16
Towards M- and F-theory

As we saw in chapter 12, there is an extremely tantalising picture of the fate of string theory at strong coupling, obtained using certain 'duality' transformations. In fact, D-branes were rather useful, as they allowed for an explicit constructive method for finding evidence of the products of duality, for example exhibiting stable states which must exist – with special properties – on both sides of the duality.

One major task is to try to understand how to write better formulations of the physics of strong coupling. There are two main goals to be achieved by this. The first is simply to find better ways of finding new and interesting backgrounds (vacua) for string theory, with techniques which allow for better handing of strongly coupled regions of the solution. The second is to attempt to find the 'correct' manner in which to describe the complete M-theory from which all string theories are supposed to arise as weakly coupled limits.

Both '*Matrix theory*'[157] and '*F-theory*'[199] are ideas in these directions, putting together the strongly coupled brane and string data in ways which allow for new geometric ways of describing and connecting string vacua, and giving insights into the next generation of formulations of the physics. In this chapter we shall uncover aspects of both, while learning much more about the properties of various branes.

16.1 The type IIB string and F-theory

One of the remarkable dualities which we observed in chapter 12 was the 'self-duality' of the type IIB superstring theory. Its fullest expression is in terms of a rich family of transformations which generate the group $SL(2,\mathbb{Z})$. The consequences of this duality group are profound, and we shall uncover some of them in this chapter.

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16.1.1 $SL(2,\mathbb{Z})$ duality

Recall that we saw that the coupling inverted under the 'duality transformation': $g_s \to 1/g_s$, or $\Phi \to -\Phi$, since $g_s = e^{\Phi}$. The fundamental string was exchanged with the D1-brane, from which it follows that the NS–NS two-form potential and the R–R two-form potential (to which those strings couple electrically, respectively), are also exchanged.

In fact, as has been discussed earlier as well, this is all part of a larger duality, whose complete transformation group is $SL(2,\mathbb{Z})$, which is parametrised by matrices of the form:

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ab - cd = 1; \quad a, b, c, d \in \mathbb{Z}. \tag{16.1}$$

Combining the R–R scalar $C_{(0)}$ and the dilaton into a complex coupling $\tau = C_{(0)} + ie^{-\Phi}$, the duality group acts on it as:

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \tag{16.2}$$

and acts on the two-form potentials as

$$\begin{pmatrix} B_{(2)} \\ C_{(2)} \end{pmatrix} \longrightarrow (\Lambda^T)^{-1} \begin{pmatrix} B_{(2)} \\ C_{(2)} \end{pmatrix} \longrightarrow \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} B_{(2)} \\ C_{(2)} \end{pmatrix}. \tag{16.3}$$

So the basic strong weak coupling duality we discovered first is the case

$$\Lambda = S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{16.4}$$

for which we get $\tau \to -1/\tau$, $B_{(2)} \to -C_{(2)}$, $C_{(2)} \to B_{(2)}$. While all of this is taking place, the R–R four-form $C_{(4)}$ is invariant, which has remarkable consequences for the D3-branes which couple to it, as we shall see later in this and other chapters.

In fact, at low energy and tree level, the $SL(2,\mathbb{Z})$ symmetry is only $SL(2,\mathbb{R})$, as the integer restriction to the former case is only visible beyond tree level. The quantisation of the charges of the D-instanton (and by supersymmetry, their action) which couple electrically to $C_{(0)}$ arises in the quantum theory, as we saw in chapter 8. It is very instructive to rewrite the low energy supergravity action (7.42) in a manifestly $SL(2,\mathbb{R})$ invariant way, with the understanding that at this level we can restrict to integers by hand. We work in Einstein frame metric, defined by $G^E_{\mu\nu} = e^{-\Phi/2} G^s_{\mu\nu}$, and find that it is useful to define a field strength doublet $\widehat{G}_{(3)} = (H_{(3)}, G_{(3)})$ and a matrix

$$\mathcal{M} = \frac{1}{\tau_2} \begin{pmatrix} |\tau|^2 & -\mathrm{Re}\tau \\ -\mathrm{Re}\tau & 1 \end{pmatrix} = e^{\Phi} \begin{pmatrix} |\tau|^2 & -C_{(0)} \\ -C_{(0)} & 1 \end{pmatrix}, \qquad \tau = \tau_1 + i\tau_2, \tag{16.5}$$

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and the action is:

$$S_{\mathrm{IIB}} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\left(R + \frac{1}{4}\mathrm{Tr}[\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}] - \frac{1}{12}\widetilde{G}^T_{(3)}\mathcal{M}\widetilde{G}_{(3)}\right.$$
$$\left. - \frac{1}{480}G^2_{(5)}\right) - \frac{\epsilon_{ij}}{4\kappa^2}\int C_{(4)}\wedge G^{[i]}_{(3)}\wedge G^{[j]}_{(3)}, \quad (16.6)$$

(where ϵ_{ij} is antisymmetric with $\epsilon_{12} = 1$) and the $SL(2,\mathbb{R})$ invariance is under:

$$\mathcal{M} \to \Lambda\mathcal{M}\Lambda^T; \qquad \widetilde{G}_{(3)} \to (\Lambda^T)^{-1}\widetilde{G}_{(3)}. \quad (16.7)$$

In fact, $\mathcal{M}$ parametrises the coset $SL(2,\mathbb{R})/SO(2)$, (the dimension of the coset, $3-1=2$ corresponds correctly to the number of scalars) and the kinetic term for the scalars can also be written as

$$-\frac{\partial_\mu\bar{\tau}\partial^\mu\tau}{2(\mathrm{Im}\tau)^2},$$

showing that the metric on the coset space is essentially*† $(\mathrm{Im}\tau)^{-2}$.

16.1.2 The (p,q) strings

We saw in chapter 11 that we can construct a family of strings as bound states of fundamental strings (denoted $(1,0)$) and D1-branes or 'D-strings' (denoted $(0,1)$). It is instructive to construct the supergravity solutions corresponding to these bound states[133]. The metric resembles the Einstein frame version of the D-string metric which we wrote in chapter 10, reproduced here (lying along x_1):

$$\begin{aligned} ds^2 &= H_1^{-3/4}(-dt^2 + dx_1^2) + H_1^{1/4}\sum_{i=2}^{9} dx_i^2, \\ e^\Phi &= g_s H_1^{-1/2}, \qquad C_{(2)} = g_s^{-1}H_1^{-1}dt\wedge dx_1, \\ H_1 &= 1 + \left(\frac{r_1}{r}\right)^6, \end{aligned} \quad (16.8)$$

* This form should also be familiar from chapter 2 when we discovered how to write modular invariant partition functions.

† As an aside, it is worth noting that this is the simplest non-trivial example of a supergravity model for which we find that the scalars are valued on a coset G/H for some non-compact G and compact H. This example will be embedded in more complicated examples later. For example, we have already seen a five dimensional example at the end of chapter 12, arising from compactifying on T^5 to five dimensions. There the scalars live on the coset $E_{6(6)}/USp(8)$, and there are $78-36=42$ of them.

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where r_1^6 is given in equation (10.36) (where we choose $N = 1$ for a single brane), which is normalised so that the $C_{(2)}$ charge of the D1-brane is $\mu_1 = (2\pi\alpha')^{-1}$. It is possible to use the $SL(2,\mathbb{R})$ transformations to write a more general solution[133], which has an asymptotic value of $C_{(0)}$ which is non-zero as well, which we shall call $c_0 = \theta/2\pi$, giving us an asymptotic coupling $\tau_0 = c_0 + i/g_s$. Such a solution is to be interpreted as being in a different vacuum from the usual case where we just have the string coupling switched on.

Defining the asymptotic value of $\mathcal{M}$ to be $\mathcal{M}_0$, (made out of τ_0 in the obvious way, in view of equation (16.5) we define for the (p, q) case:

$$\Delta_{p,q} = (p \quad q)\mathcal{M}_0^{-1}\begin{pmatrix} p \\ q \end{pmatrix} = g_s(p - qc_0)^2 + g_s^{-1}q^2, \tag{16.9}$$

and we get the same form for the metric above, but with

$$\begin{aligned} H_1 &= 1 + \Delta_{p,q}\left(\frac{r_1}{r}\right)^6, \\ C_{(2)}^{[i]} &= \frac{(\mathcal{M}_0^{-1})_{ij}q^{[j]}}{\Delta_{p,q}^{1/2}}(g_s H_1)^{-1}, \\ \tau &= \frac{pc_0 - q|\tau_0|^2 + ipH_1^{1/2}g_s^{-1}}{p - qc_0 + iqH_1^{1/2}g_s^{-1}}, \end{aligned} \tag{16.10}$$

where $q^{[1]} = p$, $q^{[2]} = q$, $C_{(2)}^{[1]} = B_{(2)}$ and $C_{(2)}^{[2]} = C_{(2)}$. The special case $(1, 0)$ is the solution for the fields around the fundamental string[163]. We see from the first line in the above that the tension of the string solution is in fact

$$\begin{aligned} \tau_{p,q}^1 &= \frac{1}{2\pi\alpha'}\sqrt{(p - qc_0)^2 + g_s^{-2}q^2} = \sqrt{([p - qc_0]\tau_{1,0})^2 + (q\tau_{0,1})^2} \\ &= \frac{1}{2\pi\alpha'}|p - q\tau_0|. \end{aligned} \tag{16.11}$$

Notice that we have reproduced the formula (11.12), but generalised to include non-zero asymptotic $C_{(0)}$, denoted c_0. This is a generalisation to a different vacuum than the previous case. In fact, it is interesting to notice that various values of c_0, g_s give interesting patterns for the lightest string, which determines what we would call the perturbative string spectrum!

