)$, $M_1 = \mathrm{L}(\mathcal{R}_1)$, $M_2 = \mathrm{L}(\mathcal{R}_2)$, $M = \mathrm{L}(\mathcal{R})$, and denote by $E_1 : M_1 \to B$, $E_2 : M_2 \to B$, $E : M \to B$ the canonical faithful normal conditional expectations. The following conditions are equivalent:*

(1) *$\mathcal{R}$ splits as the free product $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$;*

(2) *$(M, E) = (M_1, E_1) *_B (M_2, E_2)$.*

We start by proving the following intermediate result in the framework of type II_1 equivalence relations.

Theorem 6.3. *Let $\mathcal{R}$ be a countable (not necessarily ergodic) probability measure preserving equivalence relation on a standard probability space (X,μ) which splits as a free product $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$, where $\mathcal{R}_i$ is a countable type II_1 subequivalence relation for all $i \in \{1,2\}$.*

Let $A \subset \mathrm{L}(\mathcal{R})$ be a Cartan subalgebra. Then $A \preceq_{\mathrm{L}(\mathcal{R})} \mathrm{L}^\infty(X)$.

Proof. Let $B = \mathrm{L}^\infty(X)$, $M_1 = \mathrm{L}(\mathcal{R}_1)$, $M_2 = \mathrm{L}(\mathcal{R}_2)$, and $M = \mathrm{L}(\mathcal{R})$, so that $M = M_1 *_B M_2$. Let $A \subset M$ be a Cartan subalgebra.

First, assume that both $\mathcal{R}_1$ and $\mathcal{R}_2$ are amenable and thus hyperfinite by [CFW81]. Since both $\mathcal{R}_1$ and $\mathcal{R}_2$ are, moreover, of type II_1, they are necessarily generated by a free pmp action of $\mathbf{Z}$. Hence $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ is generated by a free pmp action of $\mathbf{F}_2$ and so $M \cong B \rtimes \mathbf{F}_2$. Then [PV11, Theorem 1.6] shows that $A \preceq_M B$.

Next, assume that $\mathcal{R}_1$ or $\mathcal{R}_2$ is nonamenable. Without loss of generality, we may assume that $\mathcal{R}_1$ is nonamenable. Choose a measurable subset $\mathcal{U} \subset X$ such that $\mu(\mathcal{U}) > 0$ and $\mathcal{R}_1|\mathcal{U}$ is nowhere amenable. Denote by $\mathcal{V} \subset X$ the $\mathcal{R}$-saturated measurable subset of $\mathcal{U}$ in X. Since $\mathcal{R}|\mathcal{V} = (\mathcal{R}_1|\mathcal{V}) * (\mathcal{R}_2|\mathcal{V})$, we may assume that $\mu(\mathcal{V}) = 1$.

Since $\mathcal{U}$ is a complete section for $\mathcal{R}$, it follows from [Alv10, Théorème 44] that we can write $\mathcal{R}|\mathcal{U} = \mathcal{S}_1 * \mathcal{S}_2$ where $\mathcal{S}_1 = \mathcal{R}_1|\mathcal{U}$ and $\mathcal{S}_2$ is a type II_1 subequivalence relation of $\mathcal{R}|\mathcal{U}$ which contains $\mathcal{R}_2|\mathcal{U}$.

Write $q = \mathbf{1}_{\mathcal{U}} \in B$. By [BO08, Corollary F.8], choose a projection $p \in A$ and a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* = q$. Then $vAv^* \subset qMq$ is a Cartan subalgebra by [Pop06b, Lemma 3.5]. We can thus apply Theorem 5.1 to $\mathcal{M} = \mathrm{L}(\mathcal{S}_1) *_{\mathrm{L}^\infty(\mathcal{U})} \mathrm{L}(\mathcal{S}_2)$, $\mathcal{A} = vAv^*$ and $p = 1$. Then we obtain that $vAv^* \preceq_{qMq} Bq$; hence $A \preceq_M B$. □

Proof of Theorem C. Write $B = \mathrm{L}^\infty(X)$, $M_1 = \mathrm{L}(\mathcal{R}_1)$, $M_2 = \mathrm{L}(\mathcal{R}_2)$, and $M = \mathrm{L}(\mathcal{R})$, so that $M = M_1 *_B M_2$. Define on the standard infinite measure space $(X \times \mathbf{R}, m)$ the countable infinite measure preserving equivalence relations $\mathrm{c}(\mathcal{R}_1)$, $\mathrm{c}(\mathcal{R}_2)$ and $\mathrm{c}(\mathcal{R})$, which are the Maharam extensions [Mah64] of the countable nonsingular equivalence relations $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}$, respectively. Observe that both $\mathrm{c}(\mathcal{R}_1)$ and $\mathrm{c}(\mathcal{R}_2)$ are of type II and $\mathrm{c}(\mathcal{R}) = \mathrm{c}(\mathcal{R}_1) * \mathrm{c}(\mathcal{R}_2)$.

