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A KARAMATA METHOD I. ELEMENTARY PROPERTIES AND APPLICATIONS

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ABSTRACT. In this paper we present a new approach to classical Karamata's results concerning the Hardy-Littlewood tauberian theorem.

Introduction

In his proof of the Hardy-Littlewood theorem Karamata [4] used a very simple method, which works in a more general setting. This generalized theorem (called here Karamata's theorem) can be applied to obtain certain results on mean values of some arithmetical functions studied in the number theory. As an example we give a new analytic proof of the prime number theorem. It is short and requires only some simple estimates for the Riemann zeta function.

H. Weyl and J. Karamata brought attention to the role the approximation lemma (Lemma 1.3 below is its more general version) plays in the study of Riemann integrable functions. It shows how the Lebesgue integral allows us to get past the problems of Riemann's integral theory. By this lemma we obtain the Proposition 1.4 which yields some further applications.

Unfortunately the asymptotic formulas we get here give no estimates for the remainder. However, our method has an advantage of being simple and widely applicable.

1. **Karamata's tauberian theorem.** Let $(\lambda_n)_{n\in\mathbf{N}}$ be an increasing sequence of nonnegative real numbers. For a fixed sequence (λ_n), we define a class $\mathbf{E}$ of nonnegative arithmetical functions:

$\mathbf{E} := \{a : \mathbf{N} \longrightarrow [0,+\infty)$: for every $s \in \mathbf{C}$ with Re $(s) > 0$ the series $\sum_i^\infty a(n)e^{-\lambda} n^s$ converges and its sum is an analytic function $d(s)$, such that for every $p \in \mathcal{P}$ there exists a limit

$$c_\alpha(p) := \lim_{\sigma\to 0+} d(p\sigma)/d(\sigma)\};$$

here $\mathcal{P}$ denotes the set of all prime numbers.

LEMMA 1.1. *We have* $\mathbf{E} = \bigcup_{0\leqq\alpha\leqq+\infty} \mathbf{E}_\alpha$, *where, for* $0 \leqq \alpha \leqq +\infty$, *we put*

$$\mathbf{E}_\alpha := \{a \in \mathbf{E} : c_\alpha(n) := \lim_{\alpha\to 0+} d(n\sigma)/d(\sigma) = n^{-\alpha} \text{ for all } n \in \mathbf{N}\}.$$

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PROOF. Fix $a \in \mathbf{E}$. Let $n = p^k$, where $p \in \mathcal{P}, k \in \mathbf{N}$. Then

$$(c_\alpha(p))^k = \cdot \lim_{\sigma \to 0+} \frac{d(p^k\sigma)}{d(p^{k-1}\sigma)} \cdot \frac{d(p^{k-1}\sigma)}{d(p^{k-2}\sigma)} \cdot \dots \cdot \frac{d(p\sigma)}{d(\sigma)}$$
$$= \lim_{\sigma \to 0+} \frac{d(n\sigma)}{d(\sigma)} = c_\alpha(n).$$

Now, let $n = n_1 \cdot \dots \cdot n_m$ with $n_k = p_k^{l_k}, p_k \in \mathcal{P}, l_k \in \mathbf{N}$. Then

$$c_\alpha(n_1) \cdot \dots \cdot c_\alpha(n_m) = \lim_{\sigma \to 0+} \frac{d(n_1 \cdot \dots \cdot n_m\sigma)}{d(n_1 \cdot \dots \cdot n_{m-1}\sigma)} \cdot \dots \cdot \frac{d(n_1\sigma)}{d(\sigma)}$$
$$= \lim_{\sigma \to 0+} \frac{d(n\sigma)}{d(\sigma)} = c_\alpha(n),$$

and for $n, m \in \mathbf{N}, c_\alpha(n)c_\alpha(m) = c_\alpha(nm)$. Moreover, $c_\alpha(n+1) \leqq c_\alpha(n)$. By a theorem of Erdös [1], there exists $\beta \in [-\infty, +\infty)$ such that $c(n) = n^\beta$. Since $c_\alpha(n) \leqq 1$, we have $\beta = -\alpha$, where $\alpha \in [0, +\infty]$. Thus, $a \in \mathbf{E}_\alpha$ and $\mathbf{E} \subset \bigcup \mathbf{E}_\alpha$, which ends the proof. ∎

