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LIE n-HIGHER DERIVATIONS AND LIE n-HIGHER DERIVABLE MAPPINGS

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Abstract

Let $\mathcal{A}$ be a unital torsion-free algebra over a unital commutative ring $\mathcal{R}$. To characterise Lie n-higher derivations on $\mathcal{A}$, we give an identity which enables us to transfer problems related to Lie n-higher derivations into the same problems concerning Lie n-derivations. We prove that: (1) if every Lie n-derivation on $\mathcal{A}$ is standard, then so is every Lie n-higher derivation on $\mathcal{A}$; (2) if every linear mapping Lie n-derivable at several points is a Lie n-derivation, then so is every sequence $\{d_m\}$ of linear mappings Lie n-higher derivable at these points; (3) if every linear mapping Lie n-derivable at several points is a sum of a derivation and a linear mapping vanishing on all $(n-1)$th commutators of these points, then every sequence $\{d_m\}$ of linear mappings Lie n-higher derivable at these points is a sum of a higher derivation and a sequence of linear mappings vanishing on all $(n-1)$th commutators of these points. We also give several applications of these results.

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1. Introduction

Let $\mathcal{A}$ be a unital algebra over a unital commutative ring $\mathcal{R}$. A linear mapping δ on $\mathcal{A}$ is called a *derivation* if $\delta(xy) = \delta(x)y + x\delta(y)$, a *Jordan derivation* if $\delta(x \circ y) = \delta(x) \circ y + x \circ \delta(y)$, a *Lie derivation* if $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ and a *Lie triple derivation* if $\delta([[x, y], z]) = [[\delta(x), y], z] + [[x, \delta(y)], z] + [[x, y], \delta(z)]$, in each case for each x, y, z in $\mathcal{A}$, where $x \circ y = xy + yx$ and $[x, y] = xy - yx$. A derivation δ is called an *inner derivation* if there exists some a in $\mathcal{A}$ such that $\delta(x) = ax - xa$ for each x in $\mathcal{A}$. Define a sequence of polynomials as follows:

$$p_1(x_1) = x_1 \quad \text{and} \quad p_n(x_1, x_2, \dots, x_n) = [p_{n-1}(x_1, x_2, \dots, x_{n-1}), x_n]$$

for each $x_1, x_2, \dots, x_n \in \mathcal{A}$ and each positive integer $n \geq 2$. Thus, $p_2(x_1, x_2) = [x_1, x_2]$, $p_3(x_1, x_2, x_3) = [[x_1, x_2], x_3]$ and $p_n(x_1, x_2, \dots, x_n) = [\dots[[x_1, x_2], x_3], \dots, x_n]$. For $n \geq 2$, $p_n(x_1, x_2, \dots, x_n)$ is also called an $(n-1)$th *commutator* of $x_1, x_2, \dots, x_n \in \mathcal{A}$.

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A linear mapping δ on $\mathcal{A}$ is called a *Lie n-derivation* ($n \geq 2$) if

$$\delta(p_n(x_1, x_2, \dots, x_n)) = \sum_{i=1}^{n} p_n(x_1, \dots, x_{i-1}, \delta(x_i), x_{i+1}, \dots, x_n) \tag{1.1}$$

for each $x_1, x_2, \dots, x_n \in \mathcal{A}$. Thus, δ is a Lie derivation when $n = 2$ and a Lie triple derivation when $n = 3$. The notion of a Lie n-derivation was first proposed by Abdullaev [1]. He described the form of Lie n-derivations of a certain von Neumann algebra (or of its skew-adjoint part). Lie n-derivations on various unital algebras are considered in [2, 4, 15, 18].

Let $R_{\mathcal{A}}$ be a nonempty subset of $\mathcal{A}^n$. A linear mapping δ on $\mathcal{A}$ is *Lie n-derivable on* $R_{\mathcal{A}}$ if it satisfies (1.1) for each $(x_1, x_2, \dots, x_n) \in R_{\mathcal{A}}$. A Lie n-derivation δ on $\mathcal{A}$ is *standard* if $\delta = h + \tau$, where h is a derivation on $\mathcal{A}$ and τ is a linear mapping from $\mathcal{A}$ into its centre $\mathcal{Z}(\mathcal{A})$ vanishing on all $(n-1)$th commutators of $\mathcal{A}$.

