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AN n-VERTEX THEOREM FOR CONVEX SPACE CURVES

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The classical four-vertex theorem states that a simple closed convex C^2 curve in the Euclidean plane has at least four vertices (points of extreme curvature). This theorem has many generalizations with regard to both the curve and the topological space and for a history of the subject, we refer to [**4**] and [**1**]. The particular generalization of concern, credited to H. Mohrmann, is the following n-vertex theorem.

Let a simple closed C^3 curve on a closed convex surface be intersected by a suitable plane in n points. Then the curve has at least n inflections (vertices).

The closed convex surface in the preceding is defined as having at most two points in common with any straight line. Presently, we extend this result to curves on more general convex surfaces in a real projective three-space P^3. A convex space curve Γ lies on the boundary of its convex hull (in a suitable affine restriction of P^3) and meets any line in at most two points. The curve Γ is then inflectional if the only exceptional points of Γ are inflection points and a point $p \in \Gamma$ is an inflection if near p, Γ lies on one side of the osculating plane at p.

In Section 1, we present basic definitions and an overview of the proof of the n-vertex theorem for inflectional convex space curves. In Section 2, we present the proof.

1. Preliminaries. Let $p, q, \ldots, L, M, \ldots$ and $\alpha, \beta, \ldots$ denote the points, lines and planes of P^3 respectively. Let $\langle p, L, \alpha, \ldots \rangle$ denote the flat of P^3 spanned by $p, L, \alpha, \ldots$. We assume that P^3 has the usual topology.

Let $T \subset P^3$ be an oriented line. For $r \neq t$ in T, let $[r, t]$ and (r, t) denote respectively the closed and open oriented segments of T from r to t. We put

$$[r, t) = [r, t]\backslash\{t\} \quad \text{and} \quad (r, t] = [r, t]\backslash\{r\}.$$

Let $U(s) = (r, t)$ be a neighbourhood of s in T. We set $U^-(s) = (r, s)$, $U^+(s) = (s, t)$ and $U'(s) = U^-(s) \cup U^+(s)$. Finally if $(s, t) \subset (r, u)$, we write $s < t$ in (r, u).

A *curve* Γ is a continuous map from T into P^3. A line, denoted by $\Gamma_1(t)$, is the *tangent* of Γ at t if

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$$\Gamma_1(t) = \lim_{t \neq s \to t} \langle \Gamma(t), \Gamma(s) \rangle$$

and a plane, denoted by $\Gamma_2(t)$, is the *osculating plane* of Γ at t if

$$\Gamma_2(t) = \lim_{t \neq s \to t} \langle \Gamma_1(t), \Gamma(s) \rangle.$$

For $t \in T$, we also set $\Gamma_0(t) = \Gamma(t)$ and $\Gamma_3(t) = P^3$. If $\mathscr{M} \subseteq T$ is a segment, we call $\Gamma/_{\mathscr{M}}$ an *arc* of Γ. For convenience, we identify $\Gamma(T)$ with Γ and $\Gamma(\mathscr{M})$ with $\Gamma/_{\mathscr{M}}$. Γ is of course *simple* if $\Gamma(s) \neq \Gamma(t)$ for $s \neq t$ in T.

A *space curve* Γ has the property that $\Gamma_1(t)$ and $\Gamma_2(t)$ exist for each $t \in T$ and any plane meets Γ at a finite number of points. Henceforth Γ is a space curve.

Let $t \in T$ and $\alpha \subset P^3$. Then $|\alpha \cap \Gamma| < \infty$ and it is immediate that near $\Gamma(t)$, either Γ lies on one side of α or Γ lies on both sides of α. In case of the former [latter], we say that α *supports* [*cuts*] Γ at t. Let

$$S_i(t) = \{\alpha \subset P^3 | \alpha \cap \Gamma_{i+1}(t) = \Gamma_i(t)\}; \quad i = 0, 1, 2.$$

Then (cf. [**5**]) either all $\alpha \in S_i(t)$ support Γ at t or all $\alpha \in S_i(t)$ cut Γ at t; $i = 0, 1, 2$. We say that Γ is *inflectional* if for any $t \in T$, $\alpha \in S_0(t)$ cuts Γ at t and $\alpha \in S_1(t)$ supports Γ at t. Such a curve Γ may thus contain only two types of points $\Gamma(t)$: *regular* ($\Gamma_2(t)$ cuts Γ at t) and *inflection* ($\Gamma_2(t)$ supports Γ at t). In addition, an arc of Γ is *regular* if each of its points is regular.

An arc $\Gamma(r, t)$ is said to be of *order three*, ord $\Gamma(r, t) = 3$, if any plane meets $\Gamma(r, t)$ in at most three points. It is well known that if $\Gamma(r, t)$ is an arc of order three then $\Gamma(r, t)$ is regular and

$$\Gamma_2(s) \cap \Gamma(r, t) = \{\Gamma(s)\} \quad \text{for } s \in (r, t).$$

We now assume that Γ is *elementary* (well-behaved); that is, for each $\Gamma(t)$ there exists $\Gamma(U^-(t))$ and $\Gamma(U^+(t))$ such that both are of order three. We note (cf. [**3**]) that an elementary curve Γ has continuous tangents and continuous osculating planes, each of its regular points has a neighbourhood of order three and if in addition Γ is inflectional, then Γ has an even number of inflections.

Let $\mathscr{R}$ be a subset of P^3 disjoint from a plane β. Then $\mathscr{R}$ is a subset of the affine space $A^3 = P^3 \backslash \beta$ and we denote by $H(\mathscr{R})$, the convex hull of $\mathscr{R}$ in A^3.

Let $\beta \subset P^3$ and $\Gamma: T \to A^3 = P^3 \backslash \beta$ be a space curve with $B = H(\Gamma)$ in A^3. Then Γ is *convex* if Γ lies on the boundary $\partial(B)$ of B and $|L \cap \Gamma| \leqq 2$ for any line $L \subset P^3$.

Let α be a plane meeting a convex curve Γ at exactly n points $\Gamma(t_1)$, $\Gamma(t_2), \ldots, \Gamma(t_n)$; $t_1 < t_2 < \ldots < t_n < t_1$ in T. Let $C = \partial(\alpha \cap H(\Gamma))$, the boundary on $\alpha \cap H(\Gamma)$ in α. Then C is a convex curve and we say that $\alpha \cap \Gamma$ is *normal* if there is an orientation on C such that

$$\Gamma(t_1) < \Gamma(t_2) < \ldots < \Gamma(t_n) < \Gamma(t_1) \text{ in } C.$$

The inequality here is used in the same sense as the inequality in T.

n-VERTEX THEOREM. *Let* Γ *be a simple elementary inflectional convex space curve,* $B = H(\Gamma)$ *in* $A^3 = P^3 \backslash \beta$. *Let* $\alpha \cap \Gamma$ *be normal,* α *cut* Γ *at each point of intersection and let the osculating plane of* Γ *at each inflection meet the relative interior of* $\alpha \cap B$. *If* $|\alpha \cap \Gamma| = n$ *then* Γ *possesses at least* n *inflections.*

We now assume the hypotheses of our n-vertex theorem with

$$\alpha \cap \Gamma = \{\Gamma(t_1), \ldots, \Gamma(t_n)\}.$$

Since Γ is a closed curve and α cuts Γ at t_i for $i \in Z_n = \{1, 2, \ldots, n\}$, it follows that n is even. Let $\Gamma(r_1,)$, $\Gamma(r_2), \ldots, \Gamma(r_m)$ be the inflection points of Γ. Then m is also even and we need to show that $m \geqq n$. As we subsequently quote, m is at least four and hence we assume that $n \geqq 6$. Finally by hypothesis,

(1) $\quad \Gamma_2(r_j) \cap \text{rel int}(\alpha \cap B) \neq \emptyset$ for $j \in Z_m = \{1, 2, \ldots, m\}$.