In the case $c_0 = 0$, the fundamental string $(1, 0)$ is indeed the lightest, for small g_s, as is familiar. Generically, one can always find one such string which is the lightest, for a given value of c_0. This is the dominant string at weak coupling. However, at special values, we can obtain degeneracies. For example, notice that if $|\tau_0| = 1$, we get $\tau_{p,q} = \tau_{q,p}$. Meanwhile $\tau_{p,q} = \tau_{p,p-q}$ if $c_0 = 1/2$ and $g_s^{-2} = 3/4$. Amusingly, at $\tau_0 = e^{\pi i/3}$, all three of the 'simplest' strings are degenerate: $\tau_{1,0} = \tau_{0,1} = \tau_{1,1}$. Also, for $\tau_0 = e^{2\pi i/3}$, which differs from the previous τ_0 by one, we have $\tau_{-1,0} = \tau_{0,-1} = \tau_{-1,-1}$, which are the strings we encountered before in reverse orientation. Geometrically, $\tau = e^{\pi i/3}, e^{2\pi i/3}$ are the special 'orbifold' points of the fundamental region of the $SL(2,\mathbb{Z})$ shown in figure 3.3. This fits rather well with what we already discussed in chapter 11, where we saw that we could form a three string junction, by balancing the tensions of the three types of string. At this point of the moduli space of (p, q) string theories the junction diagram is $\mathbb{Z}_3$ symmetric.

16.1.3 String networks

Recalling the three string junction[135, 137, 140] that we encountered in section 11.4, it must have already occurred to the reader that there is an amusing construction that follows. We can make a network[138] of such string junctions, preserving some supersymmetry. Let us see how this junction must work.

First, note that when three strings meet, with charges (p_i, q_i) for the ith string, the sum of the charges must vanish:

$$\sum_{i=1}^{3} p_i = 0 = \sum_{i=1}^{3} q_i. \tag{16.12}$$

In addition, we must balance the forces exerted by each string, so as to achieve a stable configuration. Let the ith string by oriented along a unit vector $\hat{n}_i$. Then, given that it has tension τ_{p_i,q_i}, the balance condition is:

$$\sum_{i=1}^{3} \tau_{p_i,q_i} \hat{n}_i = 0. \tag{16.13}$$

Now recall that our tension formula is simply

$$\tau_{p,q} = |p + q\tau|.$$

Consider the complex number $p + q\tau$. Its modulus is the tension given above, while its argument shall be denoted $\phi(p, q, \tau)$:

$$p + q\tau = |p + q\tau| e^{i\phi(p,q,\tau)} = \tau_{p,q} e^{i\phi(p,q,\tau)}.$$

Let us now rewrite our force and charge balancing conditions in terms of this. First, the charge conditions (16.12) tell us that

$$\sum_{i=1}^{3} (p_i + iq_i) = 0,$$

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and therefore:

$$\sum_{i=1}^{3} \tau_{p_i,q_i} e^{i\phi(p_i,q_i,\tau)} = 0.$$

This is two equations, a real and imaginary part, which we can get to agree with the force balance equation (16.13) if we simply set

$$\hat{n}_i = \begin{pmatrix} \cos\phi(p_i,q_i,\tau) \\ \sin\phi(p_i,q_i,\tau) \end{pmatrix}.$$

What does this mean? Well, our result tells us that we can achieve a completely balanced string network of (p,q) strings if any string with charges (p,q) is oriented at angle $\phi(p_i,q_i,\tau)$ in the plane, i.e. pointing in the direction given by $p+q\tau$. Note that this result does not depend on the location of any string within the network, just its orientation. So we can build a string network of arbitrary size out of (p,q) strings (see figure 16.1).

This solution, and the fact that it preserves eight supercharges, is very interesting, and perhaps suggestive of something remarkable, like a new non-perturbative building block of the type IIB string theory. It is particularly suggestive because it reminds one of a number of diagrams that occur elsewhere in theoretical physics, such as planar diagrams for large N gauge theory, dual triangulations of string world sheets, etc. Speculations of this sort based on pictures alone are of course easy to do, and so it would be interesting to see if there are connections with firmer foundations which might be exploited fruitfully.

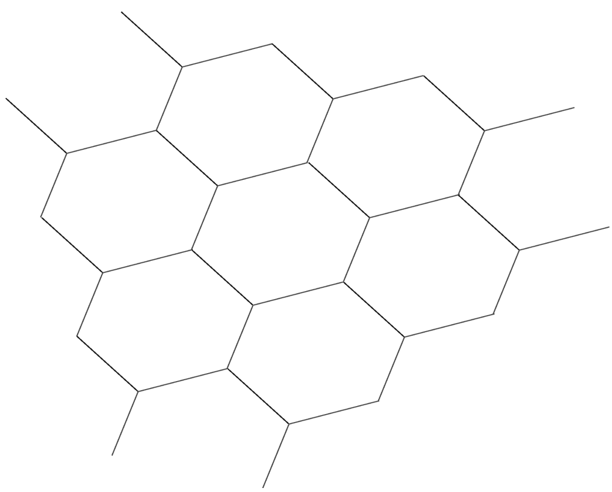

Fig. 16.1. A string network.

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16.1.4 The self-duality of D3-branes

As has been remarked upon previously, the four-form potential $C_{(4)}$ is invariant under the $SL(2,\mathbb{Z})$ duality transformation. This must mean something quite remarkable for the D3-brane which couples to it, since the world-volume action of the D3-brane couples to all of the background fields we have been discussing so far that *do* have non-trivial $SL(2,\mathbb{Z})$ duality transformation properties. In Einstein frame, the action is:

$$
\begin{aligned}
S = &-\tau_3 \int_{\mathcal{M}_4} d^4\xi\ \det{}^{1/2}[G_{ab} + e^{-\Phi/2}\mathcal{F}_{ab}] \\
&+ \mu_3 \int_{\mathcal{M}_4} \left(C_{(4)} + C_{(2)} \wedge \mathcal{F} + \frac{1}{2} C_{(0)}\, \mathcal{F} \wedge \mathcal{F}\right),
\end{aligned}
\tag{16.14}
$$

where $\mathcal{F}_{ab} = B_{ab} + 2\pi\alpha' F_{ab}$, and $\mathcal{M}_4$ is the world-volume of the D3-brane, with coordinates $\xi^0, \ldots, \xi^3$. As usual, the parameters μ_3 and τ_3 are the basic R–R charge and tension of the D3-brane:

$$
\mu_3 = \tau_3 g_\mathrm{s} = (2\pi)^{-3}(\alpha')^{-2}. \tag{16.15}
$$

Also, G_{ab} and B_{ab} are the pulls-back of the ten dimensional metric (in Einstein frame) and the NS–NS two-form potential, respectively.

Before we do anything else, let us stop to think about what is going on at low energy, in flat space. Let us also switch off the the background antisymmetric tensor fields. The theory then becomes gauge theory, in fact, the $\mathcal{N} = 4$ supersymmetric four dimensional $SU(N)$ gauge theory (if we have N branes and neglected the overall centre of mass). This theory has a number of special properties. It is supposed to be conformally invariant in the full quantum theory. That it is classically scale invariant is of course trivial. For a start, all of the fields are massless. Furthermore a quick dimensional analysis shows that the coupling g_YM has to be dimensionless, and indeed, our formula for it in terms of the closed string coupling sets it to be $g^2_\mathrm{YM} = 2\pi g_\mathrm{s}$. The theory's θ-angle is set by the R–R scalar $C_{(0)}$. The statement that it is quantum mechanically conformally invariant is highly non-trivial. This means that the β-function vanishes, or that the trace of the full energy-momentum tensor vanishes, etc. This is more involved, and we shall see that this does follow from the properties of D3-branes, in chapter 18, in remarkably interesting ways.