From now on, $\mathcal{A}$ is a *torsion-free* algebra, which means that for each positive integer n, $nx = 0$ implies that $x = 0$ for each x in $\mathcal{A}$. Let δ be a linear mapping on $\mathcal{A}$. Define a sequence $\{d_m\}$ of linear mappings on $\mathcal{A}$ by

$$d_0 = I \quad \text{and} \quad m!d_m = \delta^m \tag{1.2}$$

for each positive integer m, where I is the identity mapping on $\mathcal{A}$. If δ is a derivation, then, by [13, Section 1], $\{d_m\}$ satisfies

$$d_m(ab) = \sum_{j=0}^{m} d_j(a)d_{m-j}(b) \tag{1.3}$$

for each a, b in $\mathcal{A}$. A sequence $\{d_m\}$ of linear mappings satisfying $d_0 = I$ and (1.3) is called a *higher derivation* or *Hasse–Schmidt derivation*, after Hasse and Schmidt [6]. Similarly, we can define *Jordan higher derivations* by replacing normal multiplication in (1.3) with Jordan multiplication. A (Jordan) higher derivation $\{d_m\}$ satisfying (1.2) is called an *ordinary (Jordan) higher derivation*. Mirzavaziri [13] characterised higher derivations on algebras over a field of characteristic zero. Li *et al.* [9] found the connection between (Jordan) derivable mappings and (Jordan) higher derivable mappings. Similar to the result in [13, Section 1], we have the following proposition.

Proposition 1.1. *Let δ be a Lie n-derivation on $\mathcal{A}$ and $\{d_m\}$ be a sequence of linear mappings on $\mathcal{A}$ satisfying (1.2). Then, for each $x_1, x_2, \dots, x_n$ in $\mathcal{A}$, $\{d_m\}$ satisfies*

$$d_m(p_n(x_1, x_2, \dots, x_n)) = \sum_{i_1+i_2+\cdots+i_n=m} p_n(d_{i_1}(x_1), d_{i_2}(x_2), \dots, d_{i_n}(x_n)). \tag{1.4}$$

Proof. We prove the proposition by induction on m. It is obvious that (1.4) is satisfied for $m = 0$, since $d_0 = I$. Suppose that (1.4) is satisfied for $m = k - 1$ ($k \geq 1$). When

$m = k$, we have $k!d_k = \delta^k = \delta\delta^{k-1} = (k-1)!\delta d_{k-1}$ and

$$
\begin{aligned}
k!d_k(p_n(x_1, x_2, \ldots, x_n)) &= (k-1)!\delta d_{k-1}(p_n(x_1, x_2, \ldots, x_n)) \\
&= (k-1)!\delta\Big(\sum_{i_1+i_2+\cdot+i_n=k-1} p_n(d_{i_1}(x_1), d_{i_2}(x_2), \ldots, d_{i_n}(x_n))\Big) \\
&= (k-1)! \sum_{i_1+i_2+\cdots+i_n=k-1} \delta(p_n(d_{i_1}(x_1), d_{i_2}(x_2), \ldots, d_{i_n}(x_n))) \\
&= (k-1)! \sum_{i_1+i_2+\cdots+i_n=k-1} \sum_{j=1}^{n} p_n(d_{i_1}(x_1), \ldots, \delta d_{i_j}(x_{i_j}), \ldots, d_{i_n}(x_n))) \\
&= k! \sum_{i_1+i_2+\cdots+i_n=k-1} \sum_{j=1}^{n} p_n(d_{i_1}(x_1), \ldots, d_{i_j+1}(x_{i_j}), \ldots, d_{i_n}(x_n))) \\
&= k! \sum_{i_1+i_2+\cdots+i_n=k} p_n(d_{i_1}(x_1), d_{i_2}(x_2), \ldots, d_{i_n}(x_n)).
\end{aligned}
$$

Since $\mathcal{A}$ is torsion-free, (1.4) is satisfied for $m = k$. □

A sequence $\{d_m\}$ of linear mappings on $\mathcal{A}$ satisfying $d_0 = I$ and (1.4) is called a *Lie n-higher derivation* ($n \geq 2$). In particular, $\{d_m\}$ is a *Lie higher derivation* when $n = 2$ and a *Lie triple higher derivation* when $n = 3$. A Lie n-higher derivation is *ordinary* if it satisfies (1.2). Let $\mathrm{R}_{\mathcal{A}}$ be a nonempty subset of $\mathcal{A}^n$. A sequence $\{d_m\}$ of linear mappings is *Lie n-higher derivable on* $\mathrm{R}_{\mathcal{A}}$ if it satisfies (1.4) for each $(x_1, x_2, \ldots, x_n) \in \mathrm{R}_{\mathcal{A}}$. A Lie n-higher derivation $D = \{d_m\}$ on $\mathcal{A}$ is *standard* if $d_m = g_m + f_m$, where $\{g_m\}$ is a higher derivation on $\mathcal{A}$ and $\{f_m\}$ is a sequence of linear mappings from $\mathcal{A}$ into its centre $\mathcal{Z}(\mathcal{A})$ vanishing on all $(n-1)$th commutators of $\mathcal{A}$.