We next observe that since α cuts Γ at each t_i and (1) is a condition on the relative interior of $\alpha \cap B$, there exist planes arbitrarily close to α which also satisfy the hypotheses for α. As Γ contains a finite number of inflections, there exist such planes which do not contain any inflection point of Γ. Hence we assume

$$\{t_1, t_2, \ldots, t_n\} \cap \{r_1, r_2, \ldots, r_m\} = \emptyset.$$

We now label these points so that $t_1 < t_2 < \ldots < t_n < t_1$ and $r_1 < r_2 < \ldots < r_m < r_1$ in T. We further define $t_{i+kn} = t_i$ and $r_{j+km} = r_j$ and thus the t_i's and r_j's are defined for every integer i and j respectively. For $j \in Z_m$, $\Gamma(r_j, r_{j+1})$ is then a regular subarc which we call a *unit*.

We note that as the proof of our n-vertex theorem is not based on calculus, we require the formula (1). It will readily yield that

$$|\Gamma_2(t_i) \cap C| = 2 \quad \text{for } i \in Z_n;$$

say

$$\Gamma(t_i) \cap C = \{\Gamma(t_i), p_i\}.$$

We call p_i the *paired point* of $\Gamma(t_i)$.

In the proof, we first show [7] that α meets any unit in at most three points and then [8] determine the arrangement of $\Gamma(t_i)$ and p_i on C if α meets a unit in more than one $\Gamma(t_i)$. This arrangement is the key to understanding the behaviour of Γ. In particular [9] if α meets each unit of $\Gamma[r_j, r_{j+l}]$, $l > 1$, exactly once then the arrangement of $\Gamma(t_0)$ and p_0

on $C(r_j < t_0 < r_{j+1})$ determines the arrangement of $\Gamma(t_k)$ and p_k on $C(r_{j+k} < t_k < r_{j+k+1})$ for each k, $1 \leqq k < l$. Next [10] there is a certain arrangement of $\Gamma(t_i)$ and p_i on $C(r_k < t_i < r_{k+1})$ which implies that α meets $\Gamma[r_{k+1}, r_{k+2}]$ in at most one point.

As a consequence, we obtain [12, 15 and 16] that if α meets each unit of $\Gamma[r_j, r_{j+l}]$ then

$$|\alpha \cap \Gamma[r_j, r_{j+l}]| \leqq l + 2.$$

Finally [18] if α meets each unit of $\Gamma[r_j, r_k] \cup \Gamma[r_{k+1}, r_l]$ and $\alpha \cap \Gamma[r_k, r_{k+1}] = \emptyset$ then there are restrictions on the number of times that α may meet $\Gamma[r_j, r_k]$ and $\Gamma[r_{k+1}, r_l]$. These restrictions enable us to determine the relation between m (the number of units of Γ) and $n = |\alpha \cap \Gamma|$ and of course to show that $m \geqq n$.

2. Proof of the *n*-vertex theorem. Since Γ is convex and $\Gamma(t) \in \partial B$ for $t \in T$, there is a supporting plane $\pi(t)$ of B through $\Gamma(t)$. Since Γ is inflectional and $\pi(t)$ also supports Γ at t, we obtain that $\Gamma_1(t) \subset \pi(t)$.

For the proof of the next three results, we refer to [**2**].

1. *For* $t \in T$, $\Gamma_1(t)$ *supports the convex region* $\pi(t) \cap B$ *in* $\pi(t)$.

2. *Let* $\Gamma(u, v)$ *be a regular arc. Then*

$$\Gamma_2(u) \cap \Gamma_2(v) \cap B = \emptyset.$$

3. Four-Vertex Theorem. *A simple elementary inflectional convex space curve possesses at least four inflections.*

As indicated in Section 1, we wish to analyse the arrangement of certain points on the convex curve $C = \alpha \cap \partial B$. For this analysis, we use 2 as the primary tool.

4. Let $t \in T$. Since $|\Gamma_2(t) \cap \Gamma| < \infty$, there is a

$$U'(t) = U^-(t) \cup U^+(t)$$

such that

$$\Gamma_2(t) \cap \Gamma(U'(t)) = \emptyset.$$

Let $B_t^-[B_t^+]$ be the component of $B \backslash \Gamma_2(t)$ which contains $\Gamma(U^-(t))$ $[\Gamma(U^+(t))]$. Then B_t^- and B_t^+ are convex sets,

$$\text{cl } B_t^- = B_t^- \cup (\Gamma_2(t) \cap B) \text{ and cl } B_t^+ = B_t^+ \cup (\Gamma_2(t) \cap B).$$

Since $\Gamma_2(t)$ cuts [supports] Γ at t if $\Gamma(t)$ is a regular [inflection] point, it follows that if $\Gamma(t)$ is regular then

4.1 $\quad B_t^- \cap B_t^+ = \emptyset \quad \text{and} \quad B = B_t^- \cup (\Gamma_2(t) \cap B) \cup B_t^+$

and if $\Gamma(t)$ is an inflection then

4.2 $\quad B_t^- = B_t^+ \; [= B_t \text{ say}].$

In either case,

4.3 $\quad B \backslash (\Gamma_2(t) \cap B) = B_t^- \cup (B \backslash \text{cl } B_t^-) = B_t^+ \cup (B \backslash \text{cl } B_t^+).$

Let $\Gamma(u, v)$ be regular. By 2,

$$\Gamma_2(u) \cap \Gamma(u, v] = \emptyset = \Gamma_2(v) \cap \Gamma[u, v)$$

and thus

4.4 $\quad \Gamma(u, v] \subset B_u^+ \quad \text{and} \quad \Gamma[u, v) \subset B_v^-.$

By 2 and 4.3,

$$\Gamma_2(v) \cap B \subset B \backslash (\Gamma_2(u) \cap B) = B_u^+ \cup (B \backslash \text{cl } B_u^+).$$

Since $\Gamma_2(u) \cap B$ separates B_u^+ and $B \backslash \text{cl } B_u^+$ in B, we have either

$$\Gamma_2(v) \cap B \subset B_u^+ \quad \text{or} \quad \Gamma_2(v) \cap B \subset B \backslash \text{cl } B_u^+.$$

Thus by 4.4,

4.5 $\quad \Gamma_2(v) \cap B \subset B_u^+$

and symmetrically

$$\Gamma_2(u) \cap B \subset B_v^-.$$

By 4.3,

$$B_u^+ \subset B = B_v^- \cup (\Gamma_2(v) \cap B) \cup (B \backslash \text{cl } B_v^-).$$

Since B_u^+ is a component of $B \backslash (\Gamma_2(u) \cap B)$, 4.5 implies that either $B_u^+ \subset B_v^-$ or $B \backslash B_v^- \subset B_u^+$. If $B_u^+ \subset B_v^-$ then

$$B_v^- \cap (\Gamma_2(v) \cap B) = \emptyset$$

implies that

$$B_u^+ \cap (\Gamma_2(v) \cap B) = \emptyset;$$

a contradiction by 4.5. Thus

4.6 $\quad B \backslash B_v^- \subset B_u^+ \quad \text{and} \quad B \backslash B_u^+ \subset B_v^-.$

Finally let $t \in (u,v)$. Then by 4.4-4.6,

4.7 $\quad \Gamma_2(u) \cap B \subset B_t^-, \; \Gamma_2(v) \cap B \subset B_t^+$

and

4.8 $\quad \Gamma_2(t) \cap B \subset B_u^+ \cap B_v^-.$

5. LEMMA. *Let* $\Gamma(u, v)$ *be a regular arc and* $\gamma \cap \Gamma(u, v) = \emptyset$. *Let* $M = \langle b, c\rangle$ *be a line;* $b \in \Gamma_2(u) \cap \gamma \cap B$ *and* $c \in \Gamma_2(v) \cap \gamma \cap B$. *Let* M^* *be the closed segment of* $M \cap B$ *with end points* b *and* c. *Then*

5.1 $\Gamma_2(t) \cap M^* \neq \emptyset$ *for* $t \in [u,v]$,

5.2 $\Gamma_1(t) \cap M = \emptyset$ *for* $t \in (u, v)$,

5.3 $\langle M, \Gamma(t) \rangle$ *cuts* Γ *at* t *for* $t \in (u, v)$,

5.4 *if* $\{\Gamma(u), \Gamma(v)\} \subset \gamma\backslash M$ *then* M *separates* $\Gamma(u)$ *and* $\Gamma(v)$ *in* $\gamma \cap B$,

5.5 $\langle M, \Gamma(t) \rangle \cap \Gamma(u, v) = \{\Gamma(t)\}$ *for* $t \in (u, v)$ *and*

5.6 *if* $|\gamma \cap \Gamma[u, v]| \leqq 1$ *and* $M \cap \Gamma[u, v] = \emptyset$ *then*

$$\langle M, \Gamma(t) \rangle \cap \Gamma[u, v] = \{\Gamma(t)\} \quad \text{for } t \in [u, v].$$

Proof. 1. Let $t \in (u, v)$. By 4.7, $b \in B_t^-$ and $c \in B_t^+$. Thus $\Gamma_2(t)$ separates b and c in B and $\Gamma_2(t) \cap M^* \neq \emptyset$.