Fix now $a \in \mathbf{E}$. Let $h_\alpha(t) = 1/\Gamma(\alpha) \int_0^t (\log(1/\tau))^{\alpha-1} d\tau$ for $\alpha \in (0, +\infty), t \in [0, 1]$ and $h_{+\infty}(0) = 0, h_{+\infty}(t) = 1$ for $t \in (0, 1], h_0(t) = 0$ for $t \in [0, 1), h_0(1) = 1$. Define a linear functional $l : \mathbf{R}[x] \longrightarrow \mathbf{R}$ by $l(\sum_0^n a_k x^k) = \sum_0^n a_k c_a(k+1)$. As an immediate consequence of the formula $k^{-\alpha} = \int_0^1 t^{k-1} dh_\alpha$, we obtain the following

PROPOSITION 1.2. *If $a \in \mathbf{E}_\alpha$, then $l(p) = \int_0^1 p\, dh_\alpha$ for $p \in \mathbf{R}[x]$.*

Define $\mathbf{K}_\alpha = \mathbf{K}_\alpha(a)$, to be the set of all functions $f : [0, 1) \longrightarrow \mathbf{C}$ such that there exists

$$L(f) := \lim_{\sigma \to 0+} 1/d(\sigma) \sum_i^\infty a(n) f(e^{-\lambda_n\sigma}) e^{-\lambda_n\sigma}.$$

Observe that if $p \in \mathbf{R}[x]$, then $L(p) = 1(p)$. The crucial role in our considerations is played by the following approximation lemma.

LEMMA 1.3. *Let f: $[0, 1) \longrightarrow \mathbf{R}$ be a bounded function which is continuous almost everywhere in $[0, 1)$ (with respect to the Lebesgue measure). Then for every $\epsilon > 0$ there exist $p, q \in \mathbf{R}[x]$ such that $p \leqq f \leqq q$ and $l(q - p) \leqq \epsilon$ (if $\alpha = +\infty$ or $\alpha = 0$, then we assume that there exists $f(0+)$ or $f(1-)$, respectively).*

SKETCH OF THE PROOF. We shall restrict ourselves only to the case of $0 < \alpha < +\infty$. Let f be a bounded function, integrable in the sense of Riemann on the interval [0,1). Let f^* and f_* denote the upper regularization and the lower regularization of the function f, respectively. We have $f_* \leqq f \leqq f^*$ and $f_* = f^*$ almost everywhere. Since the function f_* is lower-semicontinuous and since the function f^* is upper-semicontinuous, there exist sequences f_k and g_k of continuous functions on the interval [0,1] such that $f_k \nearrow f_*$ and $g_k \searrow f^*$. The lemma now follows from the Lebesgue monotonic convergence theorem and the Weierstrass approximation theorem. ∎

From the above lemma we obtain the following important

PROPOSITION 1.4 (SEE [4]). *If $f\colon [0,1) \longrightarrow \mathbf{C}$ is a function such that* Re (f) *and* Im (f) *satisfy the assertion of Lemma* 1.3, *then for every α, $f \in \mathbf{K}_\alpha$ and $L(f) = \int_0^1 f\, dh_\alpha$.*

By Proposition 1.4, we get the basic Karamata tauberian theorem.

THEOREM 1.5 (KARAMATA). *If $a \in \mathbf{E}_\alpha$ for some α, then*

$$s(N) := \sum_{n \leqq N} a(n) = (1/\Gamma(1+\alpha) + o(1))d(1/\lambda_N), \text{ as } N \longrightarrow +\infty.$$

PROOF. (Karamata [4]). Consider the function $f(x) = 0$ for $x \in [0, 1/e)$, and $f(x) = 1/x$ for $x \in [1/e, 1)$. Then, by Proposition 1.4 we get $L(f) = 1/\Gamma(1+\alpha)$ and

$$s(N) = \sum_1^\infty a(n) f(e^{-\lambda_n \sigma}) e^{-\lambda_n \sigma} = (1/\Gamma(1+\alpha) + o(1))d(1/\lambda_N),$$

for $\sigma = 1/\lambda_N$. ∎

2. Let $f\colon [0,1) \longrightarrow \mathbf{C}$ be a given function. It is interesting to know whether $f \in K_\alpha$. In particular, under what conditions does an unbounded function belong to $\mathbf{K}_\alpha$? In this section we give a partial answer to these questions.