This paper is organised as follows. In Section 2, we give an identity to characterise Lie n-higher derivations on torsion-free algebras. This enables us to transfer problems related to Lie n-higher derivations into the same problems concerning Lie n-derivations. In Section 3, we show that if every Lie n-derivation of a torsion-free algebra $\mathcal{A}$ is standard, then so is every Lie n-higher derivation of $\mathcal{A}$. In Section 4, we show that if every linear mapping Lie n-derivable on a nonempty subset $\mathrm{R}_{\mathcal{A}}$ of $\mathcal{A}^n$ is a Lie n-derivation, then every sequence $\{d_m\}$ of linear mappings Lie n-higher derivable on $\mathrm{R}_{\mathcal{A}}$ is a Lie n-higher derivation. Let $\mathrm{R}_{\mathcal{A}}$ and $\tilde{\mathrm{R}}_{\mathcal{A}}$ be two nonempty subsets of $\mathcal{A}^n$. We also prove that if every linear mapping Lie n-derivable on $\mathrm{R}_{\mathcal{A}}$ is a sum of a derivation on $\mathcal{A}$ and a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ vanishing on all $(n-1)$th commutators of $\tilde{\mathrm{R}}_{\mathcal{A}}$, then every sequence $\{d_m\}$ of linear mappings Lie n-higher derivable on $\mathrm{R}_{\mathcal{A}}$ is a sum of a higher derivation on $\mathcal{A}$ and a sequence of linear mappings from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ vanishing on all $(n-1)$th commutators of $\tilde{\mathrm{R}}_{\mathcal{A}}$.

Throughout this paper, $\mathcal{A}$ is a unital torsion-free algebra over a unital commutative ring $\mathcal{R}$ and I is the identity mapping on $\mathcal{A}$, unless stated otherwise.

2. Lie *n*-higher derivations

In this section, we show that each Lie n-higher derivation can be expressed as a combination of Lie n-derivations.

PROPOSITION 2.1. *Let $\{d_m\}$ be a sequence of linear mappings on $\mathcal{A}$. Then $\{d_m\}$ is a Lie n-higher derivation if and only if $d_0 = I$ and, for each positive integer m,*

$$md_m = \sum_{k=1}^{m} \delta_k d_{m-k}, \tag{2.1}$$

where $\{\delta_m\}$ is a sequence of Lie n-derivations on $\mathcal{A}$.

PROOF. We prove the necessity by induction on m. If $\{d_m\}$ is a Lie n-higher derivation, then, according to the definition, we have that $d_0 = I$ and

$$\begin{aligned} d_1(p_n(x_1, x_2, \ldots, x_n)) &= \sum_{i_1+i_2+\cdots+i_n=1} p_n(d_{i_1}(x_1), d_{i_2}(x_2), \ldots, d_{i_n}(x_n)) \\ &= \sum_{i=1}^{n} p_n(x_1, \ldots, x_{i-1}, d_1(x_i), x_{i+1}, \ldots, x_n) \end{aligned}$$

for each $x_1, x_2, \ldots, x_n$ in $\mathcal{A}$. Take $\delta_1 = d_1$. Then δ_1 is a Lie n-derivation and $d_1 = \delta_1 d_0$. Now suppose that there are Lie n-derivations $\{\delta_m\}$ ($m \le k-1$ and $k \ge 2$) satisfying (2.1). Set $\delta_k = kd_k - \sum_{s=1}^{k-1} \delta_s d_{k-s}$. We only need to prove that δ_k is a Lie n-derivation. For each $x_1, x_2, \ldots, x_n$ in $\mathcal{A}$,