2. Let $t \in (u, v)$. Then $\Gamma_2(t) \neq \gamma$ and $\Gamma_1(t) \subset \Gamma_2(t)$ yield

$$\Gamma_1(t) \cap M \subseteq \Gamma_2(t) \cap M = \Gamma_2(t) \cap (M^*\backslash\{b,c\})$$

by 5.1 and 2. If $M^*\backslash\{c, b\} \subset \text{int } B$ then

$$\Gamma_1(t) \cap \text{int } B = \emptyset$$

yields our claim. If $M^* \subset \partial B$ then

$$\Gamma_1(t) \cap M \neq \emptyset, \Gamma_1(t) \cap \{b,c\} = \emptyset$$

and

$$\Gamma_1(t) \cap \text{int } B = \emptyset$$

clearly imply that $\langle M, \Gamma_1(t) \rangle$ is a supporting plane $\pi(t)$ of B at $\Gamma(t)$ and

$$\Gamma_1(t) \cap \text{rel int } (\pi(t) \cap B) = \emptyset;$$

a contradiction by 1.

3. Let $t \in (u, v)$. Then by 5.2, $\langle M, \Gamma(t) \rangle \in S_0(t)$. Since Γ is inflectional, 5.3 follows.

4. If $\{\Gamma(u), \Gamma(v)\} \subset \gamma\backslash M$ and $\Gamma(u)$ and $\Gamma(v)$ are in the same component of $(\gamma \cap B)\backslash M$ then $\gamma \cap \Gamma(u, v) = \emptyset$ readily implies that there is a plane $\bar{\gamma}$ such that $\bar{\gamma} \cap \gamma = M$ and $\Gamma[u, v]$ is contained in a closed half-space of P^3 bounded by $\bar{\gamma}$ and γ. In particular, $\bar{\gamma}$ may be chosen so that $\bar{\gamma}$ supports Γ at some $t \in (u, v)$; a contradiction by 5.3.

5. Let $\bar{\gamma} \supset M$ such that $\bar{\gamma}$ meets Γ at $r < s$ in (u, v). Since $|\bar{\gamma} \cap \Gamma| < \infty$, we may assume that

$$\bar{\gamma} \cap \Gamma(r, s) = \emptyset.$$

By 5.1, $M = \langle b', c'\rangle$ where

$$b' = \Gamma_2(r) \cap M^* \quad \text{and} \quad c' = \Gamma_2(s) \cap M^*$$

and hence $\Gamma(r, s)$ satisfies the hypotheses of 5 with $\bar{\gamma}$ and M. But then

$$\{\Gamma(r), \Gamma(s)\} \subset \bar{\gamma} \backslash M$$

and 5.4 imply that $M = \bar{\gamma} \cap \gamma$ separates $\Gamma(r)$ and $\Gamma(s)$ in $\bar{\gamma} \cap B$. Hence $\Gamma(r)$ and $\Gamma(s)$ lie in distinct components of $B \backslash \gamma$ and

$$\emptyset \neq \gamma \cap \Gamma(r, s) \subset \gamma \cap \Gamma(u, v);$$

a contradiction.

6. If $|\gamma \cap \Gamma[u, v]| \leqq 1$ and $M \cap \Gamma[u, v] = \emptyset$ then we argue as in the proof of 5.5 with $r < s$ in $[u, v]$.

As a point of interest, we note the geometric interpretation of 5. Let t move from u to v in $[u, v]$. By 5.1, $\Gamma_2(t) \cap M$ moves (continuously and) monotonically from b to c in M^*. By 5.5 and 5.6, $\langle M, \Gamma(t) \rangle$ rotates monotonically about M meeting $\Gamma[u, v]$ in exactly one point.

We also note that 5 will imply that ord $\Gamma(u, v) = 3$. This is a direct generalization of the following: if the line through the end points of a simple closed strictly convex arc does not meet the arc elsewhere then any line meets the arc in at most two points.

6. We recall that α cuts Γ at t_i, $\Gamma(t_i) \in C$ for $i \in Z_n$ and

$$\{t_1, \ldots, t_n\} \cap \{r_1, \ldots, r_m\} = \emptyset.$$

For $p \neq q$ in C; $C[p, q]$, $C(p, q)$, $C[p, q)$ and $C(p, q]$ are the respectively closed, open and "half-closed, half-open" oriented arcs of C from p to q and $p' < q'$ in $C(p, q)$ when $C(p', q') \subset C(p, q)$. Since Γ is simple, we set

$$C[t_i, t_k) = C[\Gamma(t_i), \Gamma(t_k)).$$

For $j \in Z_m$, $\Gamma(r_j)$ is an inflection point and

$$\Gamma_2(r_j) \cap \text{rel int } (\alpha \cap B) \neq \emptyset.$$

Hence $\Gamma_2(r_j)$ meets C at say q_j and q_j'. We label q_j and q_j' so that

$$C(q_j, q_j') = B_{r_j} \cap C.$$

Since $\Gamma(r_j, r_{j+1})$ is regular,

$$\Gamma_2(r_j) \cap B \subset B_{r_{j+1}} \quad \text{and} \quad \Gamma_2(r_{j+1}) \cap B \subset B_{r_j}$$

by 4.5. Hence our labelling yields

6.1 $\quad q_j < q_{j+1}' < q_{j+1} < q_j'$ in C

and

6.2 $\quad B_{r_j} \cap B_{r_{j+1}} \cap C = C(q_j, q_{j+1}') \cup C(q_{j+1}, q_j').$

Let $t_i \in (r_j, r_{j+1})$. By 4.8,

$$\Gamma_2(t_i) \cap B \subset B_{r_j} \cap B_{r_{j+1}},$$

a convex set situated between $\Gamma_2(r_j)$ and $\Gamma_2(r_{j+1})$ in B. Thus

$$B_{r_j} \cap B_{r_{j+1}} \setminus (\Gamma_2(t_i) \cap B)$$

consists of two components. By 2, $\Gamma_2(t_i) \cap \partial B$ will intersect at least twice any closed curve on ∂B which meets both $\Gamma_2(r_j)$ and $\Gamma_2(r_{j+1})$. Since $C = \alpha \cap \partial B$ is such a plane convex curve,

$$\Gamma_2(t_i) \cap C = \{\Gamma(t_i), p_i\}$$

as previously defined.

Since $\Gamma(t_i)$ is regular,

$$C(t_i, p_i) = B_{t_i}^{-} \cap C \quad \text{if and only if} \quad C(p_i, t_i) = B_{t_i}^{+} \cap C.$$

As

$$\{\Gamma(t_i), p_i\} = \Gamma_2(t_i) \cap C \subset B_{r_j} \cap B_{r_{j+1}} \cap C,$$

we obtain from 4.7 and 6.2 that if

$$C(t_i, p_i) = B_{t_i}^{-} \cap C$$

then

6.3 $\quad \Gamma(t_i) < q_j' < q_j < p_i < q_{j+1}' < q_{j+1} \quad \text{in } C$

and if

$$C(p_i, t_i) = B_{t_i}^{-} \cap C$$

then

6.4 $\quad p_i < q_j' < q_j < \Gamma(t_i) < q_{j+1}' < q_{j+1} \quad \text{in } C.$

7. LEMMA. *For* $j \in Z_m$, $|\alpha \cap \Gamma(r_j, r_{j+1})| \leqq 3$.

Proof. Let α meet Γ at $t_1 < t_2 < t_3 < t_4$ in (r_j, r_{j+1}) and set

$$M = \langle \Gamma(t_1), \Gamma(t_4) \rangle.$$

As $|\alpha \cap \Gamma| < \infty$, we may assume that

$$\alpha \cap \Gamma(t_2, t_3) = \emptyset.$$

Since $\alpha \cap \Gamma$ is normal, $C[t_2, t_3] \subset C[t_1, t_4]$ and M does not separate $\Gamma(t_2)$ and $\Gamma(t_3)$ in $\alpha \cap B$. Since $\Gamma(r_j, r_{j+1})$ is regular, 4.4 yields that

$$\Gamma(t_1) \in B_{t_2}^{-} \cap B_{t_3}^{-} \quad \text{and} \quad \Gamma(t_4) \in B_{t_2}^{+} \cap B_{t_3}^{+}.$$

Thus both $\Gamma_2(t_2)$ and $\Gamma_2(t_3)$ meet $M \cap B$ and $\Gamma(t_2, t_3)$ satisfies 5 with M and α. Then

$$\{\Gamma(t_2), \Gamma(t_3)\} \subset \alpha \setminus M$$

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and 5.4 imply that M separates $\Gamma(t_2)$ and $\Gamma(t_3)$ in $\alpha \cap B$; a contradiction.