Moreover, if we write $\mathrm{c}(B) = \mathrm{L}^\infty(X \times \mathbf{R})$, we canonically have

$$\mathrm{c}(M_1) = \mathrm{L}(\mathrm{c}(\mathcal{R}_1)), \quad \mathrm{c}(M_2) = \mathrm{L}(\mathrm{c}(\mathcal{R}_2)), \quad \mathrm{c}(M) = \mathrm{L}(\mathrm{c}(\mathcal{R})) \quad \text{and} \quad \mathrm{c}(M) = \mathrm{c}(M_1) *_{\mathrm{c}(B)} \mathrm{c}(M_2).$$

Let $A \subset M$ be a Cartan subalgebra. Using Notation 2.9, we obtain that $\mathrm{c}(A) \subset \mathrm{c}(M)$ is a Cartan subalgebra. Let $q \in \mathrm{Proj}_{\mathrm{f}}(\mathrm{c}(B))$ such that $\mathrm{Tr}(q) = 1$. Up to cutting down by the central support of q in $\mathrm{c}(M)$, we may assume that q has central support equal to 1 in $\mathrm{c}(M)$. By [HV13, Lemma 2.1], there exists $p \in \mathrm{Proj}_{\mathrm{f}}(\mathrm{c}(A))$ and a partial isometry $v \in \mathrm{c}(M)$ such that $p = v^*v$ and $q = vv^*$. Observe that $v\mathrm{c}(A)v^* \subset q\mathrm{c}(M)q$ is still a Cartan subalgebra by [Pop06b, Lemma 3.5]. In order to show that A and B are unitarily conjugate inside M, using Theorem 2.1 and Proposition 2.10, it suffices to show that $v\mathrm{c}(A)v^* \preceq_{\mathrm{c}(M)} \mathrm{c}(B)q$.

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Let $\mathcal{U} \subset X \times \mathbf{R}$ be a measurable subset such that $\mathbf{1}_\mathcal{U} = q$. Since $\mathbf{1}_\mathcal{U}$ has central support equal to 1 in $\mathrm{c}(M)$, $\mathcal{U}$ is a complete section for $\mathrm{c}(\mathcal{R})$. By [Alv10, Théorème 44], we can write $\mathrm{c}(\mathcal{R})|\mathcal{U} = \mathcal{S}_1 * \mathcal{S}_2$ where $\mathcal{S}_1 = \mathrm{c}(\mathcal{R}_1)|\mathcal{U}$ and $\mathcal{S}_2$ is a subequivalence relation of $\mathrm{c}(\mathcal{R})|\mathcal{U}$ which contains $\mathrm{c}(\mathcal{R}_2)|\mathcal{U}$. In particular, both $\mathcal{S}_1$ and $\mathcal{S}_2$ are type II_1 equivalence relations on the standard probability space $(\mathcal{U}, m|\mathcal{U})$.

Let $\mathcal{A} = v\mathrm{c}(A)v^*$ and $\mathcal{B} = \mathrm{L}^\infty(\mathcal{U})$. Observe that $q\mathrm{c}(M)q = \mathrm{L}(\mathrm{c}(\mathcal{R})|\mathcal{U}) = \mathrm{L}(\mathcal{S}_1 * \mathcal{S}_2)$ and $\mathcal{A}$ is a Cartan subalgebra in $\mathrm{L}(\mathcal{S}_1 * \mathcal{S}_2)$. Then Theorem 6.3 implies that $\mathcal{A} \preceq_{\mathrm{L}(\mathcal{S}_1 * \mathcal{S}_2)} \mathrm{L}^\infty(\mathcal{U})$; that is, $v\mathrm{c}(A)v^* \preceq_{\mathrm{c}(M)} \mathrm{c}(B)q$. This finishes the proof of Theorem C. □

7. Proof of Theorem D

We start by proving Theorem D in the infinite measure preserving case. More precisely, we deduce the following result from its finite measure preserving counterpart proven in [Io12a, Theorem 1.1].