$$\begin{aligned} \delta_k(p_n(x_1, x_2, \ldots, x_n)) &= k \sum_{i_1+i_2+\cdots+i_n=k} p_n(d_{i_1}(x_1), d_{i_2}(x_2), \ldots, d_{i_n}(x_n)) \\ &\quad - \sum_{s=1}^{k-1} \delta_s \sum_{i_1+i_2+\cdots+i_n=k-s} p_n(d_{i_1}(x_1), d_{i_2}(x_2), \ldots, d_{i_n}(x_n)) \\ &= \sum_{i_1+i_2+\cdots+i_n=k} (i_1 + i_2 + \cdots + i_n) p_n(d_{i_1}(x_1), d_{i_2}(x_2), \ldots, d_{i_n}(x_n)) \\ &\quad - \sum_{s=1}^{k-1} \sum_{i_1+i_2+\cdots+i_n=k-s} \sum_{j=1}^{n} p_n(d_{i_1}(x_1), \ldots, \delta_s d_{i_j}(x_j), \ldots, d_{i_n}(x_n)) \\ &= \sum_{j=1}^{n} \Big(\sum_{i_1+\cdots+i_j+\cdots+i_n=k} p_n(d_{i_1}(x_1), \ldots, i_j d_{i_j}(x_j), \ldots, d_{i_n}(x_n)) \\ &\quad - \sum_{s=1}^{k-1} \sum_{i_1+\cdots+i_j+\cdots+i_n=k-s} p_n(d_{i_1}(x_1), \ldots, \delta_s d_{i_j}(x_j), \ldots, d_{i_n}(x_n)) \Big) \\ &= \sum_{s=1}^{k-1} Q_j \quad \text{(say)}. \end{aligned}$$

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Then

$$
\begin{aligned}
Q_j &= p_n(x_1,\dots,kd_k(x_j),\dots,x_n) + \sum_{\substack{i_1+\cdots+r_j+\cdots+i_n=k\\ 1\le r_j\le k-1}} p_n(d_{i_1}(x_1),\dots,r_jd_{r_j}(x_j),\dots,d_{i_n}(x_n))\\
&\qquad - \sum_{s=1}^{k-1} p_n(x_1,\dots,\delta_s d_{k-s}(x_j),\dots,x_n)\\
&\qquad - \sum_{\substack{i_1+\cdots+r_j+\cdots+i_n=k\\ 1\le r_j\le k-1}} \sum_{s=1}^{r_j} p_n(d_{i_1}(x_1),\dots,\delta_s d_{r_j-s}(x_j),\dots,d_{i_n}(x_n))\\
&= p_n\Big(x_1,\dots,kd_k(x_j)-\sum_{s=1}^{k-1}\delta_s d_{k-s}(x_j),\dots,x_n\Big)\\
&\qquad + \sum_{\substack{i_1+\cdots+r_j+\cdots+i_n=k\\ 1\le r_j\le k-1}} p_n\Big(d_{i_1}(x_1),\dots,r_jd_{r_j}(x_j)-\sum_{s=1}^{r_j}\delta_s d_{r_j-s}(x_j),\dots,d_{i_n}(x_n)\Big)\\
&= p_n(x_1,\dots,\delta_k(x_j),\dots,x_n),
\end{aligned}
$$

so $\delta_k(p_n(x_1,x_2,\dots,x_n)) = \sum_{j=1}^{n} p_n(x_1,\dots,\delta_k(x_j),\dots,x_n)$ and δ_k is a Lie n-derivation.

Now we prove the sufficiency. Since $d_0 = I$, (1.4) holds for $m = 0$. Suppose that (1.4) is satisfied for each d_m ($m \le k-1$ and $k \ge 1$). By (2.1), for each $x_1, x_2, \dots, x_n$ in $\mathcal{A}$,

$$
\begin{aligned}
kd_k(p_n(x_1,x_2,\dots,x_n)) &= \sum_{s=1}^{k}\delta_s \sum_{i_1+i_2+\cdots+i_n=k-s} p_n(d_{i_1}(x_1),d_{i_2}(x_2),\dots,d_{i_n}(x_n))\\
&= \sum_{s=1}^{k}\sum_{i_1+i_2+\cdots+i_n=k-s}\sum_{j=1}^{n} p_n(d_{i_1}(x_1),\dots,\delta_s d_{i_j}(x_j),\dots,d_{i_n}(x_n))\\
&= \sum_{\substack{i_1+\cdots+r_j+\cdots+i_n=k\\ r_j\ge 1}}\sum_{s=1}^{r_j}\sum_{j=1}^{n} p_n(d_{i_1}(x_1),\dots,\delta_s d_{r_j-s}(x_j),\dots,d_{i_n}(x_n))\\
&= \sum_{\substack{i_1+\cdots+r_j+\cdots+i_n=k\\ r_j\ge 1}}\sum_{j=1}^{n} p_n(d_{i_1}(x_1),\dots,r_jd_{r_j}(x_j),\dots,d_{i_n}(x_n))\\
&= \sum_{r_1+\cdots+r_j+\cdots+r_n=k}\sum_{j=1}^{n} r_j p_n(d_{r_1}(x_1),\dots,d_{r_j}(x_j),\dots,d_{r_n}(x_n))\\
&= k\sum_{r_1+\cdots+r_j+\cdots+r_n=k} p_n(d_{r_1}(x_1),\dots,d_{r_j}(x_j),\dots,d_{r_n}(x_n)).
\end{aligned}
$$