Another property that this theory is supposed to have is exact $SL(2,\mathbb{Z})$ 'S-duality', generalising the following electromagnetic duality which one would expect for the Abelian case:

$$
\begin{aligned}
S &= -\tau_3 \int d^4x \mathcal{L} \\
\mathcal{L} &= -\frac{1}{4} e^{-\Phi} F_{\mu\nu}F^{\mu\nu} + \frac{1}{4} C_{(0)} F_{\mu\nu}{}^*F^{\mu\nu}.
\end{aligned}
\tag{16.16}
$$

We have the electromagnetic field $\vec{E}$ and the magnetic induction $\vec{B}$ arising from $F_{\mu\nu}$ as $E_i = F_{i0}$ and $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$. $F_{\mu\nu}$ satisfies a Bianchi identity $\partial_{[\lambda}F_{\mu\nu]} = 0$. The (source-free) field equations are given in terms of another antisymmetric tensor $\widetilde{F}_{\mu\nu}$ as $\partial_{[\lambda}{}^*\widetilde{F}_{\mu\nu]} = 0$. In the absence of $C_{(0)}$, the theta-angle, this would simply be the $F_{\mu\nu}$ we first thought of, but more generally it is[276]:

$$\widetilde{F}_{\mu\nu} \equiv -2\frac{\delta\mathcal{S}}{\delta F_{\mu\nu}} \tag{16.17}$$

and from it we get the electric induction $\vec{D}$ from $D_i = \widetilde{F}_{i0}$ and the magnetic field $\vec{H}$ as $H_i = \frac{1}{2}\epsilon_{ijk}\widetilde{F}_{jk}$. These are related to the previous fields as:

$$\begin{aligned}\vec{D} &= \frac{\partial\mathcal{L}}{\partial\vec{E}} = e^{-\Phi}\vec{E} + C_{(0)}\vec{B}\\ \vec{H} &= \frac{\partial\mathcal{L}}{\partial\vec{B}} = e^{-\Phi}\vec{B} - C_{(0)}\vec{E}.\end{aligned} \tag{16.18}$$

In components, the Bianchi identities and field equations are the familiar ones:

$$\begin{aligned}\nabla\cdot\vec{B} &= 0, \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\\ \nabla\cdot\vec{D} &= 0, \qquad \nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t}.\end{aligned} \tag{16.19}$$

These 'constituitive relations' may be written in terms of our earlier defined matrix $\mathcal{M}$:

$$\begin{pmatrix}\vec{H}\\ \vec{E}\end{pmatrix} = \mathcal{M}\begin{pmatrix}\vec{B}\\ \vec{D}\end{pmatrix}. \tag{16.20}$$

The $SL(2,\mathbb{Z})$ duality transformations are then easily written as:

$$\begin{pmatrix}\vec{H}\\ \vec{E}\end{pmatrix} \to (\Lambda^T)^{-1}\begin{pmatrix}\vec{H}\\ \vec{E}\end{pmatrix}, \qquad \begin{pmatrix}\vec{B}\\ \vec{D}\end{pmatrix} \to \Lambda\begin{pmatrix}\vec{B}\\ \vec{D}\end{pmatrix}, \tag{16.21}$$

which leave the relations (16.20) invariant, in view of the transformation of $\mathcal{M}$ given in equation (16.7).

Going to the full Born–Infeld Lagrangian, it has been shown (we will not do it here) that the duality still holds. Furthermore, inclusion of the coupling to the two-form potential preserves the $SL(2,\mathbb{Z})$ duality, provided that they transform according to equation (16.3).

Considering two D3-branes gives an $SU(2)$ gauge group, (neglecting the overall $U(1)$) and the S-duality is still supposed to hold, but with the dual theory having the dual $SO(3)$ gauge group. More generally, in this

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'Montonen–Olive duality'[277], gauge group G is replaced by a gauge group G^* whose weight lattice is dual to that of G. This is not a subject we shall go into here, although it is a beautiful one[277].

Note, however, that we can translate the expected spectrum of BPS monopoles and dyons in the gauge theory to the case here. Recall from section 13.5 that if the branes separate by some distance L, these are made by stretching the (p, q) strings between them, and ending on the D3-branes' surface, the $SU(2)$ having been broken to a $U(1)$, and the Higgs vev is set by L. Observe that we can surround a string with an S^7. This means that the point at the end of the string can be surrounded by an S^8. Meanwhile, we can locate the D3-brane world-volume as a point in $\mathbb{R}^6$, and so it can be surrounded by an S^5. Finally, to specify the location of the endpoint inside the worldvolume, we can surround it by an S^2. So the source equation for the string in ten dimensions is supplemented by a contribution from the D3-brane action[276]:

$$d^*\widetilde{G}^{[i]}_{(3)} = \mu_1^{[i]}\delta^8(\mathbf{x}) + \sum_\alpha \frac{\delta S^\alpha}{\delta C^{[i]}_{(2)}} \wedge \delta^6(\mathbf{x}), \tag{16.22}$$

where $C^{[1]}_{(2)} = B_{(2)}$ and $C^{[2]}_{(2)} = C_{(2)}$, the NS–NS and R–R form potentials respectively, and $\mu_1^{[i]}$ are the charge per unit length of the fundamental string and D-string. Also α labels each D3-brane. Here, the Hodge dual is performed in ten dimensions, and so on both sides we have something which can be integrated over S^8 in order to measure the charge. Performing the integral, and observing how the R–R and NS–NS potentials couple in the action (16.14), we have explicitly:

$$0 = \mu_1^{[1]} + \int_{S^2} \mathcal{F}, \qquad 0 = \mu_1^{[2]} + \int_{S^2} {}^*\widetilde{F}. \tag{16.23}$$

This shows that the charges of the string endpoints are correlated with the spacetime charges of the strings, allowing them to furnish the complete set of (p, q) dyons in the field theory, and the $SL(2, \mathbb{Z})$ strong/weak coupling duality descends correctly to these states as well, and they have masses $m_{p,q} = \tau_{p,q}L$.

16.1.5 (p, q) Fivebranes

In a very similar way to the construction of the supergravity solution for the (p, q) strings, a family of $(p. q)$ fivebranes may be constructed, filling out the expectation that such objects ought to exist in view of ten dimensional string/fivebrane duality, hence sourcing the doublet of two form potentials magnetically. The solution may be written in Einstein frame as:

$$ds^2 = H_5^{1/4}(-dt^2 + \sum_{i=1}^{5} dx_i^2) + H_5^{-3/4} \sum_{i=6}^{9} dx_i^2,$$
$$H_5 = 1 + \Delta_{p,q}\left(\frac{r_5}{r}\right)^2, \tag{16.24}$$

with $\Delta_{p,q}$ given in equation (16.9), and expressions for $C_{(6)}^{[i]}$ and τ similar to the ones written for the (p,q)-strings in equation (16.10). The of these solutions therefore comes out to be:

$$\tau_{p,q}^5 = \sqrt{([p - qc_{(0)}]\tau_{1,0}^5)^2 + (q\tau_{0,1}^5)^2}, \tag{16.25}$$

the expected analogous equation to the (p,q) string tension (16.11).

16.1.6 $SL(2,\mathbb{Z})$ and D7-branes

Let us consider the case of the action (16.6) with all of the higher rank potentials switched off. Furthermore, let us worry only about non-trivial structure in the x_8 and x_9 directions, leaving the directions $t, x_1, \ldots, x_8$ untouched. Let us write a complex coordinate $z = x_8 + ix_9$, in terms of which the action and equations of motion from varying it with respect to $\bar{\tau}$ are:

$$S = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\left(R - \frac{\partial\tau\bar{\partial}\bar{\tau}}{2(\mathrm{Im}\tau)^2}\right),$$
$$\partial\bar{\partial}\tau + \frac{2\partial\tau\bar{\partial}\tau}{\bar{\tau} - \tau} = 0. \tag{16.26}$$

A simple trial solution to this which preserves half the supersymmetries is to ask that τ is in fact holomorphic: $\partial\tau(z,\bar{z}) = 0$. Now recall that a D7-brane carries the magnetic charge of $C_{(0)}$. Notice further that we have its $C_{(8)}$ charge is $\mu_7 = (2\pi)^{-7}(\alpha')^{-4}$, which happens to match the normalisation of our action, $1/(2\kappa^2)$, and so in circling a single D7-brane once, $C_{(0)}$ should change by precisely 1 in order to register the correct amount of D7-brane charge (recall that we integrate ${}^*dC_{(8)}$ around the S^1 to measure a D7-brane's charge).

Using this information, a suitable choice for a D7-brane located at $z = 0$ would seem to be:

$$\tau(z) = \frac{1}{2\pi i}\log(z), \tag{16.27}$$

since circling the origin will produce a jump $\tau \to \tau + 1$. This is a good description of the object for a range of distances, but there are problems.