8. LEMMA. *If* $\Gamma(t_i, t_{i+1})$ *is regular then either*

8.1 $$\Gamma(t_i) < p_i < \Gamma(t_{i+1}) < p_{i+1} \quad or$$

$$p_i < \Gamma(t_i) < p_{i+1} < \Gamma(t_{i+1}) \quad in\ C.$$

If $\Gamma(t_i, t_{i+2})$ *is regular then*

8.2 $$\Gamma(t_i) < p_i < \Gamma(t_{i+1}) < p_{i+2} < \Gamma(t_{i+2}) < p_{i+1} \quad in\ C.$$

Proof. If $\Gamma(t_i, t_{i+1})$ is regular then by 2, $M = \langle p_i, p_{i+1} \rangle$ is a line and

$$M \cap \{\Gamma(t_i), \Gamma(t_{i+1})\} = \emptyset.$$

Thus

$$\alpha \cap \Gamma(t_i, t_{i+1}) = \emptyset$$

and 5.4 yield that M separates $\Gamma(t_i)$ and $\Gamma(t_{i+1})$ in $\alpha \cap B$. This implies 8.1.

If $\Gamma(t_i, t_{i+2})$ is regular then $\Gamma_2(t_{i+1}) \cap \alpha$ separates $\Gamma(t_i)$ and $\Gamma(t_{i+2})$ in $\alpha \cap B$ by 4.4. As in the preceding, we also obtain that $\langle p_i, p_{i+1} \rangle$ separates $\Gamma(t_i)$ and $\Gamma(t_{i+1})$ in $\alpha \cup B$ and $\langle p_{i+1}, p_{i+2} \rangle$ separates $\Gamma(t_{i+1})$ and $\Gamma(t_{i+2})$ in $\alpha \cap B$. Finally as $\alpha \cap \Gamma$ is normal,

$$\Gamma(t_i) < \Gamma(t_{i+1}) < \Gamma(t_{i+2}) \quad \text{in } C$$

which now yields 8.2.

9. LEMMA. *Let* $r_{j-1} < t_{i-1} < r_j < t_i < r_{j+1}$ *in* T. *Then*

$$C(t_{i-1}, p_{i-1}) = B^-_{t_{i-1}} \cap C$$

if and only if

$$C(t_i, p_i) = B^-_{t_i} \cap C.$$

Proof. We first observe that by 6.3 and 6.4, $C(q_j, q'_j)$ contains each of the points q_{j-1}, q'_{j-1}, $\Gamma(t_{i-1})$, p_{i-1}, $\Gamma(t_i)$, p_i, q_{j+1} and q'_{j+1}; furthermore,

$$\Gamma(t_{i+1}) < p_{i-1} \quad \text{in} \quad C(q_j, q'_j) \Leftrightarrow C(t_{i-1}, p_{i-1}) = B^-_{t_{i-1}} \cap C$$

and

$$p_i < \Gamma(t_i) \text{ in } C(q_j, q'_j) \Leftrightarrow C(t_i, p_i) = B^-_{t_i} \cap C.$$

Since $\Gamma(r_{j-1}, r_j) \cup \Gamma(r_j, r_{j+1})$ is regular, 4.4 yields

$$\Gamma(r_{j-1}, r_j) \cup \Gamma(r_j, r_{j+1}) \subset B_{r_j},$$

a component of $B \backslash \Gamma_2(r_j)$. As α cuts Γ at t_{i-1} and t_i,

$$\alpha \cap \Gamma(t_{i-1}, t_i) = \emptyset$$

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implies

$$\Gamma(t_{i-1}, t_i) \subset B^*,$$

a component of $B \setminus \alpha$. Altogether

$$\Gamma(t_{i-1}, r_j) \cup \Gamma(r_j, t_i) \subset Q = B_{r_j} \cap B^* \cap \partial B,$$

an open quadrant of ∂B bounded by $\alpha \cap B_{r_j} \cap \partial B$ and $\Gamma_2(r_j) \cap B^* \cap \partial B$. We note that

$$(\alpha \cap \partial B) \cap B_{r_j} = C \cap B_{r_j} = C(q_j, q_j')$$

and $(\Gamma_2(r_j) \cap \partial B) \cap B^*$ is an open arc with end points q_j and q_j' and containing $\Gamma(r_j)$. Let $A[A']$ be the open subarc of $(\Gamma_2(r_j) \cap \partial B) \cap B^*$ with end points $\Gamma(r_j)$ and $q_j[\Gamma(r_j)$ and $q_j']$. Thus

$$\partial Q = C[q_j, q_j'] \cup A \cup \Gamma(r_j) \cup A'.$$

Since $\Gamma(t_{i-1}) \in C(q_j, q_j')$ and $\Gamma(t_{i-1}, r_j)$ is a simple arc on Q, $\Gamma(t_{i-1}, r_j)$ decomposes Q into two open surfaces Q_{i-1} and Q_{i-1}' with the property that

$$\partial Q_{i-1} = C[q_j, t_{i-1}] \cup \Gamma[t_{i-1}, r_j] \cup A$$

and

$$\partial Q_{i-1}' = C[t_{i-1}, q_j'] \cup \Gamma[t_{i-1}, r_j] \cup A'.$$

Similarly, $\Gamma(t_i) \in C(q_i, q_j')$ yields that $\Gamma(r_j, t_i)$ decomposes Q into two open surfaces Q_i and Q_i' with

$$\partial Q_i = C[q_j, t_i] \cup \Gamma[r_j, t_i] \cup A$$

and

$$\partial Q_i' = C[t_i, q_j'] \cup \Gamma[r_j, t_i] \cup A'.$$

Then $\Gamma(t_{i-1}, r_j) \cap \Gamma(r_j, t_i) = \emptyset$ and

$$Q = Q_{i-1} \cup \Gamma(t_{i-1}, r_j) \cup Q_{i-1}' = Q_i \cup \Gamma(r_j, t_i) \cup Q_i'$$

yield that either

$$Q_i \subset Q_{i-1} \quad \text{and} \quad Q_{i-1}' \subset Q_i'$$

or

$$Q_{i-1} \subset Q_i \quad \text{and} \quad Q_i' \subset Q_{i-1}'.$$

Suppose $\Gamma(t_{i-1}) < p_{i-1}$ in $C(q_j, q_j')$. Let M be a line connecting a point on $C(q_j', q_j) \subset C(p_{i-1}, t_{i-1})$ with a point on $C(t_{i-1}, p_{i-1}) \subset C(q_j, q_j')$. Then M meets both

$$\langle q_j, q_j' \rangle = \Gamma_2(r_j) \cap \alpha \quad \text{and} \quad \langle \Gamma(t_{i-1}), p_{i-1} \rangle = \Gamma_2(t_{i-1}) \cap \alpha$$

in int B. Thus

$$\Gamma_2(r_j) \cap M \neq \emptyset$$

and $\Gamma(t_{i-1}, r_j)$ satisfied 5 with M and α. Since

$$\Gamma_1(r_j) \cap \text{int } B = \emptyset,$$

$\Gamma_2(r_j) \cap M \in \text{int } B$ implies that

$$\Gamma_1(r_j) \cap M = \emptyset \quad \text{and} \quad \langle M, \Gamma(r_j) \rangle \neq \Gamma_2(r_j).$$

This and 5.6 then yield

i) $\langle M, \Gamma(r_j) \rangle$ cuts Γ at r_j

and

ii) $\langle M, \Gamma(r_j) \rangle \cap \Gamma[t_{i-1}, r_j] = \{\Gamma(r_j)\}$.