Since $\mathcal{A}$ is torsion-free, (1.4) is satisfied for d_k. □

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REMARK 2.2. Let $\mathcal{D}$ be the set of all Lie n-higher derivations $\{d_m\}$ on $\mathcal{A}$ where $d_0 = I$, and let Δ be the set of all sequences $\{\delta_m\}$ of Lie n-derivations on $\mathcal{A}$ with $\delta_0 = 0$. Then there is a one-to-one correspondence between $\mathcal{D}$ and Δ, which we can describe explicitly as follows. For $\{\delta_m\}$ in Δ, set $d_0 = I$. The recurrence relation

$$md_m = \sum_{k=0}^{m} \delta_k d_{m-k} \tag{2.2}$$

defines a unique sequence $\{d_m\}$ of linear mappings on $\mathcal{A}$ and, by the sufficiency of Proposition 2.1, $\{d_m\} \in \mathcal{D}$. Conversely, for each $\{d_m\}$ in $\mathcal{D}$, set $\delta_0 = 0$ and consider $\delta_m = md_m - \sum_{k=0}^{m-1} \delta_k d_{m-k}$. This is just another version of (2.2) and defines a unique sequence $\{\delta_m\}$ of linear mappings on $\mathcal{A}$. By the necessity of Proposition 2.1, $\{\delta_m\} \in \Delta$.

COROLLARY 2.3. *With the notation of Remark 2.2, suppose that $\{d_m\} \in \mathcal{D}$ corresponds to $\{\delta_m\} \in \Delta$. Then $\{d_m\}$ is ordinary if and only if $\delta_m = 0$ for each $m \geq 2$. In this case, $m!d_m = d_1^m$ for each positive integer m.*

PROOF. The sufficiency is obvious. We prove the necessity by induction on m. Since $\{d_m\}$ is ordinary, there exists a Lie n-derivation δ on $\mathcal{A}$ satisfying $m!d_m = \delta^m$ for each positive integer m. Then $d_1 = \delta$ and $2d_2 = \delta^2$. By Remark 2.2, $\delta_m = md_m - \sum_{k=0}^{m-1} \delta_k d_{m-k}$ for each positive integer m. Then $\delta_1 = d_1 = \delta$ and $\delta_2 = 0$. Suppose that $\delta_m = 0$ for each integer $2 \leq m \leq k-1$ and $k \geq 3$. Then $\delta_k = kd_k - \sum_{s=0}^{k-1} \delta_s d_{k-s} = kd_k - \delta d_{k-1}$. Thus, $(k-1)!\delta_k = k!d_k - \delta(k-1)!d_{k-1} = \delta^k - \delta\delta^{k-1} = 0$. Since $\mathcal{A}$ is torsion-free, $\delta_k = 0$. □

The proof of the next corollary is similar to the proof of [13, Theorem 2.3].

COROLLARY 2.4 [13]. *Let $\mathcal{A}$ be an algebra over a field of characteristic zero, I the identity on $\mathcal{A}$ and $\{d_m\}$ a Lie n-higher derivation on $\mathcal{A}$ with $d_0 = I$. Then there is a sequence $\{\delta_m\}$ of Lie n-derivations on $\mathcal{A}$ such that*

$$d_m = \sum_{i=1}^{m} \Bigg(\sum_{r_1+r_2+\cdots+r_i=m} \Bigg(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \Bigg) \delta_{r_1}\delta_{r_2}\cdots\delta_{r_i} \Bigg), \quad m = 1, 2, \ldots.$$

3. Characterisations of standard forms of Lie n-higher derivations

THEOREM 3.1. *If every Lie n-derivation of $\mathcal{A}$ is standard, then every Lie n-higher derivation of $\mathcal{A}$ is standard.*

PROOF. Let $\{d_m\}$ be a Lie n-higher derivation on $\mathcal{A}$ with $d_0 = I$. By Proposition 2.1, there is a sequence $\{\delta_m\}$ of Lie n-derivations on $\mathcal{A}$ satisfying (2.1). By assumption, $\delta_m = h_m + \tau_m$ for each nonnegative integer m, where h_m is a derivation and τ_m is a linear mapping from $\mathcal{A}$ into the centre $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$ with $\tau_m(p_n(x_1, x_2, \ldots, x_n)) = 0$ for each $x_1, x_2, \ldots, x_n$ in $\mathcal{A}$.