Theorem 7.1. *Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ be an amalgamated product group such that Σ is finite and for all $i \in \{1, 2\}$, Γ_i is infinite. Let $(\mathcal{B}, \mathrm{Tr})$ be a type I von Neumann algebra endowed with a semifinite faithful normal trace. Let $\Gamma \curvearrowright (\mathcal{B}, \mathrm{Tr})$ be a trace-preserving action such that for all $i \in \{1, 2\}$, the crossed product von Neumann algebra $\mathcal{B} \rtimes \Gamma_i$ is of type II. Put $\mathcal{M} = \mathcal{B} \rtimes \Gamma$. Let $p \in \mathrm{Proj}_\mathrm{f}(\mathcal{B})$ and $\mathcal{A} \subset p\mathcal{M}p$ be any regular amenable von Neumann subalgebra.*

Then for every nonzero projection $e \in \mathcal{A}' \cap p\mathcal{M}p$, we have $\mathcal{A}e \preceq_{p\mathcal{M}p} p\mathcal{B}p$.

Proof. For every subset $\mathcal{F} \subset \Gamma$, denote by $P_\mathcal{F}$ the orthogonal projection from $\mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$ onto the closed linear span of $\{xu_g : x \in \mathcal{B} \cap \mathrm{L}^2(\mathcal{B}, \mathrm{Tr}), g \in \mathcal{F}\}$. Since $\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})'' = p\mathcal{M}p$, Proposition 2.4 (see also [HV13, Lemma 2.7]) provides a central projection $z \in \mathcal{Z}(p\mathcal{M}p)$ and a net of unitaries $w_k \in \mathcal{U}(\mathcal{A}z)$ such that:

- $\lim_k \|P_\mathcal{F}(w_k)\|_{2,\mathrm{Tr}} = 0$ for all finite subset $\mathcal{F} \subset \Gamma$;
- For every $\varepsilon > 0$, there exists a finite subset $\mathcal{F} \subset \Gamma$ such that $\|a - P_\mathcal{F}(a)\|_{2,\mathrm{Tr}} \leqslant \varepsilon$ for all $a \in \mathrm{Ball}(\mathcal{A}(p - z))$.

We prove by contradiction that $z = 0$. So, assume that $z \neq 0$. Recall that $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$. Hence the subgroup $\Sigma_0 = \bigcap_{g \in \Gamma} g\Sigma g^{-1} < \Sigma$ is finite and normal in Γ. Define the quotient homomorphism $\rho : \Gamma \to \Gamma/\Sigma_0$ and put $\Lambda = \Gamma/\Sigma_0$, $\Lambda_i = \Gamma_i/\Sigma_0$ for $i \in \{1, 2\}$, $\Upsilon = \Sigma/\Sigma_0$ so that $\Lambda = \Lambda_1 *_\Upsilon \Lambda_2$. We get that $\bigcap_{s \in \Lambda} s\Upsilon s^{-1} = \{e\}$; hence $\mathrm{L}(\Lambda)$ is a II_1 factor which does not have property Gamma by [Io12a, Corollary 6.2].

Define the unitary $W \in \mathcal{U}(\mathrm{L}^2(\mathcal{B}, \mathrm{Tr}) \otimes \ell^2(\Gamma) \otimes \ell^2(\Lambda))$ by

$$W(\xi \otimes \delta_g \otimes \delta_s) = \xi \otimes \delta_g \otimes \delta_{\rho(g^{-1})s}, \quad \forall \xi \in \mathrm{L}^2(\mathcal{B}, \mathrm{Tr}), \forall g \in \Gamma, \forall s \in \Lambda.$$

Next, define the *dual coaction* $\Delta_\rho : \mathcal{M} \to \mathcal{M} \mathbin{\overline{\otimes}} \mathrm{L}(\Lambda)$ by $\Delta_\rho(x) = W^*(x \otimes 1)W$ for all $x \in \mathcal{M}$. Observe that Δ_ρ is a trace-preserving $*$-embedding which satisfies $\Delta_\rho(bu_g) = bu_g \otimes v_{\rho(g)}$ for all $b \in \mathcal{B}$ and all $g \in \Gamma$.

For every subset $\mathcal{F} \subset \Gamma$, denote by $Q_{\rho(\mathcal{F})}$ the orthogonal projection from $\mathrm{L}^2(\mathrm{L}(\Lambda))$ onto the closed linear span of $\{v_{\rho(g)} : g \in \mathcal{F}\}$. Observe that $(1 \otimes Q_{\rho(\mathcal{F})})(\Delta_\rho(x)) = \Delta_\rho(P_{\Sigma_0\mathcal{F}}(x))$ for all $x \in \mathcal{M}$. Since Δ_ρ is $\|\cdot\|_{2,\mathrm{Tr}}$-preserving and since Σ_0 is finite, for any finite subset $\mathcal{F} \subset \Gamma$, we have

$$\lim_k \|(1 \otimes Q_{\rho(\mathcal{F})})(\Delta_\rho(w_k))\|_2 = \lim_k \|\Delta_\rho(P_{\Sigma_0\mathcal{F}}(w_k))\|_2 = 0.$$