At first, observe that for every $t \geqq 1$ the function $f(x) = x^{t-1} \in \mathbf{K}_\alpha$ for all α. Thus we derive the following

PROPOSITION 2.1. *If $a \in \mathbf{E}_\alpha$ then for every $t \geqq 1$*

$$\lim_{\sigma \to 0+} d(t\sigma)/d(\sigma) \equiv t^{-\alpha}. \tag{1}$$

Let us define a function $c\colon [1, +\infty) \longrightarrow [0, 1]$ by

$$c(t) = \sup \{ d(t\sigma)/d(\sigma) : \sigma \in (0, +\infty) \}.$$

PROPOSITION 2.2. *The function $c(t)$ has the following properties*

(a) $c(t) \equiv 1$ for $\alpha = 0$ and for $\lambda_1 = 0$;

(b) $c(t_1 t_2) \leqq c(t_1)c(t_2)$;

(c) $c(t) \leqq (c(t\gamma p))^{1/p}(c(t\delta q))^{1/q}$, where $\gamma, \delta, p, q > 0, \gamma + \delta = 1$, $1/p + 1/q = 1$, $t\gamma p, t\delta q \geqq 1$;

(c′) $c(t) \leqq (c(1 + (t-1)p))^{1/p}$ for $p \geqq 1$.

PROOF. It is easy to prove (a) and (b) and therefore we omit the details. (c') is a conseqence of (c). In order to prove (c) observe that, by Hölder's inequality, we have

$$\begin{aligned} d(t\sigma) &= \sum_1^\infty (a(n))^{1/p} e^{-\lambda_n \gamma t\sigma} (a(n))^{1/q} e^{-\lambda_n \delta t\sigma} \\ &\leqq (d(t\gamma p\sigma))^{1/p} (d(t\delta q\sigma))^{1/p} \end{aligned}$$

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whence we obtain (c).

Define now $\lambda = \sup\{\log c(t)/\log t : t > 1\}$. Since $\log c(e^t)$ is a subadditive function, it is easy to check (see [5] p. 410) that

$$\lambda = \lim_{t \to 1+} \log c(t)/\log t.$$

By Proposition 2.2 (c'), we get

$$\lambda = \inf\{\log c(t)/(t-1) : t > 1\} \leqq \log c(2). \tag{2}$$

Let

$$\mu = \limsup_{t \to +\infty} \log c(t)/\log t$$

and $\beta = -\mu$. From (1) and (2) we get, for $0 < \alpha < +\infty$ and $\lambda_1 > 0 : 0 < -\lambda \leqq \beta \leqq \alpha$. ∎

PROBLEM 2.3. Is it true that $\mu = -\alpha$ for $0 < \alpha < +\infty$ and $\lambda_1 > 0$?

EXAMPLE 2.4. Let $\lambda_n = n$ and $0 < \alpha \leqq 1$. If $a \in \mathbf{E}_\alpha$ and the series

$$\sum_1^\infty a(n)e^{-n\sigma}$$

has the form:

$$(1 - e^{-\sigma})^{-\alpha} \sum_1^\infty b(n)e^{-n\sigma},$$

where $b(n) \geqq 0$, then $\beta = \alpha$.