We prove the theorem by induction on m. Set $g_0 = I$ and $f_0 = 0$. Then $d_0 = g_0 + f_0$. By (2.1), it is obvious that $d_1 = \delta_1 = h_1 + \tau_1$. Set $g_1 = h_1$ and $f_1 = \tau_1$. Then $d_1 = g_1 + f_1$, $\{g_m\}_{m=0,1}$ is a higher derivation and $\{f_m\}_{m=0,1}$ is a sequence of linear mappings from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ with $f_m(p_n(x_1, x_2, \ldots, x_n)) = 0$ $(m = 0, 1)$ for each $x_1, x_2, \ldots, x_n$ in $\mathcal{A}$.

Suppose that, for each nonnegative integer $m \leq k$, there are a higher derivation $\{g_m\}_{m\leq k}$ and a sequence $\{f_m\}_{m\leq k}$ of linear mappings from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ with $f_m(p_n(x_1, x_2, \ldots, x_n)) = 0$ $(m \leq k)$ for $x_1, x_2, \ldots, x_n$ in $\mathcal{A}$ and $d_m = g_m + f_m$. By (2.1),

$$\begin{aligned}(k+1)d_{k+1} &= \sum_{s=1}^{k+1} \delta_s d_{k+1-s} = \sum_{s=1}^{k+1} (h_s + \tau_s)(g_{k+1-s} + f_{k+1-s}) \\ &= \sum_{s=1}^{k+1} h_s g_{k+1-s} + \sum_{s=1}^{k+1} (h_s f_{k+1-s} + \tau_s d_{k+1-s}).\end{aligned}$$

Set $(k+1)g_{k+1} = \sum_{s=1}^{k+1} h_s g_{k+1-s}$ and $(k+1)f_{k+1} = \sum_{s=1}^{k+1}(h_s f_{k+1-s} + \tau_s d_{k+1-s})$. Then $\{g_m\}_{m\leq k+1}$ is a higher derivation by [13, Theorem 2.5], f_{k+1} is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ and $(k+1)d_{k+1} = (k+1)g_{k+1} + (k+1)f_{k+1}$. For each $x_1, x_2, \ldots, x_n$ in $\mathcal{A}$,

$$\begin{aligned}(k+1)f_{k+1}(p_n(x_1, x_2, \ldots, x_n)) &= \sum_{s=1}^{k+1} \tau_s\Big(\sum_{i_1+i_2+\cdots+i_n=k+1-s} p_n(d_{i_1}(x_1), d_{i_2}(x_2), \ldots, d_{i_n}(x_n))\Big) \\ &= \sum_{s=1}^{k+1} \sum_{i_1+i_2+\cdots+i_n=k+1-s} \tau_s(p_n(d_{i_1}(x_1), d_{i_2}(x_2), \ldots, d_{i_n}(x_n))) \\ &= 0.\end{aligned}$$

Since $\mathcal{A}$ is torsion-free, $d_{k+1} = g_{k+1} + f_{k+1}$ and $f_{k+1}(p_n(x_1, x_2, \ldots, x_n)) = 0$ for each $x_1, x_2, \ldots, x_n$ in $\mathcal{A}$. □

Remark 3.2. The above theorem generalises [5, Propositions 3.1 and 3.2], which state that if $\mathcal{A}$ is an algebra over a field of characteristic zero and every Lie derivation (respectively Lie triple derivation) of $\mathcal{A}$ is standard, then every Lie higher derivation (respectively Lie triple higher derivation) of $\mathcal{A}$ is standard.

In [2, 4, 18], the authors discussed sufficient conditions for Lie n-derivations to be standard on $\mathcal{A}$, where $\mathcal{A}$ is a triangular ring, a von Neumann algebra without abelian central summands of type I_1 or a unital algebra with a wide idempotent. From Theorem 3.1, we can obtain sufficient conditions for Lie n-higher derivations to be standard on $\mathcal{A}$.

4. Characterisations of Lie n-higher derivations by local actions

Theorem 4.1. *Let $\mathrm{R}_{\mathcal{A}}$ be a nonempty subset of $\mathcal{A}^n$. If every mapping Lie n-derivable on $\mathrm{R}_{\mathcal{A}}$ is a Lie n-derivation, then every sequence $\{d_m\}$ of linear mappings Lie n-higher derivable on $\mathrm{R}_{\mathcal{A}}$ with $d_0 = I$ is a Lie n-higher derivation.*

Proof. Suppose that $\{d_m\}$ is Lie n-higher derivable on $\mathrm{R}_{\mathcal{A}}$ and $d_0 = I$. Let $\{\delta_m\}$ be a sequence of linear mappings on $\mathcal{A}$ with $\delta_0 = 0$ and $\delta_m = md_m - \sum_{k=0}^{m-1} \delta_k d_{m-k}$.