Since $\Upsilon < \Lambda$ is a finite subgroup, this implies that $\Delta_\rho(\mathcal{A}z) \npreceq_{\mathcal{M} \overline{\otimes} \mathrm{L}(\Lambda)} q\mathcal{M}q \mathbin{\overline{\otimes}} \mathrm{L}(\Upsilon)$ for all $q \in \mathrm{Proj}_\mathrm{f}(\mathcal{B})$.

Put $\widetilde{\Lambda} = \Lambda *_\Upsilon (\Upsilon \times \mathbf{F}_2) = \Lambda_1 *_\Upsilon \Lambda_2 *_\Upsilon (\Upsilon \times \mathbf{F}_2)$ and consider the malleable deformation (α_t) on $\mathrm{L}(\widetilde{\Lambda})$ from § 3.1. Define $N < \Lambda$ as the normal subgroup generated by $\{t\Lambda t^{-1} : t \in \mathbf{F}_2\}$ so that $\mathrm{L}(\widetilde{\Lambda}) = \mathcal{N} \rtimes \mathbf{F}_2$ with $\mathcal{N} = \mathrm{L}(N)$. Applying the Popa–Vaes dichotomy result [PV11, Theorem 1.6] to each of the inclusions

$$(\mathrm{id} \otimes \alpha_t)(\Delta_\rho(\mathcal{A}z)) \subset p\mathcal{M}p \mathbin{\overline{\otimes}} \mathrm{L}(\widetilde{\Lambda}) = (p\mathcal{M}p \mathbin{\overline{\otimes}} \mathcal{N}) \rtimes \mathbf{F}_2 \quad \text{with } t \in (0, 1),$$

we obtain that at least one of the following holds true:

(1) either there exists $t \in (0, 1)$ such that $(\mathrm{id} \otimes \alpha_t)(\Delta_\rho(\mathcal{A}z)) \preceq_{p\mathcal{M}p \overline{\otimes} \mathrm{L}(\widetilde{\Lambda})} p\mathcal{M}p \mathbin{\overline{\otimes}} \mathcal{N}$;

(2) or for all $t \in (0, 1)$, $(\mathrm{id} \otimes \alpha_t)(\Delta_\rho(p\mathcal{M}p))$ is amenable relative to $p\mathcal{M}p \mathbin{\overline{\otimes}} \mathcal{N}$ inside $p\mathcal{M}p \mathbin{\overline{\otimes}} \mathrm{L}(\widetilde{\Lambda})$.

We will prove below that each case leads to a contradiction.

In case (1), by [Io12a, Theorem 3.2] and since $\Delta_\rho(\mathcal{A}z) \not\preceq_{p\mathcal{M}p \overline{\otimes} \mathrm{L}(\Lambda)} p\mathcal{M}p \mathbin{\overline{\otimes}} \mathrm{L}(\Upsilon)$ and $\mathcal{N}_{p\mathcal{M}pz}(\mathcal{A}z)'' = p\mathcal{M}pz$, there exists $i \in \{1, 2\}$ such that $\Delta_\rho(p\mathcal{M}pz) \preceq_{p\mathcal{M}p \overline{\otimes} \mathrm{L}(\Lambda)} p\mathcal{M}p \mathbin{\overline{\otimes}} \mathrm{L}(\Lambda_i)$. In order to get a contradiction, we will need the following.

Claim. *Let $e \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{M})$, $\mathcal{Q} \subset e\mathcal{M}e$ be any von Neumann subalgebra and $\mathcal{S}$ any nonempty collection of subgroups of Γ. If $\mathcal{Q} \not\preceq_{\mathcal{M}} q(\mathcal{B} \rtimes H)q$ for all $H \in \mathcal{S}$ and all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$, then $\Delta_\rho(\mathcal{Q}) \not\preceq_{\mathcal{M} \overline{\otimes} \mathrm{L}(\Lambda)} q\mathcal{M}q \mathbin{\overline{\otimes}} \mathrm{L}(\rho(H))$ for all $H \in \mathcal{S}$ and all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$.*