As neither of the two points of $M \cap C$ belongs to $C[q_j, t_{i-1}]$, we have

iii) $\langle M, \Gamma(r_j) \rangle \cap C[q_j, t_{i-1}] = M \cap C[q_j, t_{i-1}] = \emptyset$.

Similarly, any line through $\Gamma(r_j)$ in $\Gamma_2(r_j)$ which meets $\alpha \cap \text{int } B$ does not meet the convex arc A. Thus $M \cap \langle q_j, q'_j \rangle \in \text{int } B$ yields

iv) $\langle M, \Gamma(r_j) \rangle \cap A = \langle M \cap \langle q_j, q'_j \rangle, \Gamma(r_j) \rangle \cap A = \emptyset$.

Now ii), iii) and iv) imply that

$$\langle M, \Gamma(r_j) \rangle \cap \partial Q_{i-1} = \langle M, \Gamma(r_j) \rangle \cap \Gamma[t_{i-1}, r_j] = \{\Gamma(r_j)\}.$$

Thus

$$\langle M, \Gamma(r_j) \rangle \cap Q_{i-1} = \emptyset$$

and Q_{i-1} is contained in a component of $B \backslash \langle M, \Gamma(r_j) \rangle$. If $Q_i \subset Q_{i-1}$ then $\Gamma(t_{i-1}, r_j) \cup \Gamma(r_j, t_i)$ is contained in a component of $B \backslash \langle M, \Gamma(r_j) \rangle$ and $\langle M, \Gamma(r_j) \rangle$ necessarily supports Γ at r_j; a contradiction by i).

We have thus obtained that $\Gamma(t_{i-1}) < p_{i-1}$ in $C(q_j, q'_j)$ implies $Q_{i-1} \subset Q_i$. If $p_{i-1} < \Gamma(t_{i-1})$ in $C(q_j, q'_j)$ then replacing Q_{i-1} with Q'_{i-1} in the preceding yields $Q'_{i-1} \subset Q'_i$. Thus

$$\Gamma(t_{i-1}) < p_{i-1} \text{ in } C(q_j, q'_j) \Leftrightarrow Q_{i-1} \subset Q_i.$$

By a similar argument for $\Gamma(t_i)$ and p_i,

$$\Gamma(t_i) < p_i \text{ in } C(q_j, q'_j) \Leftrightarrow Q_i \subset Q_{i-1}.$$

Combining the various equivalences now yields our claim.

We note that if $r_1 < t_1 < r_2 \leqq \ldots \leqq r_i < t_i < r_{i+1}$ in $T (i \geqq 2)$ then by 9,

$$C(t_1, p_1) = B^-_{t_1} \cap C \text{ if and only if } C(t_i, p_i) = B_{t_i} \cap C.$$

10. LEMMA *Let* $r_{j-1} < t_{i-1} < r_j < t_i < r_{j+1}$ *in* T *and*

$$C[t_i, t_{i+1}) \subset C(t_{i-1}, p_{i-1}) = B^-_{t_{i-1}} \cap C.$$

Then

$$C[t_{i+1}, p_{i-1}) \subset C(t_i, p_i) = B^-_{t_i} \cap C$$

and

$$\alpha \cap \Gamma(r_j, r_{j+1}) = \{\Gamma(t_i)\}.$$

Proof. From 9, and 6.3,

$$C(t_i, p_i) = B^-_{t_i} \cap C$$

and both $\Gamma(t_{i-1}) < p_{i-1}$ and $p_i < \Gamma(t_i)$ in $C(q_j, q'_j) = B_{r_j} \cap C$. Thus

$$C[t_i, t_{i+1}) \subset C(t_{i-1}, p_{i-1})$$

yields

$$\Gamma(t_{i-1}) < \Gamma(t_i) < \Gamma(t_{i+1}) \leqq p_{i-1} \quad \text{in } C(q_j, q'_j)$$

and

$$p_i < \Gamma(t_i) < \Gamma(t_{i+1}) \leqq p_{i-1} \quad \text{in } C(q_j, q'_j).$$

By the latter statement, $C[t_{i+1}, p_{i-1}) \subset C(t_i, p_i)$.

Since $r_j < t_i < r_{j+1}$ and $\Gamma(r_j, r_{j+1})$ is regular,

$$B^-_{t_i} \cap B^+_{t_i} = \emptyset \quad \text{by 4.1 and}$$

$$\Gamma(t_i, r_{j+1}) \subset B^+_{t_i} \quad \text{by 4.4.}$$

Thus

$$\Gamma(t_{i+1}) \in B^-_{t_i} \text{ implies } t_{i+1} \notin (t_i, r_{j+1}).$$

Then $t_{i-1} < r_j < t_i < r_{j+1} < t_{i+1}$ in T and

$$\alpha \cap \Gamma(r_j, r_{j+1}) = \{\Gamma(t_i)\}.$$

11. LEMMA. *Let* $1 \leqq l \leqq m$,

$$\alpha \cap \Gamma(r_1, r_2) = \{\Gamma(t_1), \Gamma(t_2), \Gamma(t_3)\}$$

and

$$\alpha \cap \Gamma(r_k, r_{k+1}) \neq \emptyset$$

for $k = 1, \ldots, l$. *Then*

11.1 $\alpha \cap \Gamma(r_k, r_{k+1}) = \{\Gamma(t_{k+2})\}$ *for* $k = 2, \ldots, l$,

11.2 $C(t_{l+2}, p_{l+2}) = B^-_{t_{l+2}} \cap C$

and

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11.3 $\Gamma(t_1) < p_1 < \Gamma(t_2) < p_3 \leqq \ldots \leqq p_{l+2} < \Gamma(t_{l+2})$ *in* C.

Proof. Since $\alpha \cap \Gamma$ is normal and $n \geqq 6$,

i) $\Gamma(t_1) < p_1 < \Gamma(t_2) < p_3 < \Gamma(t_3) < \Gamma(t_4) < \ldots < \Gamma(t_n)$ in C

by 8.2. For $l = 1$, the relation 11.1 is void and 11.3 is contained in i). Since $\Gamma(t_2, t_3)$ is regular, $\Gamma(t_2) \in B_{t_3}^-$ by 4.4 and thus

$$C(t_3, t_3) = B_{t_3}^- \cap C$$

from i).

Assume that $l \geqq 2$ and that our assertions are proved up to $l - 1$; that is,

ii) $\alpha \cap \Gamma(r_k, r_{k+1}) = \{\Gamma(t_{k+2})\}$ for $k = 2, \ldots, l - 1$,

iii) $C(t_{l+1}, p_{l+1}) = B_{t_{l+1}}^- \cap C$

and

iv) $\Gamma(t_1) < p_1 < \Gamma(t_2) < p_3 \leqq \ldots \leqq p_{l+1} < \Gamma(t_{l+1})$ in C.

Since $\alpha \cap \Gamma(r_l, r_{l+1}) \neq \emptyset$, ii) yields

$$r_{l-1} < t_{l+1} < r_l < t_{l+2} < r_{l+1} \text{ in } T.$$

As $\Gamma(t_1) < \Gamma(t_2) < \Gamma(t_{l+1}) < \Gamma(t_{l+2}) < \Gamma(t_{l+3})$ in C,

$$C[t_{l+2}, t_{l+3}) \subset C(t_{l+1}, t_1) \subset C(t_{l+1}, p_{l+1}) = B_{t_{l+1}}^- \cap C$$

by iv) and iii). Thus by 10 with $j = l$ and $i = l + 2$,

$$\alpha \cap \Gamma(r_l, r_{l+1}) = \{\Gamma(t_{l+2})\}$$

and

$$\Gamma(t_{l+3}, p_{l+1}) \subset C(t_{l+2}, p_{l+2}) = B_{t_{l+2}}^- \cap C.$$

Finally, $\Gamma(t_1) < p_1 < p_{l+1} < \Gamma(t_{l+1}) < \Gamma(t_{l+2})$ in C and $p_{l+1} \leqq p_{l+2} < \Gamma(t_{l+2})$ in C yield 11.3.