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By Proposition 2.1, it is sufficient to show that each δ_m is a Lie n-derivation. We prove this by induction on m.

Clearly $\delta_1 = d_1$ is Lie n-derivable on $\mathrm{R}_{\mathcal{A}}$ and it is a Lie n-derivation by assumption. Suppose that δ_m is a Lie n-derivation for each $m \le k$. Then, proceeding in the same way as in the proof of Proposition 2.1, for each $(x_1, x_2, \dots, x_n) \in \mathrm{R}_{\mathcal{A}}$,

$$\begin{aligned}\delta_{k+1}(p_n(x_1, x_2, \dots, x_n)) &= (k+1) \sum_{i_1+i_2+\cdots+i_n=k+1} p_n(d_{i_1}(x_1), d_{i_2}(x_2), \dots, d_{i_n}(x_n)) \\ &\quad - \sum_{s=0}^{k} \delta_s \sum_{i_1+i_2+\cdots+i_n=k+1-s} p_n(d_{i_1}(x_1), d_{i_2}(x_2), \dots, d_{i_n}(x_n)) \\ &= \sum_{j=1}^{n} p_n(x_1, \dots, \delta_{k+1}(x_j), \dots, x_n).\end{aligned}$$

Thus, δ_{k+1} is a Lie n-derivation. □

Remark 4.2. The above theorem prompts us to find conditions under which every linear mapping Lie n-derivable on some nonempty subset $\mathrm{R}_{\mathcal{A}}$ is a Lie n-derivation. While results of this type do not seem to be known, there are a number of papers which prove that for $n = 2$ or 3, every linear mapping δ Lie n-derivable on some nonempty subset $\mathrm{R}_{\mathcal{A}}$ has the form $\delta = h + \tau$, where h is a derivation on $\mathcal{A}$ and τ is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ vanishing on all $(n-1)$th commutators of $\mathrm{R}_{\mathcal{A}}$ (see [3, 7, 8, 10, 12, 14, 16, 17]). In view of these results, we obtain the following theorem.

Theorem 4.3. *Let $\mathrm{R}_{\mathcal{A}}$ and $\tilde{\mathrm{R}}_{\mathcal{A}}$ be two nonempty subsets of $\mathcal{A}^n$. If every linear mapping δ Lie n-derivable on $\mathrm{R}_{\mathcal{A}}$ has the form $\delta = h + \tau$, where h is a derivation on $\mathcal{A}$ and τ is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ vanishing on all $(n-1)$th commutators of $\tilde{\mathrm{R}}_{\mathcal{A}}$, then every sequence $\{d_m\}$ of linear mappings Lie n-higher derivable on $\mathrm{R}_{\mathcal{A}}$ with $d_0 = I$ has the form $d_m = g_m + f_m$, where $\{g_m\}$ is a higher derivation on $\mathcal{A}$ and $\{f_m\}$ is a sequence of linear mappings from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ such that each f_m vanishes on all $(n-1)$th commutators of $\tilde{\mathrm{R}}_{\mathcal{A}}$.*

Proof. Suppose that $\{d_m\}$ is Lie n-higher derivable on $\mathrm{R}_{\mathcal{A}}$ and $d_0 = I$. Let $\{\delta_m\}$ be a sequence of linear mappings on $\mathcal{A}$ with $\delta_0 = 0$ and $\delta_m = md_m - \sum_{k=0}^{m-1} \delta_k d_{m-k}$.

We prove the theorem by induction on m. Set $g_0 = I$ and $f_0 = 0$; then $d_0 = g_0 + f_0$. It is obvious that $\delta_1 = d_1$ is Lie n-derivable on $\mathrm{R}_{\mathcal{A}}$. By assumption, δ_1 has the form $\delta_1 = h_1 + \tau_1$, where h_1 is a derivation of $\mathcal{A}$ and τ_1 is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ vanishing on all $(n-1)$th commutators of $\tilde{\mathrm{R}}_{\mathcal{A}}$. Set $g_1 = h_1$ and $f_1 = \tau_1$. Then $d_1 = g_1 + f_1$, $\{g_m\}_{m=0,1}$ is a higher derivation and $\{f_m\}_{m=0,1}$ is a sequence of linear mappings from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ such that each f_m $(m = 0, 1)$ vanishes on all $(n-1)$th commutators of $\tilde{\mathrm{R}}_{\mathcal{A}}$.