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Near the origin, $\mathrm{Im}\tau$ is becoming large and negative, which cannot make sense, since it should be positive, given that it is the inverse string coupling. So there the solution breaks down, but this is perhaps fine, since we can simply use open string perturbation theory there, in the spirit of previous brane solutions which break down near the origin. We can generalise this trivially to many branes located at points z_i by writing:

$$\tau(z) = \frac{1}{2\pi i}\log(z - z_i). \tag{16.28}$$

Unfortunately, at large z, the solution is not very good either. If there was a four dimensional problem (i.e. with only one other spatial direction) this solution would be a 'cosmic string', and as such, the energy per unit length diverges for this solution, and so we cannot also solve the gravity equations.

Recall however that τ is allowed to jump by an $SL(2,\mathbb{Z})$ transformation. This can be exploited[279], since now τ is not just any number. The inequivalent values of it are restricted to lie in the fundamental domain $\mathcal{F}$ in figure 3.3. So the energy density is now controlled by:

$$\frac{1}{2\kappa^2}\int d^2z\left(\frac{1}{2}\frac{\partial\tau\bar{\partial}\bar{\tau}}{(\tau-\bar{\tau})^2}\right) = \frac{1}{2\kappa^2}\int d^2z\left(\frac{1}{2}\partial\bar{\partial}\log(\tau-\bar{\tau})\right), \tag{16.29}$$

but we can convert this to an integral over the fundamental domain in the τ plane via:

$$d\tau d\bar{\tau} = dz d\bar{z}\frac{\partial\tau}{\partial z}\frac{\partial\bar{\tau}}{\partial\bar{z}}$$

to give:

$$\frac{1}{2\kappa^2}\int_{\mathcal{F}} d^2\tau\left(\frac{1}{2}\partial\bar{\partial}\log(\tau-\bar{\tau})\right), \tag{16.30}$$

and we can integrate by parts to perform a boundary integral over the edge of the domain to give $2\pi/12$ for the integral, which is the mass density in units of $1/2\kappa^2$. Actually, we have assumed that we have flat space for the solution. This is not correct, really, since the energy density in the τ field ought to have a non-trivial back reaction on the geometry. Let us attempt to find a solution which looks like the following (inspired by the structure of the case $p = 7$ in equation (10.38)):

$$ds^2 = -dt^2 + \sum_{i=1}^{7} dx_i^2 + H_7(z,\bar{z})dzd\bar{z}. \tag{16.31}$$

In fact, the equations of motion for the τ field are not modified by this ansatz, since they would have included contributions from the combination $(-G)^{1/2}G^{z\bar{z}}$, which remains unchanged with the above ansatz. The

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only non-trivial equation which results from this is

$$R_{00} - \frac{1}{2}G_{00}R = -\frac{1}{H_7}\frac{1}{8\tau_2^2}G_{00}\partial\tau h\bar{\partial}\bar{\tau}. \tag{16.32}$$

In fact, this can be written as:

$$\partial\bar{\partial}\log H_7 = \frac{\partial\tau\bar{\partial}\bar{\tau}}{\tau_2^2} = \partial\bar{\partial}\log\tau_2. \tag{16.33}$$

This is just Poisson's equation in two dimensions. The source is $\partial\bar{\partial}\log\tau_2$, and its energy density of $2\pi/12$ is the total charge in the problem. An obvious long distance solution is:

$$\log H_7 = -\frac{1}{12}\log|z|.$$

Looking back at the metric, we see that the z-plane has metric $ds^2 \sim |z^{-1/12}dz|^2$. We can change variables to $\tilde{z} = z^{1-1/12}$, and see that the metric is flat $ds^2 \sim |d\tilde{z}|^2$, but there is a deficit angle of $2\pi/12$, since as we do a complete circle in z, $\tilde{z}$ only goes around part of the way.

It is straightforward to see that if there are N copies of this sort of solution, the result is $\log H_7 = -\frac{N}{12}\log|z|$ and so the metric is $ds^2 \sim |z^{-N/12}dz|^2$. There is a deficit angle of $2\pi N/12$. Let us consider the case of $N = 24$. Well, by a change of variables similar to what we did previously, $\tilde{z} = z^{1-N/12}$, for $N = 24$, we get $\tilde{z} = 1/z$, and then the metric is $ds^2 \sim |d\tilde{z}|^2$, but *the periodicity of z and $\tilde{z}$ are the same.* So there is no conical singularity. We have just built a familiar space, $\mathbb{CP}^1$, or in more familiar terms, S^2, which of course has 'deficit' angle 4π. This is highly suggestive, as we shall see.

Let us try to make an exact solution of the equations of motion (16.33). Actually, to be careful, we should construct a solution to which is manifestly modular invariant. A guess at a solution is obviously $\log H_7 = \tau_2$, but this fails because τ_2 is not modular invariant. Because the operator $\partial\bar{\partial}$ acts, we are free to add anything which is annihilated by this to our guess, in other words, the real part of any holomorphic function. Well, this is where our experience with modular invariance from one-loop string theory in chapter 3 suddenly becomes useful. A nice candidate is in fact to replace τ_2 with $\tau_2\eta^2\bar{\eta}^2$, where η is Dedekind's function, which we met in equation (3.58), since that combination is modular invariant, being a one-loop string partition function. Recall that $q = e^{2\pi i\tau}$. A final requirement is that we must not let the metric function H_7 go to zero. With our present prescription, it goes to zero at a generic point z_i where a seven-brane is located. This is because near there, we have the behaviour given by equation (16.27) and so $q \sim z - z_i$ with the result $H_7 \sim |(z-z_i)^{1/12}|^2$. So, multiplying in the inverse of such a factor for each of the N points, we have finally[279]:

$$H_7 = \tau_2 \eta^2 \bar{\eta}^2 \left| \prod_{i=1}^{N} (z - z_i)^{-1/12} \right|^2 . \tag{16.34}$$

16.1.7 Some algebraic geometry

Let us step back and see what we are doing. We actually are studying a background in which $\tau(z)$ and hence the string coupling varies as we move around the plane transverse to the sevenbrane. We can solve the full equations of motion if we have 24 of the branes present, and the transverse space curls up into an S^2, or $\mathbb{CP}^1$. The function τ varies over the $\mathbb{CP}^1$ and is acted on by $SL(2,\mathbb{Z})$, the physically distinct values being given by the fundamental region $\mathcal{F}$ given in figure 3.3. We can visualise this geometry by thinking of an auxiliary torus T^2 which is fibred over the $\mathbb{CP}^1$, since τ can always be thought of as the modulus of the torus. The torus can change as $\tau \to \tau+1$ as it circles a sevenbrane. However, as we shrink that circle to a point, maintaining this condition is rather singular, and the result is that a cycle of the torus must degenerate over the point. We have the idea that as we encircle the point, there is a 'monodromy', meaning that everything that can transform under $SL(2,\mathbb{Z})$ gets multiplied by the matrix

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This happens generically in 24 places, and the physics of it will become much clearer later.

We can describe this all in a rather amusing (and powerful) way, using a small amount of algebraic geometry. Consider three complex coordinates x, y, w. We will identify points as follows: $(x,y,w) \sim (\lambda x, \lambda y, \lambda w)$, for some complex number λ. The resulting four dimensional space is $\mathbb{CP}^2$. This is a generalisation of the more familiar $\mathbb{CP}^1$ which is simply the sphere, as described in insert 16.1.

Starting with our $\mathbb{CP}^2$ coordinates (x,y,w), consider the following homogeneous equation of degree three, giving the 'Weierstrass' form:

$$W(x,y,w) = y^2 w - x^3 - f x w^2 - g w^3 = 0, \tag{16.35}$$

where f and g are constants. Here, homogeneous of degree three means that $W(\lambda x, \lambda y, \lambda w) = \lambda^3 W(x,y,w)$. This equation will give us some one complex dimensional object as a subspace of $\mathbb{CP}^2$. In fact, it is a torus

Insert 16.1. S^2 **or** $\mathbb{CP}^1$ **from affine coordinates**

As a simple example of the use of affine coordinates to define something familiar, let us look at the sphere, S^2, which in this language is better called $\mathbb{CP}^1$. We start with two complex coordinates, (x, y). Our space of interest has one complex dimension made by identifying $(x, y) \sim (\lambda x, \lambda y)$. Now lets find the space we want. If $y \neq 0$, then we can set y to one by an appropriate choice of λ. Then we have one complex coordinate x, giving a plane. A plane differs from S^2 or $\mathbb{CP}^1$ by the addition of the point at infinity. Indeed, we have this point in the description. It is the case $y = 0$, for which we can set $x = 1$ by the scaling, giving our final point. In other words, we can recover the standard North and South pole preferred projections of the S^2 to a plane seen in elementary geometry: one is the x plane with $y = 1$, and the other is the y plane with $x = 1$.