We note that by reversing in 11 the orientation of both Γ and C, we arrive at a similar result. Combining these two results and introducing a suitable notation, we obtain 12.1-12.4 of the following theorem.

12. THEOREM. *Let* $1 \leqq i \leqq l \leqq m$,

$$\alpha \cap \Gamma(r_i, r_{i+1}) = \{\Gamma(t_i), \Gamma(t_{i+1}), \Gamma(t_{i+2})\}$$

and

$$\alpha \cap \Gamma(r_k, r_{k+1}) \neq \emptyset$$

for $k = 1, \ldots, l$. *Then*

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12.1 $\alpha \cap \Gamma(r_k, r_{k+1}) = \{\Gamma(t_k)\}$ *for* $k = 1, \ldots, i - 1$,

12.2 $\alpha \cap \Gamma(r_k, r_{k+1}) = \{\Gamma(t_{k+2})\}$ *for* $k = i + 1, \ldots, l$,

12.3 $C(t_1, p_1) = B^-_{t_1} \cap C$ *and* $C(t_{l+2}, p_{l+2}) = B^-_{t_{l+2}} \cap C$,

12.4 $\Gamma(t_1) < p_1 \leqq p_i < \Gamma(t_{i+1}) < p_{i+2} \leqq p_{l+2} < \Gamma(t_{l+2})$ *in C*

and

12.5 $\Gamma[r_1, r_{1+l}] \neq \Gamma$.

Proof of 12.5. By 12.1, 12.2 and 12.4, if

$$\alpha \cap \Gamma[r_1, r_{1+l}] = \alpha \cap \Gamma[t_1, t_{l+2}]$$

then $\Gamma(t_1) < p_1 < \Gamma(t_{i+1}) < p_{l+2} < \Gamma(t_{l+2})$ in C.

Suppose that $\Gamma[r_1, r_{1+l}] = \Gamma$. Then $m = l$, $n = l + 2$,

$$\alpha \cap \Gamma[r_1, r_{1+m}] = \alpha \cap \Gamma[t_1, t_n] \quad \text{and}$$

$$\Gamma(t_1) < p_1 < \Gamma(t_{i+1}) < p_n < \Gamma(t_n) \quad \text{in } C.$$

Since $n \geqq 6$, either $n \neq i + 2$ or $1 \neq i$. If $n \neq i + 2$ then

$$\alpha \cap \Gamma[r_m, r_{m+m}] = \alpha \cap \Gamma[t_n, t_1] \quad \text{and}$$

$$\Gamma(t_n) < p_n < \Gamma(t_{i+1}) < p_1 < \Gamma(t_1);$$

a contradiction. If $1 \neq i$ then

$$\alpha \cap \Gamma[r_2, r_{2+m}] = \alpha \cap \Gamma[t_2, t_1] \quad \text{and}$$

$$\Gamma(t_2) < p_2 < \Gamma(t_{i+1}) < p_1 < \Gamma(t_1) \quad \text{in } C;$$

a contradiction. Thus $\Gamma[r_1, r_{1+l}] \neq \Gamma$.

13. LEMMA. *Let* $1 \leqq l \leqq m$,

$$\alpha \cap \Gamma(r_1, r_2) = \{\Gamma(t_1), \Gamma(t_2)\},$$

$$p_1 < \Gamma(t_1) < p_2 < \Gamma(t_2) \text{ in } C \text{ and}$$

$$\alpha \cap \Gamma(r_k, r_{k+1}) \neq \emptyset \text{ for } k = 1, \ldots, l.$$

Then

13.1 $\alpha \cap \Gamma(r_k, r_{k+1}) = \{\Gamma(t_{k+1})\}$ *for* $k = 2, \ldots, l$,

13.2 $C(t_{l+1}, p_{l+1}) = B^-_{t_{l+1}} \cap C$

and

13.3 $\Gamma(t_1) < p_2 \leqq \ldots \leqq p_{l+1} < \Gamma(t_{l+1})$ *in C*.

Proof. We replace i) by $p_1 < \Gamma(t_1) < p_2 < \Gamma(t_2)$ in C and argue exactly as in the proof of Lemma 11.

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14. LEMMA. *Let* $1 \leqq l \leqq m$,

$$\alpha \cap \Gamma[r_{-2}, r_{-1}] = \{\Gamma(t_{-2}), \Gamma(t_{-1})\},$$

$$\Gamma(t_{-2}) < p_{-2} < \Gamma(t_{-1}) < p_{-1} \text{ in } C \text{ and}$$

$$\alpha \cap \Gamma(r_{-k-1}, r_{-k}) \neq \emptyset \text{ for } k = 1, \dots, l.$$

Then

14.1 $\alpha \cap \Gamma(r_{-k-1}, r_{-k}) = \{\Gamma(t_{-k-1})\}$ *for* $k = 2, \dots, l$,

14.2 $C(t_{-l-1}, p_{-l-1}) = B^{-}_{t_{-l-1}} \cap C$

and

14.3 $\Gamma(t_{-l-1}) < p_{-l-1} \leqq \dots p_{-2} < \Gamma(t_{-1})$ *in* C.

Proof. We reverse the orientation of both Γ and C and apply Lemma 13.

15. THEOREM. *Let* $1 \leqq i \leqq l \leqq m$ *and*

$$\alpha \cap \Gamma(r_k, r_{k+1}) = \begin{cases} \{\Gamma(t_k)\} & k = 1, \dots, i-1 \\ \{\Gamma(t_i), \Gamma(t_{i+1})\} \text{ for } & k = i \\ \{\Gamma(t_{k+1})\} & k = i+1, \dots, l. \end{cases}$$

Then $\Gamma[r_1, r_{1+l}] \neq \Gamma$ *and either*

15.1 $p_1 \in C(t_1, t_{l+1})$ *and*

$$C(t_k, p_k) = \begin{cases} B^{-}_{t_k} \cap C & k = 1, \dots, i \\ & \text{if} \\ B^{+}_{t_k} \cap C & k = i+1, \dots, l+1 \end{cases}$$

or

15.2 $P_{l+1} \in C(t_1, t_{l+1})$ *and*

$$C(t_k, p_k) = \begin{cases} B^{+}_{t_k} \cap C & k = 1, \dots, i \\ & \text{if} \\ B^{-}_{t_k} \cap C & k = i+1, \dots, l+1. \end{cases}$$

Proof. If $\Gamma[r_1, r_{1+l}] = \Gamma$ then $m = l$ and $n = l + 1 = |\alpha \cap \Gamma|$. Since m and n are both even, this is impossible. Next 8.1 and

$$\alpha \cap \Gamma(r_i, r_{i+1}) = \{\Gamma(t_i), \Gamma(t_{i+1})\}$$

imply that either

i) $\Gamma(t_i) < p_i < \Gamma(t_{i+1}) < p_{i+1}$ in C
or

ii) $p_i < \Gamma(t_i) < p_{i+1} < \Gamma(t_{i+1})$ in C.

If i) then

$$C(t_i, p_i) = B^-_{t_i} \cap C \text{ and } C(t_{i+1}, p_{i+1}) = B^+_{t_{i+1}} \cap C.$$

Now 9 and the hypothesis yield the characterization of $C(t_k, p_k)$ in 15.1. We also note that $\Gamma[r_1, r_{i+1}]$ then satisfies 14 and thus

$$p_1 \in C(t_1, t_{i+1}) \subset C(t_1, t_{l+1}) \text{ by 14.3.}$$

If ii) then 15.2 follows similarly by 9 and 13.3.