Proof of the Claim. Since $\mathcal{Q} \not\preceq_{\mathcal{M}} q(\mathcal{B} \rtimes H)q$ for all $H \in \mathcal{S}$ and all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$, Proposition 2.4 implies that there exists a net $v_k \in \mathcal{U}(\mathcal{Q})$ such that $\lim_k \|P_{\mathcal{F}}(v_k)\|_{2,\mathrm{Tr}} = 0$ for all subsets $\mathcal{F} \subset \Gamma$ which are small relative to $\mathcal{S}$. Observe that since Σ_0 is finite, $\Sigma_0\mathcal{F}$ is small relative to $\mathcal{S}$ for all subsets $\mathcal{F} \subset \Gamma$ which are small relative to $\mathcal{S}$. Moreover, $(1 \otimes Q_{\rho(\mathcal{F})})(\Delta_\rho(x)) = \Delta_\rho(P_{\Sigma_0\mathcal{F}}(x))$ for all $x \in \mathcal{Q}$ and all subsets $\mathcal{F} \subset \Gamma$ which are small relative to $\mathcal{S}$. Since Δ_ρ is $\|\cdot\|_{2,\mathrm{Tr}}$-preserving, for all subsets $\mathcal{F} \subset \Gamma$ which are small relative to $\mathcal{S}$, we have

$$\lim_k \|(1 \otimes Q_{\rho(\mathcal{F})})(\Delta_\rho(v_k))\|_{2,\mathrm{Tr}} = \lim_k \|\Delta_\rho(P_{\Sigma_0\mathcal{F}}(v_k))\|_{2,\mathrm{Tr}} = 0. \tag{5}$$

Denote by $\rho(\mathcal{S})$ the nonempty collection of subgroups $\rho(H) \subset \Lambda$ with $H \in \mathcal{S}$. Let $\mathcal{G} \subset \Lambda$ be any subset which is small relative to $\rho(\mathcal{S})$. Then there exist $n \geqslant 1$, $H_1, \ldots, H_n \in \mathcal{S}$ and $s_1, t_1, \ldots, s_n, t_n \in \Lambda$ such that $\mathcal{G} \subset \bigcup_{i=1}^n s_i\rho(H_i)t_i$. Choose $g_i, h_i \in \Gamma$ such that $\rho(g_i) = s_i$ and $\rho(h_i) = t_i$ and denote $\mathcal{F} = \bigcup_{i=1}^n g_iH_ih_i$. Then $\mathcal{G} \subset \rho(\mathcal{F})$. Therefore, (5) implies that $\lim_k \|(1 \otimes Q_{\mathcal{G}})(\Delta_\rho(v_k))\|_{2,\mathrm{Tr}} = 0$ for all subsets $\mathcal{G} \subset \Lambda$ which are small relative to $\rho(\mathcal{S})$. Thus, Proposition 2.4 implies that $\Delta_\rho(\mathcal{Q}) \not\preceq_{\mathcal{M} \overline{\otimes} \mathrm{L}(\Lambda)} q\mathcal{M}q \mathbin{\overline{\otimes}} \mathrm{L}(\rho(H))$ for all $H \in \mathcal{S}$ and all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$. □

We apply the Claim to $\mathcal{Q} = p\mathcal{M}pz$ and $\mathcal{S} = \{\Gamma_1, \Gamma_2\}$. In order to do that, we need to check that $p\mathcal{M}pz \not\preceq_{q\mathcal{M}q} q(\mathcal{B} \rtimes \Gamma_i)q$ for all $i \in \{1, 2\}$ and all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$. Since $\mathcal{B} \rtimes \Sigma$ is a type I von Neumann algebra and $\mathcal{B} \rtimes \Gamma_i$ is a type II von Neumann algebra, Proposition 2.6 yields the result. Therefore, by the Claim, we get that $\Delta_\rho(p\mathcal{M}pz) \not\preceq_{p\mathcal{M}p \overline{\otimes} \mathrm{L}(\Lambda)} p\mathcal{M}p \mathbin{\overline{\otimes}} \mathrm{L}(\Lambda_i)$ for all $i \in \{1, 2\}$. This is a contradiction.

In case (2), since $\mathrm{L}(\Lambda)$ does not have property Gamma, [Io12a, Theorem 5.2] shows that either there exists $i \in \{1, 2\}$ such that $\mathrm{L}(\Lambda) \preceq_{\mathrm{L}(\Lambda)} \mathrm{L}(\Lambda_i)$ or $\mathrm{L}(\Lambda)$ is amenable. Both of these cases are easily seen to lead to a contradiction. This finishes the proof of Theorem 7.1. □

Proof of Theorem D. Now let $\Gamma \curvearrowright (X, \mu)$ be any nonsingular free ergodic action on a standard measure space such that for all $i \in \{1, 2\}$, the restricted action $\Gamma_i \curvearrowright (X, \mu)$ is recurrent. Let $B = \mathrm{L}^\infty(X)$ and put $M = B \rtimes \Gamma$. Assume that $A \subset M$ is another Cartan subalgebra.