T^2. This is true for any such cubic in $\mathbb{CP}^2$, and we can see it as follows. A single complex equation in $\mathbb{CP}^2$ gives a one complex dimensional (or Riemann) surface Σ, and so all we need to do is determine its genus, or Euler number, which completely classifies it, as stated in chapter 2. After a change of variables, we can write our equation as $w^3 = x^3+y^3$. Let us first assume that x^3+y^3 does not vanish. Then our equation yields three generically distinct values of w for each (x, y), which on their own each form a $\mathbb{CP}^1$. So naively, the equation has the Euler number of three $\mathbb{CP}^1$s, which is $3 \times 2 = 6$. But there are three roots of $x^3 + y^3 = 0$, and so the equation requires that $w = 0$ in that situation. These make three points, each of which are represented three times, once on each $\mathbb{CP}^1$. Let us remove the three points from the $\mathbb{CP}^1$s, and hence the Euler number of Σ-{points} is $3(2 - 3) = -3$ and then we must add back in the missing three points, giving a total of zero, the Euler number of a torus.

We can see a torus more directly as follows. Let us first assume that $w \neq 0$, and so we can set it to unity. Then we have $y^2 = x^3 + fx + g$. The solutions for y are double valued, giving two copies of the $\mathbb{CP}^1$ given by x. (We have added the point at infinity in x.) However, there are three places where the cubic vanishes, giving us a place where y is single valued. Together with the point at infinity, this allows us to draw two branch cuts through which to join the two 'branches' of y. We connect the two $\mathbb{CP}^1$s through two separate cuts forming tubes which construct for us a torus. See figure 16.2.

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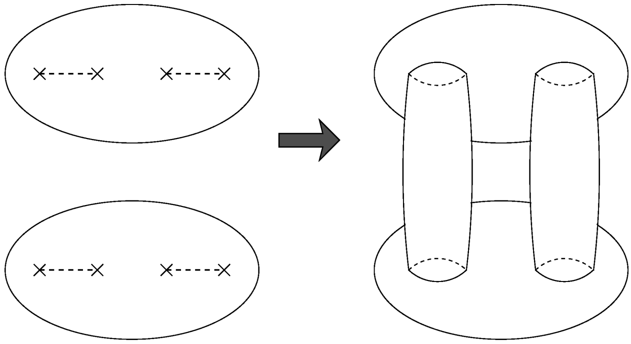

Fig. 16.2. Why a cubic gives a torus.

Let stay with the $w = 1$ or 'affine' form for a while. This form keeps us in the picture where x forms a plane over which y takes its values; y is double valued everywhere except where the cubic $x^3 + fx + g = 0$ has roots. It is an elementary fact that the nature of the roots of this cubic is determined by the discriminant which is proportional to $\Delta = 4f^3 + 27g^2$. We have three situations,

- $\Delta > 0$ There is one real root and a pair of complex ones.
- $\Delta = 0$ All of the roots are real, and at least two are equal.
- $\Delta < 0$ There are three distinct real roots.

We sketch these cases in various ways in figure 16.3 for (y, x) real.

In the case where the roots are distinct ($\Delta \neq 0$), we can make a torus as described above and depicted in figure 16.2. We can see how the generic

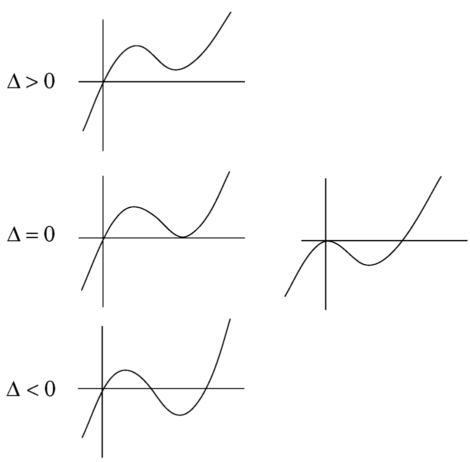

Fig. 16.3. Real cubics and their roots.

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two classes of one-cycle of the torus are made by the two classes of journey one can make through the cuts, as shown in figure 16.4. There, we have also noted that a shift and a scaling on x can be used to put a root at zero and another at unity, and then the final root is at λ, giving the form

$$y^2 = x(x-1)(x-\lambda).$$

However, consider the case when $\Delta = 0$ and two roots coincide. Then one or other class of cycle can pinch off, causing the torus to degenerate. One may ask what the complex structure τ of a torus presented in the form (16.35) might be. It is given by the famous j-function:

$$j(\tau) \equiv \frac{\left(\theta_2^8(\tau) + \theta_3^8(\tau) + \theta_3^8(\tau)\right)^3}{\eta^{24}(\tau)} = \frac{4(24f)^3}{4f^3 + 27g^2}. \tag{16.36}$$

The function $j(z)$ is a very special one. It is a modular invariant complex number, and is in fact a one-to-one map of the fundamental region $\mathcal{F}$ to the complex plane. Since the denominator is the discriminant, we see that when the torus degenerates ($\Delta = 0$), $j(\tau)$ diverges.

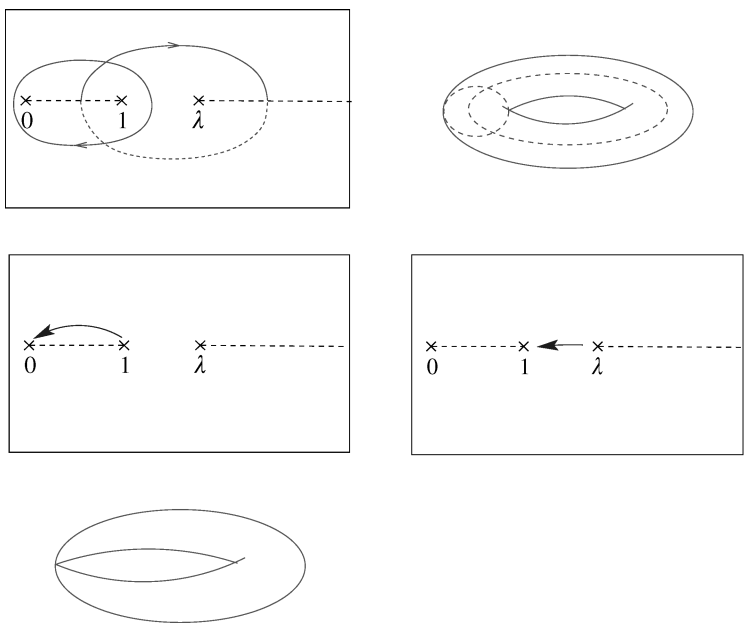

Fig. 16.4. The top sketches show one sheet of the cut complex x plane and the generic torus made from it, including the two classes of one-cycle (cf. figure 16.2). (Note that the dotted half of one of the curves is in fact on the other sheet.) The bottom sketches show how the torus can degenerate if roots collide, giving $\Delta = 0$ (cf. figure 16.3).

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16.1.8 F-theory, and a dual heterotic description

Let us return to our problem of describing seven branes. The degeneration of a torus is exactly what happens when we are located at a seven brane, since if encircling one of charge 1 produces the jump $\tau \to \tau + 1$, then shrinking the loop to a point shows that the torus associated to that point must be degenerate.

We saw that we had a sensible solution of the equations of motion if we have generically 24 sevenbranes located on a sphere. The coupling τ can be allowed to vary as we move around the sphere, with coordinate z, between the sevenbranes. We can then associate a torus with every value of $\tau(z)$, thus making a *fibred structure*[199] of T^2 over $\mathbb{CP}^1$. At the location of a sevenbrane, we must have the torus degenerate, which is a statement that our fibration has 24 places where the torus fibre degenerate (see figure 16.5). We can describe this using the language above by allowing the numbers f, g become functions $f(z), g(z)$. Then we have that $\Delta(z) = 4f^3(z) + 27g^2(z)$ must vanish in 24 places. We can achieve this by making $f(z)$ an eighth order polynomial in z and $g(z)$ a twelfth order polynomial, and so we have:

$$W(x, y, z) = y^2 - x^3 - f(z)x - g(z) = 0. \tag{16.37}$$

Now observe that there are nine coefficients to specify $f(z)$ and thirteen for g. Four of these are parameters are redundant, however. For the first, scale $f \to \lambda^2 f, g \to \lambda^3 g$ which gives no change of the torus, as is evident from equation (16.36). For the other three, recall from chapter 3 that there is an $SL(2, \mathbb{C})$ action on the $\mathbb{CP}^1$ of z that allows up to three points to be placed at positions of one's choice (typically $z = 0, 1, \infty$). So there are 18 complex parameters which go into this solution.

Mathematically, this all fits the fact that the moduli space of K3 manifolds which can be written as an *'elliptic'* (i.e. torus) fibration is 18 complex dimensional, with a local description as:

$$\mathcal{M}_{\text{K3elliptic}} = \frac{O(18,2)}{O(18) \times O(2)}. \tag{16.38}$$

Our fibration of T^2 over $\mathbb{CP}^1$ builds our friend the K3 manifold for us (see figure 16.5).