It is easy to verify the following

PROPOSITION 2.5. *Let $f_k \in \mathbf{K}_\alpha$ for $k \in \mathbf{N}$. Assume that the series*

$$\sum_{k=1}^\infty \sup_\sigma 1/d(\sigma) \sum_{n=1}^\infty a(n)\,|f_k|(e^{-\lambda_n\sigma})e^{-\lambda_n\sigma}$$

converges. Then $f \in \mathbf{K}_\alpha$ and

$$L(f) = \sum_1^\infty L(f_k).$$

COROLLARY 2.6. *Let*

$$f(t) = \sum_1^\infty b_n t^{n-1} \text{ for } t \in [0,1].$$

If the series

$$\sum_1^\infty |b_n| c(n)$$

converges, then $f \in \mathbf{K}_\alpha$ and

$$L(f) = \int_0^1 f\, dh_\alpha = \sum_1^\infty b_n n^{-\alpha}.$$

COROLLARY 2.7. *If* $\omega < \beta$ *then* $(1-t)^{-\omega} \in \mathbf{K}_\alpha$.

PROOF. We have

$$(1-t)^{-\omega} = \sum_1^\infty \binom{-\omega}{n-1}$$

and

$$\binom{-\omega}{n-1} = O(n^{\omega-1}).$$

Hence $\binom{-\omega}{n-1} n^{-\beta} = O(n^{-1-(\beta-\omega)})$, which implies that the series

$$\sum_1^\infty \left|\binom{-\omega}{n-1}\right| c(n)$$

converges. ∎

EXAMPLE 2.8. If $\lambda_n = n$ and $a(n) \in \mathbf{E}_\alpha, \beta > 1$, then

$$\sum_1^\infty a(n) \frac{x^n}{1-x^n} \sim \zeta(\alpha) \sum_1^\infty a(n) x^n \text{ as } x \longrightarrow 1-.$$

COROLLARY 2.9. *Let* $\omega < \beta$. *If* $1 \gg (1-t)^\omega f(t) \in \mathbf{K}_\alpha$, *then* $f \in \mathbf{K}_\alpha$ *and*

$$L(f) = \int_0^1 f\, dh_\alpha.$$

As an application, we obtain the following interesting

EXAMPLE 2.10. Let $s \in \mathbf{C}$ and Re $(s) < \beta$. If $\lambda_1 > 0$ then, for $N \longrightarrow +\infty$

$$\sum_{n \leqq N} a(n)\lambda_n^{-s} = (\alpha(\alpha - s)^{-1} + o(1))\lambda_N^{-s} \sum_{n \leqq N} a(n).$$

PROOF. Consider the function

$$f(t) = \frac{1}{t}\left(\log \frac{1}{t}\right)^{-s} \chi_{[1/e,1)}(t).$$

We have $O(1) = (1-t)^c f(t) \in \mathbf{K}_\alpha$, where $c = \max\,(O, \text{Re}\,(s))$, and we can apply Corollary 2.9. ∎

By substituting different functions f one can obtain other interesting asymptotic formulas. Take, for example, f to be the function

$$f(t) = \frac{1}{t}\{g(t)\}\chi_{[1/e,1)}(t),$$

where

$$g(t) = 1/\log \frac{1}{t} \quad \text{and} \quad \{x\} = x - [x].$$

By Proposition 1.4, we get the following important formula.

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PROPOSITION 2.11. *If $\lambda_1 > 0$ then, for $N \longrightarrow +\infty$, we have (for $0 < \alpha < +\infty$)*

$$\lambda_N \sum_{n \leqq N} a(n)/\lambda_n - \sum_{n \leqq N} a(n)[\lambda_N/\lambda_n] = (C(\alpha) + o(1)) \sum_{n \leqq N} a(n),$$

where $C(\alpha) = 1 + (\alpha - 1)^{-1} - \zeta(\alpha)$, for $\alpha \neq 1$ and $C(1) = 1 - \gamma$; γ denoting the Euler constant.

Finally, we give a generalization of results obtained by Kalecki [3], Mercier and Nowak [7] and Mercier [6].

PROPOSITION 2.12. *Let $f \colon [1, +\infty) \longrightarrow \mathbf{R}$ be a measurable and continuous almost everywhere function, $k, \alpha \in (0, +\infty)$, and $g \in \mathbf{E}_\alpha$. Then*

$$\sum_{n \leqq N} g(n) \left\{ f\left(\frac{N}{n}\right) \right\}^k = (C + o(1)) \sum_{n \leqq N} g(n) \text{ as } N \longrightarrow +\infty,$$

where

$$C = 1/\Gamma(\alpha) \int_1^\infty \{f(t)\}^k t^{-1-\alpha} dt.$$

3. **Some applications to number theory.** We consider two cases; $\lambda_n = \log n$ and $\lambda_n = n$.