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Since $A, B \subset M$ are both tracial von Neumann subalgebras of M with expectation, we use Notation 2.9. Define $\mathrm{c}(B) = \mathrm{L}^\infty(X \times \mathbf{R})$ and consider the Maharam extension $\Gamma \curvearrowright \mathrm{c}(B)$ of the action $\Gamma \curvearrowright B$ so that we canonically have $\mathrm{c}(M) = \mathrm{c}(B) \rtimes \Gamma$. Observe that for all $i \in \{1, 2\}$, the action $\Gamma_i \curvearrowright \mathrm{c}(B)$ is still recurrent, so that $\mathrm{c}(B) \rtimes \Gamma_i$ is a type II von Neumann algebra.

Let $p \in \mathrm{Proj}_{\mathrm{f}}(\mathrm{c}(A))$. By [HV13, Lemma 2.1], there exist $q \in \mathrm{Proj}_{\mathrm{f}}(\mathrm{c}(B))$ and a partial isometry $v \in \mathrm{c}(M)$ such that $p = v^*v$ and $q = vv^*$. Observe that $v\mathrm{c}(A)v^* \subset q\mathrm{c}(M)q$ is still a Cartan subalgebra by [Pop06b, Lemma 3.5].

By Theorem 7.1, we get $v\mathrm{c}(A)v^* \preceq_{q\mathrm{c}(M)q} \mathrm{c}(B)q$. By Proposition 2.10, this implies that $A \preceq_M B$. Since M is a factor, by [HV13, Theorem 2.5], we get that there exists a unitary $u \in \mathcal{U}(M)$ such that $uAu^* = B$. This finishes the proof of Theorem D. □

8. AFP von Neumann algebras with many nonconjugate Cartan subalgebras

Connes and Jones exhibited in [CJ82] the first examples of II_1 factors M with at least two Cartan subalgebras which are not conjugate by an automorphism of M. More concrete examples were found by Ozawa and Popa in [OP10b].

Recently, Speelman and Vaes exhibited in [SV12] the first examples of group measure space II_1 factors $M = \mathrm{L}^\infty(Y) \rtimes \Lambda$ with uncountably many nonstably conjugate Cartan subalgebras. Recall from [SV12] that two Cartan subalgebras A and B of a II_1 factor N are *stably conjugate* if there exist nonzero projections $p \in A$ and $q \in B$ and a surjective $*$-isomorphism $\alpha : pNp \to qNq$ such that $\alpha(Ap) = Bq$. Put $\mathcal{N} = N \mathbin{\overline{\otimes}} \mathbf{B}(\ell^2)$, $\mathcal{A} = A \mathbin{\overline{\otimes}} \ell^\infty$ and $\mathcal{B} = B \mathbin{\overline{\otimes}} \ell^\infty$. Observe that $\mathcal{A}$ and $\mathcal{B}$ are Cartan subalgebras in the type II_∞ factor $\mathcal{N}$. Moreover, we have that A and B are stably conjugate in N if and only if $\mathcal{A}$ and $\mathcal{B}$ are conjugate in $\mathcal{N}$.

Let $\Lambda \curvearrowright (Y, \nu)$ be a probability measure preserving free ergodic action as in the statement of [SV12, Theorem 2] so that the corresponding group measure space II_1 factor $N = \mathrm{L}^\infty(Y) \rtimes \Lambda$ has uncountably many nonstably conjugate Cartan subalgebras.

Put $\Gamma = \Lambda * \mathbf{Z}$ and consider the *induced* action $\Gamma \curvearrowright (X, \mu)$ with $X = \mathrm{Ind}_\Lambda^\Gamma Y$. Observe that $\Gamma \curvearrowright (X, \mu)$ is an infinite measure preserving free ergodic action. Write $\mathcal{M} = \mathrm{L}^\infty(X) \rtimes \Gamma$ for the corresponding group measure space type II_∞ factor. Since $\Gamma = \Lambda * \mathbf{Z}$, we canonically have $\mathcal{M} = \mathcal{M}_1 *_{\mathcal{B}} \mathcal{M}_2$ with $\mathcal{B} = \mathrm{L}^\infty(X)$, $\mathcal{M}_1 = \mathcal{B} \rtimes \Lambda$ and $\mathcal{M}_2 = \mathcal{B} \rtimes \mathbf{Z}$. On the other hand, we also have

$$\mathcal{M} = (\mathrm{L}^\infty(Y) \rtimes \Lambda) \mathbin{\overline{\otimes}} \mathbf{B}(\ell^2(\Gamma/\Lambda)) = N \mathbin{\overline{\otimes}} \mathbf{B}(\ell^2(\Gamma/\Lambda)).$$

Therefore, we obtain the following result.