Furthermore, the reader might recognise this local structure from section 7.4. It is the local description of the moduli space of the heterotic string compactified on T^2. Let's check the counting. We get two complex parameters from the internal components of the graviton and the antisymmetric tensor: G_{ij} is symmetric and B_{ij} is antisymmetric, and $i, j = 8, 9$. Also, the rank 16 gauge group ($SO(32)$ or $E_8 \times E_8$) can have 16 Wilson

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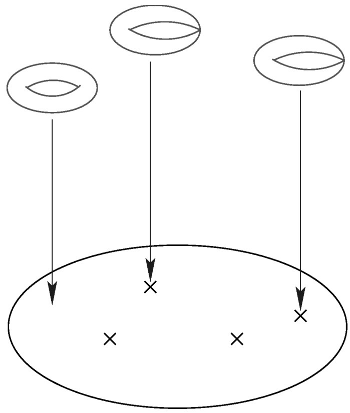

Fig. 16.5. The F-theory description of 24 sevenbranes (located at the crosses) as a torus fibration of T^2 over $\mathbb{CP}^1$. In fact, this is a description of K3 as an elliptic fibration!

lines on each direction of the torus. This gives 18 complex moduli in total, with generic gauge group $U(1)^{18} \times U(1)^2$. The extra $U(1)^2 \times U(1)^2$ supplementing the generic $U(1)^{16}$ gauge group from the current algebra sector, comes from the internal components $G_{\mu i}$, $B_{\mu i}$.

There is one more important parameter we ought to consider, the heterotic string coupling. This is identified with the size of the $\mathbb{CP}^1$ base of the fibration, which we are free to specify in making the elliptic fibration. We shall see this explicitly later. The other parameters we have naively available to us on the IIB side are not accessible. We cannot switch on either of the two-form fields since they transform under the $SL(2, \mathbb{Z})$. Furthermore, the torus fibres only have complex structure parameters; we should not think of them as tori whose Kähler structure (i.e. their size) can vary. By construction, only τ has physical meaning, at least in this type IIB picture.

The fact that the size of the $\mathbb{CP}^1$ is essentially the heterotic string coupling fits nicely with the expectation that the limit where we have a very small sphere over which the IIB coupling is varying greatly (due to the presence of 24 branes) would benefit from a weakly coupled dual string theory description.

16.1.9 (p, q) Sevenbranes

So far this duality is motivated by plausibility arguments. It would be nice to demonstrate this duality more in detail, and happily we have the tools to do it. The first thing to note is that we have 24 seven branes, but the duality to the heterotic string suggests that we only have $U(1)^{18} \times U(1)^2$ as the generic gauge group. Now a $U(1)^2$ of this (on this type IIB side) comes

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from internal components of the metric, $G_{\mu i}$, $(i = 8, 9)$ leaving a prediction that somehow, as many as six sevenbranes are not able to contribute. We can resolve this as follows. The description of $U(1)$s on the world-volume of D-branes is in terms of fundamental strings, or, more specifically, $(1, 0)$ strings, using the description of section 16.1.2. Correspondingly, since $\tau \longrightarrow \tau + 1$ as we encircle one, the monodromy matrix about the sevenbrane is

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which leaves these string charges invariant. Clearly, we have the useful idea of a (p, q) sevenbrane[199, 200], which is a sevenbrane on which a (p, q) string can end. What is the monodromy about such a brane? Well, let us imagine that we transform from $(1, 0)$ string to a (p, q) string using an $SL(2, \mathbb{Z})$ matrix $M_{(p,q)}$:

$$M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}.$$

Then the monodromy is derived by simply conjugating the problem, as follows:

$$\begin{aligned} &T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow TM^{-1}M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &TM^{-1} \begin{pmatrix} p \\ q \end{pmatrix} = M^{-1} \begin{pmatrix} p \\ q \end{pmatrix} \longrightarrow MTM^{-1} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \\ &\Rightarrow M_{(p,q)} = MTM^{-1} = \begin{pmatrix} 1 - pq & p^2 \\ -q^2 & 1 + pq \end{pmatrix}. \end{aligned} \tag{16.39}$$

This is illustrated in figure 16.6. Now the condition that two sevenbranes can both be treated in perturbation theory at the same time is if their monodromy matrices commute. In other words, if they are (p_1, q_1) and

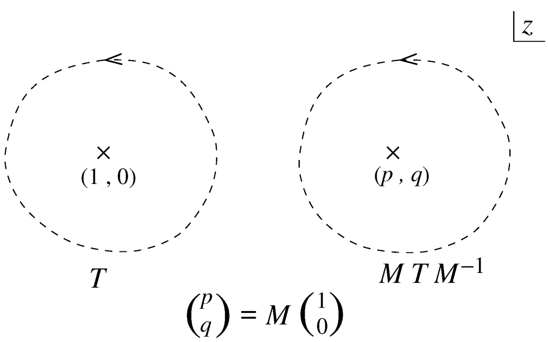

Fig. 16.6. The monodromy around a (p, q) sevenbrane, on which a (p, q) string can end, and its relation, by conjugation, to the $(1, 0)$ case.

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(p_2, q_2), then $p_1 q_2 - p_2 q_1 = 0$. Branes which satisfy this condition are said to be 'mutually local'. Furthermore, the total monodromy around all of the points in the $\mathbb{CP}^1$ must be the identity $\prod_{i=1}^{24} M_{(p_i,q_i)} = 1$. This means that all of the seven-branes definitely cannot be of type $(1, 0)$, since $T^{24} \neq 1$. The slightly weaker locality condition allows a maximum of 18 mutually local branes, and hence $U(1)^{18}$ as the generic gauge group from the sevenbranes.

16.1.10 Enhanced gauge symmetry and singularities of K3

There is even more structure to the theory than that which we have already uncovered, since as we might expect from previous examples, there are enhanced gauge symmetries. The $U(1)^{18}$ can be enhanced to any of an A–D–E family of gauge groups of the same rank, of which the A-series is most obvious. We can tune parameters such that n of the branes are coincident, giving $U(n)$ as the gauge group. Actually, it is prudent to cast this into the language of the K3 geometry. Asking that n branes coincide is equivalent to asking that n of the basic singularities that can occur in the fibre coincide. What really happens is that the singularity becomes of a stronger type, measured by n.

In fact, we already know the description from chapter 13. We should think of the whole of the K3 as developing a singularity, and not just the fibre. We have already encountered the A–D–E singularities of K3 before, and it is instructive to observe how they are to be found in this elliptic description. In the purely brane description, an enhanced gauge symmetry arises because a fundamental string stretched between the branes becomes of zero length and hence there are extra massless sectors. The origin of this string in the F-theory description is as a the base of a $\mathbb{CP}^1$ fibred over the line which is the string. This $\mathbb{CP}^1$ shrinks to zero size when the seven branes coincide. See figure 16.7.

This is precisely the same description of the ALE singularity which we encountered in chapter 13. It is easy now to see how the other A–D–E singularities are described. It is in terms of n $\mathbb{CP}^1$s, c_i, with a set of intersection numbers $c_i \cdot c_j$ giving the Dynkin diagram of the appropriate group. The reader may wish to turn to insert 4.3 for the ADE Dynkin diagrams, showing the topology of the intersections of the $\mathbb{CP}^1$s (represented by the circles).‡

‡ Alternatively, the reader may examine figure 13.2 in chapter 13, where we established the connection between the Dynkin diagrams and the $\mathbb{CP}^1$s underlying an ALE singularity, but they must remember to delete the crossed circle to get the Dynkin diagrams.

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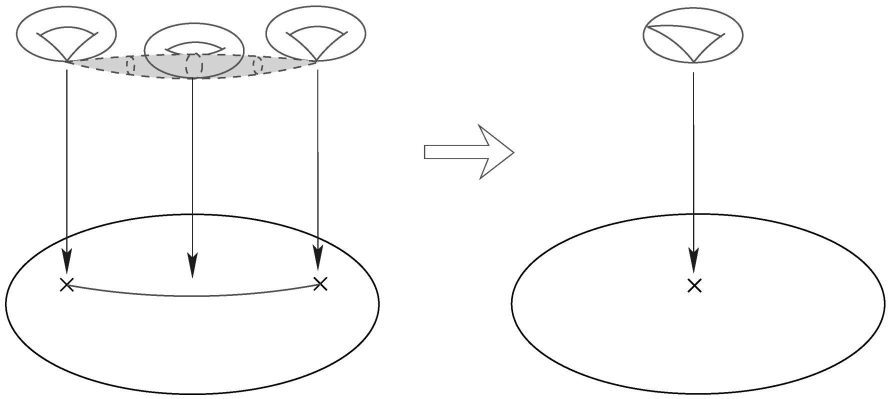

Fig. 16.7. When branes collide: A fundamental string stretching between them goes to zero length when they become coincident. This lifts to a $\mathbb{CP}^1$ which stretches between the locations of the two sevenbranes, which shrinks to zero size when the sevenbranes coincide. The resulting fibre is more singular.