16. THEOREM. *Let* $1 \leqq h < i \leqq l \leqq m$,

$$\alpha \cap \Gamma(r_h, r_{h+1}) = \{\Gamma(t_h), \Gamma(t_{h+1})\},$$

$$|\alpha \cap \Gamma(r_i, r_{i+1})| = 2 \textit{ and}$$

$$\alpha \cap \Gamma(r_k, r_{k+1}) \neq \emptyset \textit{ for } k = 1, \ldots, l.$$

Then

16.1 $$\alpha \cap \Gamma(r_k, r_{k+1}) = \begin{cases} \{\Gamma(t_k)\} & k = 1, \ldots, h-1 \\ \{\Gamma(t_{k+1})\} & k = h+1, \ldots, i-1 \\ \{\Gamma(t_{i+1}), \Gamma(t_{i+2})\} & k = i \\ \{\Gamma(t_{k+2})\} & k = i+1, \ldots, l, \end{cases} \textit{ if}$$

16.2 $$C(t_k, p_k) = \begin{cases} B^-_{t_k} \cap C & k = 1, \ldots, h, i+2, \ldots, l+2 \\ & \textit{if} \\ B^+_{t_k} \cap C & k = h+1, \ldots, i+1 \end{cases}$$

16.3 $\Gamma(t_1) < p_1 < \Gamma(t_{h+1}) < \Gamma(t_{i+1}) < p_{l+2} < \Gamma(t_{l+2})$ *in* C

and

16.4 $\Gamma[r_1, r_{1+l}] \neq \Gamma.$

Proof. We note that if 16.1 as well as

i) $\Gamma(t_h) < p_h < \Gamma(t_{h+1}) < p_{h+1}$ in C

and

ii) $p_{i+1} < \Gamma(t_{i+1}) < p_{i+2} < \Gamma(t_{i+1})$ in C

are proved then 16.2 and 16.3 follow by applying 15.1 to $\Gamma[r_1, r_{h+1}]$, 9 to $\Gamma[r_{h+1}, r_i]$ and 15.2 to $\Gamma[r_i, r_{l+1}]$.

Since $\Gamma(t_h, t_{h+1})$ is regular, either i) or

$$p_h < \Gamma(t_h) < p_{h+1} < \Gamma(t_{h+1}) \text{ in } C$$

by 8.1. In case of the latter, $\Gamma[r_h, r_{l+1}]$ satisfies 13 and hence

$$|\alpha \cap \Gamma(r_i, r_{i+1})| = 1$$

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by 13.1; a contradiction. Now i) and 14.1 applied to $[r_1, r_{h+1}]$ yield that

$$\alpha \cap \Gamma(r_k, r_{k+1}) = \{\Gamma(t_k)\} \text{ for } k = 1, \ldots, h - 1.$$

For $k = h + 1, \ldots, i - 1$, $|\alpha \cap \Gamma(r_k, r_{k+1})| \leqq 2$ by 7 and 12. Arguing as in the preceding,

$$|\alpha \cap \Gamma(r_h, r_{h+1})| = |\alpha \cap \Gamma(r_i, r_{i+1})| = 2$$

implies that $|\Gamma(r_k, r_{k+1})| = 2$ is impossible. Hence

$$\alpha \cap \Gamma(r_k, r_{k+1}) = \{\Gamma(t_{k+1})\} \text{ for } k = h + 1, \ldots, i - 1$$

and

$$\alpha \cap \Gamma(r_i, r_{i+1}) = \{\Gamma(t_{i+1})\ \Gamma(t_{i+2})\}.$$

Since $|\alpha \cap \Gamma(r_h, r_{h+1})| = 2$, $\Gamma(r_1, r_{i+1}]$ cannot satisfy 14. Thus by 8.1, ii) is true and $\Gamma[r_i, r_{l+1}]$ satisfies 13. Now 13.1 yields 16.1.

Suppose that $\Gamma[r_1, r_{1+l}] = \Gamma$ and choose a j such that $h + 1 \leqq j \leqq i$. Then by 16.2,

$$C(t_j, p_j) = B_{t_j}^{+} \cap C.$$

Since $l = m$, $\Gamma(r_j, r_{j+m})$ satisfies 16 with $j \leqq i \leqq h \leqq j + (m - 1)$. Thus by 16.2 and the new sequence of numbering,

$$C(t_j, p_j) = B_{t_j}^{-} \cap C;$$

a contradiction.

17. In this section, we collect and condense our results. An arc $\Gamma[r_k, r_{k+1})$ with the property $\alpha \cap \Gamma[r_k, r_{k+1}) = \emptyset$ is a *0-arc*. A subarc $\Gamma[r_i, r_{j+1})$ is *maximal* if α meets each of its units and $\Gamma[r_{i-1}, r_i)$ and $\Gamma[r_{j+1}, r_{j+2})$ are either 0-arcs or contained in $\Gamma[r_i, r_{j+1})$. By 7, 12 and 16, we can classify a maximal $\Gamma[r_i, r_{j+1})$ as follows:

1-*arc*: $|\alpha \cap \Gamma(r_k, r_{k+1})| = 1$ for every $k \in I = \{i, i + 1, \ldots, j\}$;

2-*arc*: $|\alpha \cap \Gamma(r_k, r_{k+1})| = 2$ for exactly one $k \in I$;

3-*arc*: $\begin{cases} |\alpha \cap \Gamma(r_k, r_{k+1})| = 2 \text{ for exactly two } k \in I \\ \text{or} \\ |\alpha \cap \Gamma(r_k, r_{k+1})| = 3 \text{ for exactly one } k \in I. \end{cases}$

Thus if $\Gamma[r_i, r_{i+l})$ is an x-arc then

17.1 $\quad |\alpha \cap \Gamma[r_i, r_{i+l})| = l + x - 1;\ x = 1, 2, 3.$

We next note that a 2-arc necessarily satisfies 15. Accordingly, a 2-arc is a *2′-arc* [*2″-arc*] if 15.1 [15.2] is valid.

If $\Gamma[r_i, r_{i+m})$ is a 1-arc then clearly $m = n$. If a maximal subarc is not a 1-arc then it is a proper subarc of Γ by 12.5, 15 and 16.4 and hence Γ necessarily contains 0-arcs. Thus to prove $m \geqq n$, we need to show that

Γ does not contain "too many" 3-arcs and 2-arcs and "too few" 1-arcs and 0-arcs.

18. LEMMA. *Let $\Gamma[r_1, r_k)$ and $\Gamma(r_{k+1}, r_l)$ be maximal. If $\Gamma(r_1, r_k)$ is a 3-arc or a 2''-arc then $\Gamma[r_{k+1}, r_l)$ is a 1-arc or a 2''-arc.*

Proof. Let

$$\alpha \cap \Gamma(r_1, r_k) = \{\Gamma(t_1), \dots, \Gamma(t_i)\}.$$

Then by 12, 15.2 and 16,

i) $\quad C(t_i, p_i) = B_{t_i}^- \cap C$ and $p_i \in C(t_1, t_i)$.

Let

$$\alpha \cap \Gamma(r_{k+1}, r_l) = \{\Gamma(t_{i+1}), \dots, \Gamma(t_j)\}$$

and suppose that $\Gamma[r_{k+1}, r_l)$ is a 3-arc or a 2'-arc. Then by 12, 15.1 and 16,

ii) $\quad C(t_{i+1}, p_{i+1}) = B_{t_{i+1}}^- \cap C$ and $p_{i+1} \in C(t_{i+1}, t_j)$.