EXAMPLE 3.1. Using Euler's identity (for $\sigma > 1$)

$$\zeta(\sigma) = \prod_p (1 - p^{-\sigma})^{-1}$$

gives

$$\log \zeta(\sigma) = -\sum_p \log(1 - p^{-\sigma}) = \sum_p \sum_{k=1}^{\infty} p^{-k\sigma}/k$$

$$= \sum_p p^{-\sigma} + \sum_p \sum_{k=2}^{\infty} p^{-k\sigma}/k = \sum_p p^{-\sigma} + O(1).$$

Since $\zeta(1 + \sigma) \sim 1/\sigma$ as $\sigma \longrightarrow 0+$, we have $\sum_p p^{-(1+\sigma)} = \log(1/\sigma) + O(1)$ and $a(n) = (1/n)\chi_{\mathcal{P}}(n) \in \mathbf{E}_0$ (for $\lambda_n = \log n$). Hence, by Theorem 1.5, we get $\sum_{p \leqq N} 1/p = (1 + o(1)) \log \log N$, as $N \longrightarrow +\infty$.

EXAMPLE 3.2. We have

$$-\zeta'/\zeta(\sigma) = \sum_1^\infty \Lambda(n) n^{-\sigma} = \sum_p \log p p^{-\sigma} + \sum_p \sum_{k=2} \log p p^{-k\sigma}$$

$$= \sum_p \log p p^{-\sigma} + O(1).$$

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Since $\zeta'(\sigma) \sim -(\sigma - 1)^{-2}$ as $\sigma \longrightarrow 1+$, we have $\sum_p (\log p)p^{-(1+\sigma)} \sim 1/\sigma$ as $\sigma \longrightarrow 0+$. Thus, $1/n \log n\chi_{\mathcal{P}}(n) \in \mathbf{E}_1$ and $\Lambda(n)/n \in \mathbf{E}_1$. By Theorem 1.5 we get the following asymptotic formulas

$$\sum_{p \leqq N} \log p/p = (1 + o(1)) \log N = \sum_{n \leqq N} \Lambda(n)/n.$$

Note that these simple formulas follow immediately from the asymptotic properties of the Riemann zeta function for $s > 1$.

By Karamata's theorem, we can make use of analytic properties of Dirichlet's series in order to obtain some information on its coefficients. This is a consequence of the following remark:

If a Dirichlet series $\sum_1^\infty a(n)n^{-s}$ with nonnegative coefficients converges in the halfplane Re $(s) > \alpha > 0$, then we can write

$$\Gamma(s) \sum_1^\infty a(n)n^{-s} = \int_0^\infty x^{s-1} \sum_1^\infty a(n)e^{-nx}\, dx.$$

Using now the inverse transformation to the Mellin transformation, we obtain for $x > 0$.

$$\sum_1^\infty a(n)e^{-nx} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(s) \sum_1^\infty a(n)e^{-nx}\, dx. \tag{3}$$

Making use of analytic properties of the sum of the series $\sum_1^\infty a(n)n^{-s}$ we can show that $a(n) \in \mathbf{E}_\alpha$, and by Theorem 1.5, we can obtain an asymptotic formula for $s(N)$.

EXAMPLE 3.3. (Prime number theorem). Consider the series $\sum_1^\infty \Lambda(n)n^{-s}$, which converges for Re $(s) = \sigma > 1$. By formula (3), for $x > 0, \sigma > 1$, we have

$$\sum_1^\infty \Lambda(n)e^{-nx} = -\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma - i\infty} \zeta'/\zeta(s)x^{-s}\, ds.$$

By a simple analytic property of the Riemann zeta function, for $\sigma > 1$, we have

$$\zeta(s) = 1 + (s-1)^{-1} - s\int_1^\infty (y - [y])y^{-s-1}\, dy.$$

This implies that ζ is a meromorphic function in the halfplane Re $(s) > 0$ and the functions $g(s) = (s-1)\zeta(s)$ and $h(s) = (s-1)^{-1} + \zeta'/\zeta(s)$ are holomorphic in this domain. Moreover, we have the obvious estimates

$$|\zeta(\sigma + it)| \leqq |t| + 2, \; |\zeta'(\sigma + it)| \leqq |t| + 2 \text{ for } |t| \geqq 1, \; \sigma \geqq 1.$$