Let us now turn to a few special points in the moduli space of this K3 description, where we will uncover some of this in detail in a more familiar setting.

16.1.11 F-theory at constant coupling

The main facility of the F-theory description is that it provides an economical geometrical way of describing the physics of type IIB vacua with sevenbranes together with varying coupling $\tau = C_{(0)} + ie^{-\Phi}$. This goes beyond our powerful but still only perturbative description of sevenbrane vacua. When we have multiple sevenbranes in the perturbative description, we must cancel the sevenbrane charge locally (using orientifolds) so as not to source any varying coupling away from the branes which would take us outside of perturbation theory.

Nevertheless, in understanding the statements of the previous few subsections better (especially the appearance of the heterotic string!), we ought to try to make contact with the perturbative type II description. What we need to do is find a limit where the torus fibration has all of its structure trapped a few points, between which τ is a constant[197]. There are a number of ways of doing this, as can be seen by looking at the expression (16.36) for the j-function. There, we see that we have two obvious choices, either $g(z) = 0$ or $f(z) = 0$. In the first case, we have only five moduli left to describe this possibility, and we see that $j = 24^3 = 13\ 824$ for which $\tau = i$. This is one of the very special points in the moduli space

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of tori, as we have already seen. The second choice gives us the other special point. There we have only nine moduli, and we see that $j = 0$, which is indeed $\tau = e^{2\pi i/3}$, the orbifold point of $\mathcal{F}$.

Returning to the first case, our K3 is given by

$$y^2 = x^3 + f(z)x, \qquad f(z) = \prod_{i=1}^{8}(z - z_i), \tag{16.40}$$

and we have generically eight zeros, z_i, giving the discriminant $\Delta = 4\prod_{i=1}^{8}(z - z_i)^3$. The 24 branes must have split into eight groups of three sevenbranes. Recalling that a basic sevenbrane in this description has deficit angle $\pi/6$, we uncover that there is a deficit of $\pi/2$ at each of the eight points[197].

There is a way of splitting the eight points up differently. We can use up all of our remaining five moduli to have singularities at only three points, two of order three and one of order one

$$\Delta = 4(z - z_1)^6(z - z_2)^9(z - z_3)^9.$$

This gives deficit angles $3\pi/2$, $3\pi/2$ and π. These values for the deficit angles can be described as orbifold fixed points, since a $\mathbb{Z}_N$ orbifold has deficit angle $2\pi(N-1)/N$. The first two points are therefore $\mathbb{Z}_4$ fixed points, while the last is fixed under a $\mathbb{Z}_2$. We have seen this description before in chapter 7. We really have $T^2/\mathbb{Z}_4$. Let us see what has happened to the constant fibre, by studying the monodromy around its base points. We have $f = (z - z_1)^2(z - z_2)^3(z - z_3)^3$. Looking at a $\mathbb{Z}_4$ fixed point (at $z = z_2$ or z_3) as we encircle it once $z \to e^{2\pi i}z$, we see that $f \to e^{6\pi i}f$. Looking at the form of the defining cubic in equation (16.40), we see that the K3 remains invariant if we also send

$$x \to e^{3\pi i}x = -x \qquad y \to e^{\frac{9\pi i}{2}}y = iy.$$

So we see that the fibre has a $\mathbb{Z}_4$ orbifold action on it as well, and is therefore $T^2/\mathbb{Z}_4$. In fact there is another simple description of this same fact. The case $\tau = i$ is the unique point which is invariant under $S : \tau \to -1/\tau$, where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

is the standard $SL(2, \mathbb{Z})$ representation. The element S is of order four, and so the case $\tau = i$ is the situation of a square torus with a $\mathbb{Z}_4$ symmetry. Looking above the $\mathbb{Z}_2$ point (at z_1), we have

$$f \to e^{4\pi i}f, \quad x \to e^{2\pi i}x = x, \quad y \to e^{3\pi i}y = -y,$$

and so we have a $\mathbb{Z}_2$ symmetry, generated by $S^2 = -1$.

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We observe that our torus fibre is also a $T^2/\mathbb{Z}_4$, with the correct correlation of its order four and order two points with the order four and order two points in the base, and so we discover that our K3 is in fact $T^4/\mathbb{Z}_4$, an orbifold description which we encountered previously in section 7.6.5. It should now be easy to anticipate what happens for branch two, using the knowledge we developed about K3's orbifold limits in section 7.6.5, or about the other special point of $\mathcal{F}$, the moduli space of the torus T^2 in insert 3.3. With $f = 0$, let us write

$$y^2 = x^3 + g(z), \quad g(z) = \prod_{i=1}^{12}(z - z_i), \tag{16.41}$$

and so we have we have generically twelve zeros, z_i, with $\Delta = 27\prod_{i=1}^{12}(z - z_i)^2$. The 24 branes are grouped into 12 pairs, with deficit angle $\pi/3$. Again, we cannot write this as an orbifold in general, but if we use up all of our moduli we can place them at three points z_1, z_2, z_3 in two distinct ways:

$$\Delta = 27(z - z_1)^6(z - z_2)^8(z - z_3)^{10},$$

or

$$\Delta = 27(z - z_1)^8(z - z_2)^8(z - z_3)^8.$$

The first way has gives a $\mathbb{Z}_2$ fixed point again, accompanied by a $\mathbb{Z}_3$ and a $\mathbb{Z}_6$. These are of course the fixed points of $T^2/\mathbb{Z}_6$. The second grouping has three $\mathbb{Z}_3$ points, which are the fixed points of $T^2/\mathbb{Z}_3$. The monodromy around a $\mathbb{Z}_6$ point in the first case gives a K3 invariant under

$$g \to e^{10\pi i}g, \quad x \to e^{\frac{10\pi i}{3}}x = e^{\frac{4\pi i}{3}}x, \quad y \to e^{5\pi i}y = -y,$$

which is again a $\mathbb{Z}_6$ action. Once again, we can also deduce this from that fact that the torus $\tau = e^{2\pi i/3}$ is the special point invariant under

$$ST = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \tag{16.42}$$

which is of order six, $(ST)^6 = 1$. Above the $\mathbb{Z}_3$ point we get

$$g \to e^{8\pi i}g, \quad x \to e^{\frac{8\pi i}{3}}x = e^{\frac{2\pi i}{3}}x, \quad y \to e^{4\pi i}y = y, \tag{16.43}$$

which is a $\mathbb{Z}_3$ action, generated by $(ST)^2$. Lastly, over the $\mathbb{Z}_2$ point, we have

$$g \to e^{6\pi i}g, \quad x \to e^{\frac{6\pi i}{3}}x = x, \quad y \to e^{3\pi i}y = -y, \tag{16.44}$$

which is a $\mathbb{Z}_2$ action, generated by $(ST)^3 = S^2 = -1$. All of this information is simply the expression of the fact that K3 is now in its $T^4/\mathbb{Z}_6$ orbifold limit.

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For the other grouping, things are even simpler, as all of the points are the same[198]. The monodromy around any of them gives that which we saw in equation (16.43), a $\mathbb{Z}_3$ action, showing that this limit represents K3 in its $T^4/\mathbb{Z}_3$ orbifold limit.

The missing orbifold is of course $T^4/\mathbb{Z}_2$. This is achieved by the symmetric choice of placing equal groups of branes at each of four orbifold points in the base, giving $T^4/\mathbb{Z}_2$ since in that case each singularity has deficit angle π. Slightly more generically, this can be achieved by asking that $f^3 = \alpha g^2$, for some parameter α. This does not fix τ's constant value, as should be clear from the j-function in equation (16.36). This is extremely useful, since we are then free to take the type IIB string coupling all the way to zero to achieve our goals of making contact with weakly coupled descriptions. This gives us:

$$\Delta = (4\alpha^3 + 27) \prod_{i=1}^{4} (z - z_i)^6.$$

The monodromy around one of these points is $\mathbb{Z}_2$, which is generated by $S^2 = -1$, as is clear from

$$g \to e^{6\pi i} g, \quad f \to e^{4\pi i} f, \quad x \to e^{\frac{6\pi i}{3}} x = x, \quad y \to e^{3\pi i} y = -y. \quad (16.45)$$

The next matter to consider is the precise way of identifying the A–D–E singularity which a fibre can develop over a point. This is a matter requiring some mathematical care and sophistication, and so as not to stray too far afield, we will not embark on such a discussion. We will simply note that this has been classified by Kodaira[183] in terms of the order, as polynomials in z, of the quantities $(f(z), g(z), \Delta(z))$ that we have been working with. Table 16.1 lists all of the types of singularity and the enhanced gauge symmetry they give[200].