Theorem 8.1. *The amalgamated free product type II_∞ factor $\mathcal{M} = \mathcal{M}_1 *_{\mathcal{B}} \mathcal{M}_2$ has uncountably many nonconjugate Cartan subalgebras.*

This result shows that the condition in Theorem D imposing recurrence of the action $\Gamma_i \curvearrowright (X, \mu)$ for all $i \in \{1, 2\}$, is indeed necessary.

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References

Ada94 S. Adams, *Indecomposability of equivalence relations generated by word hyperbolic groups*, Topology **33** (1994), 785–798.

Alv10 A. Alvarez, *Théorème de Kurosh pour les relations d'équivalence boréliennes*, Ann. Inst. Fourier (Grenoble) **60** (2010), 1161–1200.

Ana95 C. Anantharaman-Delaroche, *Amenable correspondences and approximation properties for von Neumann algebras*, Pacific J. Math. **171** (1995), 309–341.

BO08 N. P. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, in *Graduate Studies in Mathematics*, Vol. 88 (American Mathematical Society, Providence, RI, 2008).

CH10 I. Chifan and C. Houdayer, *Bass–Serre rigidity results in von Neumann algebras*, Duke Math. J. **153** (2010), 23–54.

CS13 I. Chifan and T. Sinclair, *On the structural theory of* II_1 *factors of negatively curved groups*, Ann. Sci. Éc. Norm. Supér. (4) **46** (2013), 1–33.

CSU13 I. Chifan, T. Sinclair and B. Udrea, *On the structural theory of* II_1 *factors of negatively curved groups,* II. *Actions by product groups*, Adv. Math. **245** (2013), 208–236.

Con73 A. Connes, *Une classification des facteurs de type* III, Ann. Sci. Éc. Norm. Supér. (4) **6** (1973), 133–252.

Con76 A. Connes, *Classification of injective factors*, Ann. of Math. **104** (1976), 73–115.

Con94 A. Connes, *Noncommutative geometry* (Academic Press, San Diego, CA, 1994).

CFW81 A. Connes, J. Feldman and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergodic Theory Dynam. Systems **1** (1981), 431–450.

CJ82 A. Connes and V. F. R. Jones, *A* II_1 *factor with two non-conjugate Cartan subalgebras*, Bull. Amer. Math. Soc. **6** (1982), 211–212.

CT77 A. Connes and M. Takesaki, *The flow of weights on factors of type III*, Tohoku Math. J. **29** (1977), 473–575.

Dyk93 K. Dykema, *Free products of hyperfinite von Neumann algebras and free dimension*, Duke Math. J. **69** (1993), 97–119.

FT01 A. J. Falcone and M. Takesaki, *Non-commutative flow of weights on a von Neumann algebra*, J. Funct. Anal. **182** (2001), 170–206.

FM77 J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras.* I *and* II, Trans. Amer. Math. Soc. **234** (1977), 289–324; 325–359.

Gab00 D. Gaboriau, *Coût des relations d'équivalence et des groupes*, Invent. Math. **139** (2000), 41–98.

Hou10 C. Houdayer, *Strongly solid group factors which are not interpolated free group factors*, Math. Ann. **346** (2010), 969–989.

HR11 C. Houdayer and É. Ricard, *Approximation properties and absence of Cartan subalgebra for free Araki–Woods factors*, Adv. Math. **228** (2011), 764–802.

HV13 C Houdayer and S. Vaes, *Type* III *factors with unique Cartan decomposition*, J. Math. Pures Appl. (9) **100** (2013), 564–590.

Io12a A. Ioana, *Cartan subalgebras of amalgamated free product* II_1 *factors*, arXiv:1207.0054.

Io12b A. Ioana, *Classification and rigidity for von Neumann algebras*, in *Proceedings of the 6th European Congress of Mathematics (Krakow, 2012)* (European Mathematical Society, Zürich), to appear, arXiv:1212.0453.

IPP08 A. Ioana, J. Peterson and S. Popa, *Amalgamated free products of w-rigid factors and calculation of their symmetry groups*, Acta Math. **200** (2008), 85–153.

Jun07 K. Jung, *Strongly* 1*-bounded von Neumann algebras*, Geom. Funct. Anal. **17** (2007), 1180–1200.