As $r_{k-1} < t_i < r_k < r_{k+1} < t_{i+1} < r_{k+2}$ in T, we obtain that

$$\Gamma(t_i) < p_i \text{ in } C(q_k, q'_k) = B_{r_k} \cap C$$

and

$$p_{i+1} < \Gamma(t_{i+1}) \text{ in } C(q_{k+1}, q'_{k+1}) = B_{r_{k+1}} \cap C$$

by i), ii) and 6.3. These results combined now yield by 6.1,

$$q_k < \Gamma(t_i) < \Gamma(t_{i+1}) < q'_{k+1} < q_{k+1} < p_{i+1} < p_i < q'_k \text{ in } C.$$

From 6.2,

$$B_{r_k} \cap B_{r_{k+1}} \cap C = C(q_k, q'_{k+1}) \cup C(q_{k+1}, q'_k)$$

and thus

$$\{\Gamma(t_i), \Gamma(t_{i+1}), p_{i+1}, p_i\} \subset B_{r_k} \cap B_{r_{k+1}} \cap C.$$

We choose points

$$b \in \Gamma_2(r_k) \cap \alpha \cap \text{int } B = \langle q_k, q'_k \rangle \cap \text{int } B,$$

$$c \in \Gamma_2(r_{k+1}) \cap \alpha \cap \text{int } B = \langle q_{k+1}, q'_{k+1} \rangle \cap \text{int } B$$

and set $M = \langle b, c \rangle$. By 2, $b \neq c$ and M is a line. It meets both

$$\langle \Gamma(t_i), p_i \rangle = \Gamma_2(t_i) \cap \alpha \text{ and}$$

$$\langle \Gamma(t_{i+1}), p_{i+1} \rangle = \Gamma_2(t_{i+1}) \cap \alpha$$

in int B; furthermore,

$$M \cap C[q_k, q'_{k+1}] = \emptyset$$

and M does not separate $\Gamma(t_i)$ and $\Gamma(t_{i+1})$ in $\alpha \cap B$. Finally, we note that M meets neither $\Gamma_1(r_k)$ nor $\Gamma_1(r_{k+1})$; for example,

$$M \cap \Gamma_1(r_k) = \{b\} \cap \Gamma_1(r_k) = \{b\} \cap \text{int } B \cap \Gamma_1(r_k) = \emptyset.$$

Let t move from t_i to r_k in $[t_i, r_k]$. By 5.5 and 5.6, the plane $\langle M, \Gamma(t) \rangle$ rotates monotonically about M. Similarly as t moves r_k to r_{k+1} in $[r_k, r_{k+1}]$ or from r_{k+1} to t_{i+1} in $[r_{k+1}, t_{i+1}]$, $\langle M, \Gamma(t) \rangle$ rotates monotonically about M. Since

$$M \cap \Gamma_1(r_k) = M \cap \Gamma_1(r_{k+1}) = \emptyset,$$

Γ is cut at r_k by $\langle M, \Gamma(r_k) \rangle$ and at r_{k+1} by $\langle M, \Gamma(r_{k+1}) \rangle$. Thus the preceding combined yields that $\langle M, \Gamma(t) \rangle$ rotates monotonically about M as t moves from t_i to t_{i+1} in $[t_i, t_{i+1}]$. Since $\langle M, \Gamma(t) \rangle \neq \alpha$ for $t \in (t_i, t_{i+1})$, it follows that M separates $\Gamma(t_i)$ and $\Gamma(t_{i+1})$ in $\alpha \cap B$; a contradiction.

19. The n-vertex theorem has been formulated in Section 1. As stated there, we assume that $\{t_1, \dots, t_n\} \cap \{r_1, \dots, r_m\} = \emptyset$.

Let m_x denote the number of x-arcs of Γ; $x = 0, 1, 2, 3$. If $m_0 = 0$ then Γ is a 1-arc and $m = n$. Let $m_0 > 0$. We have to prove $m \geqq n$.

Let the m_x x-arcs consist of

$$k(x, 1), k(x, 2), \dots, k(x, m_x)$$

units respectively; $x = 1, 2, 3$. Then

$$\Gamma = \bigcup_{j=1}^{m} \Gamma[r_j, r_{j+1})$$

implies

$$m = m_0 + \sum_{j=1}^{m_1} k(1, j) + \sum_{j=1}^{m_2} k(2, j) + \sum_{j=1}^{m_3} k(3, j).$$

By 17.1, α meets an x-arc consisting of $k(x, j)$ units in $k(x, j) + x - 1$ points; $x = 1, 2, 3$. Thus $n = |\alpha \cap \Gamma|$ yields

$$\begin{aligned} n &= \sum_{j=1}^{m_1} k(1, j) + \sum_{j=1}^{m_2} (k(2, j) + 1) + \sum_{j=1}^{m_3} (k(3, j) + 2) \\ &= \sum_{j=1}^{m_1} k(1, j) + \sum_{j=1}^{m_2} k(2, j) + \sum_{j=1}^{m_3} k(3, j) + m_2 + 2m_3 \\ &= (m - m_0) + m_2 + 2m_3 \\ &= m - (m_0 - m_2 - 2m_3). \end{aligned}$$

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Hence our claim is equivalent to

19.1 $m_0 \geqq m_2 + 2m_3$.

Let $\Gamma[r_i, r_{j+1})$ be an x-arc; $x = 1, 2, 3$. As $m_0 > 0$,

$$\Gamma[r_i, r_{j+1}) \neq \Gamma$$

and hence $\Gamma[r_{j+1}, r_{j+2})$ is a 0-arc by definition. Since Γ is a closed curve, there is at least one 0-arc for each x-arc ($x = 1, 2, 3$) and thus

19.2 $m_0 \geqq m_1 + m_2 + m_3$.

If now $m_1 \geqq m_3$ then 19.1 is true. Accordingly, we may assume that $m_3 > 0$.

Let $\mathscr{S}$ denote the collection of m_3 connected subarcs obtained by omitting the 3-arcs from Γ. Let $\widetilde{\mathscr{S}}[\mathscr{S}^*]$ denote the set of those elements of $\mathscr{S}$ which [do not] contain 1-arcs and let $\widetilde{m}_x[m_x^*]$ denote the number of x-arcs in the elements of $\widetilde{\mathscr{S}}[\mathscr{S}^*]$; $x = 0, 1, 2$. Thus $m_3 = |\widetilde{\mathscr{S}}| + |\mathscr{S}^*|$, $m_2 = \widetilde{m}_2 + m_2^*$, $m_1 = \widetilde{m}_1$ and $m_0 = \widetilde{m}_0 + m_0^*$. We note that

19.3 $|\mathscr{S}^*| \geqq m_3 - m_1$

is trivially true if $m_1 > m_3$ and it follows from our construction if $m_3 > m_1$.

Let $\Gamma[r_i, r_l) \in \mathscr{S}$. By the preceding, the number of 0-arcs on $\Gamma[r_i, r_l)$ is greater by at least one than the number of 1-arcs and 2-arcs on $\Gamma[r_i, r_l)$. Thus

19.4 $\widetilde{m}_0 \geqq \widetilde{m}_1 + \widetilde{m}_2 + |\widetilde{\mathscr{S}}|$ and $m_0^* \geqq m_2^* + |\mathscr{S}^*|$.

We now claim

19.5 if $\Gamma[r_i, r_l) \in \mathscr{S}^*$ then the number of 0-arcs on $\Gamma[r_i, r_l)$ is greater by at least two than the number of 2-arcs on $\Gamma[r_i, r_l)$.

If 19.5 has been proved then $m_0^* \geqq m_2^* + 2|\mathscr{S}^*|$. Hence by 19.3 and 19.4,

$$\begin{aligned} m_0 = \widetilde{m}_0 + m_0^* &\geqq \widetilde{m}_1 + \widetilde{m}_2 + |\widetilde{\mathscr{S}}| + m_2^* + |\mathscr{S}^*| + |\mathscr{S}^*| \\ &= m_1 + m_2 + (\,|\widetilde{\mathscr{S}}| + |\mathscr{S}^*|\,) + |\mathscr{S}^*| \\ &\geqq m_1 + m_2 + m_3 + (m_3 - m_1) \\ &= m_2 + 2m_3. \end{aligned}$$

This is 19.1 and thus $m \geqq n$.

It remains to prove 19.5. Let $y[z]$ denote the number of 0-arcs [2 arcs] on $\Gamma[r_i, r_l] \in \mathscr{S}^*$. Thus $y \geqq z + 1$. If $y = z + 1$ then the 0-arcs and 2-arcs alternate on $\Gamma[r_i, r_l)$. Since $\Gamma(r_i)$ is the terminal point of a 3-arc and $\Gamma[r_i, r_{i+1})$ is a 0-arc, $\Gamma(r_{i+1})$ is the initial point of a 2″-arc by 18.

Furthermore, 18 also implies that each 2-arc on $\Gamma[r_i, r_l)$ is a 2″-arc. As $\Gamma(r_{l-1})$ is the terminal point of a 2″-arc and $\Gamma[r_{l-1}, r_l)$ is a 0-arc, $\Gamma(r_l)$ is not the initial point of a 3-arc again by 18. This is a contradiction and thus $y \geqq z + 2$.

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