Now suppose that for each nonnegative integer $m \le k$, δ_m is Lie n-derivable on $\mathrm{R}_{\mathcal{A}}$, $\delta_m = h_m + \tau_m$ and $d_m = g_m + f_m$, where $\{h_m\}_{m\le k}$ is a sequence of derivations, $\{g_m\}_{m\le k}$ is a higher derivation and $\{\tau_m\}_{m\le k}$, $\{f_m\}_{m\le k}$ are two sequences of linear mappings from

$\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ vanishing on all $(n-1)$th commutators of $\tilde{\mathrm{R}}_{\mathcal{A}}$. As in the proof of Theorem 4.1, we show that, for each $(x_1, x_2, \ldots, x_n) \in \mathrm{R}_{\mathcal{A}}$,

$$\delta_{k+1}(p_n(x_1, x_2, \ldots, x_n)) = \sum_{j=1}^{n} p_n(x_1, \ldots, \delta_{k+1}(x_j), \ldots, x_n).$$

Thus, δ_{k+1} is Lie n-derivable on $\mathrm{R}_{\mathcal{A}}$. But δ_{k+1} has the form $\delta_{k+1} = h_{k+1} + \tau_{k+1}$, where h_{k+1} is a derivation of $\mathcal{A}$ and τ_{k+1} is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ vanishing on all $(n-1)$th commutators of $\tilde{\mathrm{R}}_{\mathcal{A}}$. Now

$$\begin{aligned}(k+1)d_{k+1} &= \sum_{s=1}^{k+1} \delta_s d_{k+1-s} = \sum_{s=1}^{k+1}(h_s + \tau_s)(g_{k+1-s} + f_{k+1-s}) \\ &= \sum_{s=1}^{k+1} h_s g_{k+1-s} + \sum_{s=1}^{k+1}(h_s f_{k+1-s} + \tau_s d_{k+1-s}).\end{aligned}$$

Set $(k+1)g_{k+1} = \sum_{s=1}^{k+1} h_s g_{k+1-s}$ and $(k+1)f_{k+1} = \sum_{s=1}^{k+1}(h_s f_{k+1-s} + \tau_s d_{k+1-s})$. Then $\{g_m\}_{m \le k+1}$ is a higher derivation by [13, Theorem 2.5], f_{k+1} is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ and $(k+1)d_{k+1} = (k+1)g_{k+1} + (k+1)f_{k+1}$. As in the proof of Theorem 3.1, we see that, for each $(x_1, x_2, \ldots, x_n) \in \tilde{\mathrm{R}}_{\mathcal{A}}$, $(k+1)f_{k+1}(p_n(x_1, x_2, \ldots, x_n)) = 0$. Since $\mathcal{A}$ is torsion-free, we have $d_{k+1} = g_{k+1} + f_{k+1}$ and $f_{k+1}(p_n(x_1, x_2, \ldots, x_n)) = 0$ for each $(x_1, x_2, \ldots, x_n) \in \tilde{\mathrm{R}}_{\mathcal{A}}$. □

Remark 4.4. Take $\mathcal{A}$ to be a generalised matrix algebra, a triangular algebra, a unital prime algebra with nontrivial idempotents, an algebra of all bounded linear operators, a von Neumann algebra or a $\mathcal{J}$-subspace lattice algebra (see [11]). Define two nonempty subsets of $\mathcal{A}^n$ by

$$\begin{aligned}\mathrm{R}_{\mathcal{A}}(n,t) &= \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathcal{A} \text{ with } x_1 x_2 = t\}, \\ \tilde{\mathrm{R}}_{\mathcal{A}}(n,t) &= \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathcal{A} \text{ with } p_n(x_1, x_2, \ldots, x_n) = t\}\end{aligned}$$

for some positive integer n and some element t in $\mathcal{A}$. Let P be an idempotent of $\mathcal{A}$. Under certain conditions, [3, 7, 8, 10, 12, 14, 16, 17] obtained respective characterisations of Lie derivable mappings on $\mathrm{R}_{\mathcal{A}}(2, P)$, Lie derivable mappings on $\mathrm{R}_{\mathcal{A}}(2, 0)$, Lie derivable mappings on $\tilde{\mathrm{R}}_{\mathcal{A}}(2, 0)$, Lie triple derivable mappings on $\mathrm{R}_{\mathcal{A}}(3, P)$ or Lie triple derivable mappings on $\mathrm{R}_{\mathcal{A}}(3, 0)$, as sums of derivations and central mappings vanishing on all commutators of the respective subsets. By Theorem 4.3, we can characterise Lie higher derivable mappings and Lie triple higher derivable mappings on these same subsets.

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