By an elementary inequality (see [8]),

$$5 + 8\cos\phi + 4\cos 2\phi + \cos 3\phi = (1 + \cos\phi)(1 + 2\cos\phi)^2 \geqq 0$$

whence we deduce that $\zeta^5(\sigma)|\zeta(\sigma + it)|^8|\zeta(\sigma + 2it)|^4|\zeta(\sigma + 3it)| \geqq 1$ for $\sigma \geqq 1$.

This implies that $\zeta(s) \neq 0$ for $\sigma \geqq 1$ and we obtain a uniform estimate of $1/\zeta(s)$. We have $|\zeta(\sigma+it)|^{-1} \leqq \zeta^{5/8}(\sigma)|\zeta(\sigma+2it)|^{1/2}|\zeta(\sigma+3it)|^{1/8}$,

$$\begin{aligned} 1/\zeta(\sigma+it) &= 1/\zeta(\sigma+1+it) + \int_{\sigma}^{\sigma+1} \zeta'/\zeta^2(r+it)\,dr \\ &= O(|t|^{1+1+1/4}\int_{\sigma}^{\sigma+1}(r-1)^{-1/2}\,dr) = O(|t|^{9/4}(\sigma-1)^{-1/4}), \end{aligned} \tag{2}$$

uniformly for $|t| \geqq 1, \sigma > 1$. Using again (4), we get

$$1/\zeta(\sigma+it) = O(|t|^{11/2}\int_{\sigma}^{\sigma+1}(r-1)^{-1/2}\,dr) = O(|t|^{11/2}) = O(|t|^6)$$

uniformly for $|t| \geqq 1, \sigma > 1$ and $\sigma \geqq 1$. Thus we have the estimate $h(\sigma+it) = O(|t|^7)$. By this result and the equality $\Gamma(\sigma+it) = O((1+|t|)^{-k})$ for every fixed $k \in \mathbf{N}$, uniformly for $1 \leqq \sigma \leqq \sigma_1$, we can write:

$$\begin{aligned} \sum_1^{\infty} \Lambda(n)e^{-nx} &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)(s-1)^{-1}x^{-s}\,ds - \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)h(s)x^{-s}\,ds \\ &= e^{-x}/x - \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\Gamma(s)h(s)x^{-s}\,ds = e^{-x}/x + I(x). \end{aligned}$$

By the Riemann-Lebesgue lemma we get $I(x) = o(1/x)$, whence

$$\sum_1^{\infty}\Lambda(n)e^{-nx} = (1+o(1))/x$$

as $x \longrightarrow O+$ and $\Lambda(n) \in \mathbf{E}_1$ (for $\lambda_n = n$). Now, by Theorem 1.5 we obtain

$$\sum_{n \leqq N}\Lambda(n) = (1+o(1))N \text{ as } N \longrightarrow +\infty.$$

REMARK 3.4. It is well-known that the last formula is equivalent to the relationships

$$\sum_{p \leqq N}\log p \sim N$$

or

$$\sum_{p \leqq N} 1 \sim N/\log N. \tag{5}$$

Now, it is easy to verify that (5) is equivalent to the formula

$$\sum_p e^{-p\sigma} = (1+o(1))\left(\sigma\log\frac{1}{\sigma}\right)^{-1} \text{ as } \sigma \longrightarrow O+.$$