Kes59 H. Kesten, *Full Banach mean values on countable groups*, Math. Scand. **7** (1959), 146–156.

Kri76 W. Krieger, *On ergodic flows and the isomorphism of factors*, Math. Ann. **223** (1976), 19–70.

Mah64 D. Maharam, *Incompressible transformations*, Fund. Math. **56** (1964), 35–50.

OP10a N. Ozawa and S. Popa, *On a class of* II_1 *factors with at most one Cartan subalgebra*, Ann. of Math. (2) **172** (2010), 713–749.

OP10b N. Ozawa and S. Popa, *On a class of* II_1 *factors with at most one Cartan subalgebra* II, Amer. J. Math. **132** (2010), 841–866.

Pop06a S. Popa, *On a class of type* II_1 *factors with Betti numbers invariants*, Ann. of Math. (2) **163** (2006), 809–899.

Pop06b S. Popa, *Strong rigidity of* II_1 *factors arising from malleable actions of w-rigid groups* I, II, Invent. Math. **165** (2006), 369–408; 409–451.

Pop07 S. Popa, *Deformation and rigidity for group actions and von Neumann algebras*, in *Proceedings of the International Congress of Mathematicians (Madrid, 2006),* Vol. I (European Mathematical Society, Zürich, 2007), 445–477.

Pop08 S. Popa, *On the superrigidity of malleable actions with spectral gap*, J. Amer. Math. Soc. **21** (2008), 981–1000.

PV11 S. Popa and S. Vaes, *Unique Cartan decomposition for* II_1 *factors arising from arbitrary actions of free groups*, Acta Math., to appear, arXiv:1111.6951.

PV12 S. Popa and S. Vaes, *Unique Cartan decomposition for* II_1 *factors arising from arbitrary actions of hyperbolic groups*, J. Reine Angew. Math., to appear, arXiv:1201.2824.

Shl00 D. Shlyakhtenko, *Prime type* III *factors*, Proc. Natl. Acad. Sci. USA **97** (2000), 12439–12441.

SV12 A. Speelman and S. Vaes, *A class of* II_1 *factors with many non conjugate Cartan subalgebras*, Adv. Math. **231** (2012), 2224–2251.

Tak02 M. Takesaki, *Theory of operator algebras.* I, in *Encyclopaedia of mathematical sciences, Vol. 124*, Operator Algebras and Non-commutative Geometry, vol. 5 (Springer, Berlin, 2002).

Tak03 M. Takesaki, *Theory of operator algebras.* II, in *Encyclopaedia of mathematical sciences,* Vol. 125, Operator Algebras and Non-commutative Geometry, vol. 6 (Springer, Berlin, 2003).

Ued99 Y. Ueda, *Amalgamated free products over Cartan subalgebra*, Pacific J. Math. **191** (1999), 359–392.

Ued11 Y. Ueda, *Factoriality, type classification and fullness for free product von Neumann algebras*, Adv. Math. **228** (2011), 2647–2671.

Ued13 Y. Ueda, *Some analysis on amalgamated free products of von Neumann algebras in non-tracial setting*, J. Lond. Math. Soc. **88** (2013), 25–48.

Vae07 S. Vaes, *Rigidity results for Bernoulli actions and their von Neumann algebras (after S. Popa)*, Séminaire Bourbaki, exposé 961. Astérisque **311** (2007), 237–294.

Vae08 S. Vaes, *Explicit computations of all finite index bimodules for a family of* II_1 *factors*, Ann. Sci. Éc. Norm. Supér. (4) **41** (2008), 743–788.

Vae10 S. Vaes, *Rigidity for von Neumann algebras and their invariants*, in *Proceedings of the International Congress of Mathematicians (Hyderabad, 2010),* Vol. III (Hindustan Book Agency, New Delhi, 2010), 1624–1650.

Vae13 S. Vaes, *One-cohomology and the uniqueness of the group measure space decomposition of a* II_1 *factor*, Math. Ann. **355** (2013), 661–696.

Voi85 D.-V. Voiculescu, *Symmetries of some reduced free product* C^**-algebras.*, in *Operator algebras and their connections with topology and ergodic theory*, Lecture Notes in Mathematics, vol. 1132 (Springer, Heidelberg, 1985), 556–588.

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VDN92 D.-V. Voiculescu, K. J. Dykema and A. Nica, *Free random variables*, CRM Monograph Series 1 (American Mathematical Society, Providence, RI, 1992).

Voi96 D.-V. Voiculescu, *The analogues of entropy and of Fisher's information measure in free probability theory.* III. *The absence of Cartan subalgebras*, Geom. Funct. Anal. **6** (1996), 172–199.

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