Then

$$\left(\sum_p e^{-p\sigma}\right)^2 = \sum_n p(n)e^{-n\sigma} = (1+o(1))\left(\sigma\log\frac{1}{\sigma}\right)^{-2} \text{ as } \sigma \longrightarrow O+,$$

where

$$p(n) = \sum_{\substack{p_1+p_2=n,\\ p_1,p_2\in P}} 1.$$

Thus, we have

$$\sum_n p(2n)e^{-n\sigma} = (1+o(1))4\left(\sigma \log \frac{1}{\sigma}\right)^{-2}$$

and $p(2n) \in \mathbf{E}_2(\lambda_n = n)$. By the Karamata theorem we obtain the formula

$$\sum_{n\leqq N} p(2n) = (1+o(1))2N^2/\log^2 N.$$

From this one can deduce that

$$\liminf_{n\to\infty} p(2n)\log^2 n/4n \leqq 1 \leqq \limsup_{n\to\infty} p(2n)\log^2 n/4n.$$

EXAMPLE 3.5. Determine positive numbers $a(n)$ to satisfy the equality

$$\prod_2^\infty (1-n^{-s})^{-1} = \sum_1^\infty a(n)n^{-s}, \ \mathrm{Re}\,(s) > 1.$$

Then we get the asymptotic formula

$$\sum_{n\leqq N} a(n) = (1/2\sqrt{\pi}+o(1))\, N \exp\,(2\log^{1/2} N)\log^{-3/4} N$$

(we omit calculations).

Finally, we give certain applications of Proposition 2.11.

If a, b: $\mathbf{N} \longrightarrow \mathbf{C}$ are arithmetical functions, then it is easy to check the following identity:

$$\sum_{n\leqq N} a(n) \sum_{m\leqq N/n} b(mn) = \sum_{n\leqq N} b(n)\sum_{d|n} a(d).$$

In particular, if $b \equiv 1$, then, by setting

$$A(n) = \sum_{d|n} a(d) = (a*1)(n)$$

(here "$*$" denotes the Dirichlet convolution) we get

$$\sum_{n\leqq N} A(n) = \sum_{n\leqq N} a(n)[N/n].$$

Now, let $a \in \mathbf{E}_\alpha$, where $O < \alpha < +\infty$. By Proposition 2.11 we get the formula

$$\sum_{n\leqq N} A(n) = N\sum_{n\leqq N} a(n)/n - C(\alpha)\sum_{n\leqq N} a(n) + o\left(\sum_{n\leqq N} a(n)\right).$$

In the special case of $\alpha = 1$ we have

$$\sum_{n\leqq N} A(n) = N\sum_{n\leqq N} a(n)/n + (\gamma-1)\sum_{n\leqq N} a(n) + o\left(\sum_{n\leqq N} a(n)\right).$$

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For $a \equiv 1$, we get $\sum_{n \leqq N} d(n) = N \log N + (2\gamma - 1)N + o(N)$.

Let $\Lambda(n) = \log p$ if $n = p^k, p \in P$, and $\Lambda(n) = 0$ otherwise or equivalently

$$-\zeta'/\zeta(s) = \sum_1^\infty \Lambda(n) n^{-s}.$$

We have

$$\sum_{d|n} \Lambda(d) = \log n$$

and

$$\sum_{n \leqq N} \log n = N \log N - N + o(N).$$

Hence, we obtain

$$\sum_{n \leqq N} \Lambda(n)/n = \log N - \gamma + o(1).$$

Since

$$\sum_{k=2}^\infty \sum_p \log p p^{-k} = \sum_p \log p/p(p-1) < \infty,$$

then

$$\sum_{p \leqq N} \log p/p = \log N + C + o(1),$$

where

$$C = -\gamma - \sum_p \log p/p(p-1).$$

Using the method of summation by parts, we get

$$\sum_{p \leqq N} 1/p = \log \log N + B + o(1/\log N),$$

where B is a constant

$$\left(\text{equal to } \gamma + \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right)\right).$$

Since $\chi_p(n) \in \mathbf{E}_1$, we get

$$\sum_{n \leqq N} \sum_{p|n} 1 = \sum_{n \leqq N} \omega(n) = N \sum_{p \leqq N} 1/p + (\gamma - 1) \sum_{p \leqq N} 1 + o\left(\sum_{p \leqq N} 1\right)$$
$$= N \log \log N + BN + (\gamma - 1)N/\log N + o(N/\log N).$$

This means that

$$\sum_{n \leqq N} \omega(n) = N \log \log N + BN + (\gamma - 1)N/\log N + o(N/\log N).$$

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