Looking at table 16.1, we immediately see that the gauge groups associated to the special orbifold limits we have studied are given in table 16.1. There are a number of interesting general features of this result. The most obvious is the fact that we get exceptional gauge groups in the latter three cases. We have encountered no way of achieving this using perturbative D-branes up to now, and this remains the case. As we have already noted, although the coupling is constant in the last three models, it is not weak, and so the branes are not perturbative D-branes.

In the $\mathbb{Z}_2$ case however, we have something different[197]. We can achieve the required gauge group at weak coupling, and happily we have the freedom (by choice of α) to make the constant string coupling any value we

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Table 16.1. *Kodaira's classification of the A–D–E singularities of K3 that can occur in the Weierstrass parametrisation given in equation (16.37)*

order(f)	order(g)	order(Δ)	fibre type	singularity
≥ 0	≥ 0	0	smooth	none
0	0	n	I_n	A_{n-1}
≥ 1	1	2	II	none
1	≥ 2	3	III	A_1
≥ 2	2	4	IV	A_2
2	≥ 3	$n+6$	I_n^*	D_{n+4}
≥ 2	3	$n+6$	I_n^*	D_{n+4}
≥ 3	4	8	IV^*	E_6
3	≥ 5	9	III^*	E_7
≥ 4	5	10	II^*	E_8

like. Choosing that the string coupling is zero (i.e. $\tau \to i\infty$) implies that we have completely cancelled the sevenbrane charge locally at each of the four points. In a perturbative description, this is achieved by using an O7-plane in the neighbourhood of an appropriate amount of D7-branes. Looking back to our computations of chapter 7, we see that the O7-plane charge is -4 in units where the D7-brane charge is 1. So we need to have four D7-branes and one O7-plane for charge cancellation. Actually, we also know precisely what gauge group this would give. It is in fact $SO(8)$. This is remarkably similar to have we have in the first line of table 16.1. There are four groups of six coincident sevenbranes. If we associate four of them with ordinary D7-branes, then two of them correspond to the orientifold sitting at the $\mathbb{Z}_2$ orbifold fixed point. We have arrived at the $T^4/\mathbb{Z}_2$ orientifold of type IIB, where a $(-1)^{F_L}\Omega$ also acts internally. From our experience with T-duality of simple orientifolds (see, for example, chapter 8), we see that this is simply T-dual to the $SO(32)$ type I string theory compactified on T^2. Accordingly, the orientifold (O9-plane) of charge -16 (in D9-brane units) splits into $2^2 = 4$ O7-planes of

Table 16.2. *The results for the gauge groups in the various constant coupling F-theory K3 orbifold limits*

K3 orbifold	Gauge group
$T^4/\mathbb{Z}_2$	$SO(8)^4$
$T^4/\mathbb{Z}_3$	E_6^3
$T^4/\mathbb{Z}_4$	$E_7^2 \times SO(8)$
$T^4/\mathbb{Z}_6$	$E_8 \times E_6 \times SO(8)$

charge −4. In order to achieve local charge cancellation, the 16 D7-branes are moved into four groups of four to sit at the O7-planes.

So we have obtained the weakly coupled description we sought. Furthermore, using the result of chapter 12 that the $SO(32)$ type I string is strong/weak coupling dual to the heterotic string, we also have the bonus of proving that we have a duality to the heterotic string on T^2. Deforming away from this special point using the moduli establishes the duality at all points on the moduli space.

Incidentally, in the spirit of the discussions in chapter 12, we can even see what the 'dual' heterotic string is in this picture. In ten dimensional type I, it would have been the D1-brane. We have T-dualised on a T^2, however, and so we see that the dual string becomes a D3-brane wrapped on the T^2. It is a useful exercise to check that the resulting heterotic string's coupling is set by the area of the torus. Tuning moduli to return to the general non-orbifold situation, we see that the dual heterotic string is a D3-brane wrapped on the $\mathbb{CP}^1$. The seven dimensional heterotic string coupling is set by the size of the $\mathbb{CP}^1$ in general.

16.1.12 The moduli space of $\mathcal{N} = 2$ $SU(N)$ with $N_{\rm f} = 4$

Let us continue to focus on one of the four singular points for a while longer, placing everything at the origin $z = 0$. At weak coupling, we have seen that the branes carry an $SO(8)$ gauge symmetry and that the perturbative description is as four D7-branes and an orientifold O7-plane. Let us place a D3-brane probe into this background, oriented so that it is living in, say, the x_1, x_2, x_3 directions. This breaks half of the supersymmetries, leaving a total of eight supercharges. Observe further that when the D3-brane is located at the orientifold, the gauge theory on its world-volume is in fact $SU(2)$, since this situation is T_{89}-dual to a D5-brane in type I string theory, as we have seen. Because we have T-dualised, however, the D3-brane can move off the orientifold, and then the gauge group is $U(1)$. We can move the D7-branes to positions (z_1, z_2, z_3, z_4), which breaks the $SO(8)$ to $U(1)^4$ generically. There can be enhanced symmetry points to $U(n)$ if n of the D7-branes come together away from the O7-plane, and $SO(2n)$ if the coincide at the O7-plane.

What we have arrived at is the weakly coupled description of the Coulomb branch of the moduli space of $\mathcal{N} = 2$ four dimensional $SU(2)$ gauge theory with four flavours of quark in the fundamental. The latter come from the strings stretching between the D7-branes and the D3-branes. Their classical masses are given by the positions z_i. Moving the D3-brane from the origin is the process of giving a vacuum expectation value (vev) to the complex adjoint scalar in the $\mathcal{N} = 2$ vector multiplet,

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and the z-plane is the space of gauge inequivalent values of this vev. The origin remains as the naive classical $SU(2)$ gauge symmetry restoration, and the gauge groups associated to the D7-branes are *global* flavour symmetries in the D3-brane world-volume.

It is amusing that we have obtained this rich and beautiful theory as a piece of the F-theory background seen by probing with the D3-brane, and we can learn much about each from this. The first thing we can learn (assuming we did not now it before) is the gauge theory's β-function, encoded in the one-loop running of the gauge coupling. We can read this out from the weak coupling behaviour of the gauge coupling. Placing the orientifold at the origin, and the four D7-branes at positions we have:

$$\tau(z) = \tau_0 + \frac{1}{2\pi i}\left[\sum_{i=1}^{4}\ln(z - z_i) - 4\ln z\right],$$

and use the fact that $\tau(z) = C_{(0)} + ie^{-\Phi}$. Remember also that $g_{\rm s}(z) = e^{\Phi}(z)$ and that the Yang–Mills coupling and θ-angle are related to the string theory parameters by $g_{\rm YM}^2 = 2\pi g_{\rm s}$ and $\theta = 2\pi C_{(0)}$. The β-function for the pure glue is negative with respect to the contribution from the quarks. The quark masses are set by the positions z_i, since those positions set the length of the 3–7 strings. Notice that when all the $z_i = 0$, and we are at the $SU(2)$ point at the origin of moduli space, then we get no running of the coupling and $\tau = \tau_0$, the tree level value. This fits with the fact that the case of $N_{\rm f} = 2N_{\rm c}$ has vanishing β-function, and is in fact conformally invariant. We can also take the opposite limit, and send some of the z_i to infinity, thus reducing the number of quarks, all the way down to the case of pure glue, if we wish.

As we have seen before, we cannot trust the above one-loop expression near $z = \{0, z_i\}$, since the logarithm takes the expression large and negative, which is not acceptable behaviour for the gauge coupling. Of course, this is because we have neglected the instanton contribution, which produce non-perturbative effects which remove this singular behaviour. The beautiful results[240] of Seiberg and Witten address precisely this point, with the result that there is a complete solution of the problem in terms of the geometry of an auxiliary torus. The torus encodes the physics of the Coulomb branch, including the spectrum of masses of (p, q) dyons. The torus is singular over six points, four of them (the z_i) are the places where the quarks becomes effectively massless. The other two points originate from the single $SU(2)$ point at the origin: it has split (since instanton effects switch on to maintain positivity of the gauge coupling or, equivalently, the moduli space metric[240] and they are separated by a distance

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of order $e^{i\tau_0\pi}$, and they represent the places where $(0,1)$ monopoles and $(1,-1)$ dyons become massless.

From the point of view of the D-brane picture, it is extremely natural that an auxiliary torus appears in the description of the non-perturbative physics, as this is the torus of the underlying F-theory description. So what we learn is that the orientifold O7-plane splits into two seven-branes, of type $(0,1)$ and $(1,-1)$, beyond weak coupling, physics which is isomorphic to the removal of the gauge theory $SU(2)$ point by instanton effects[240]. We have seen that the full F-theory description, which allows the $SL(2,\mathbb{Z})$ behaviour of τ to come into play and keep it manifestly positive, maps to the same solution of the problem for the coupling in the gauge theory.

16.2 M-theory origins of F-theory

It is natural to wonder whether the appearance of the torus of F-theory is a sign of hidden twelve dimensional dynamics for which we should seek, in the spirit of the search for M-theory based on eleven dimensional dynamics seen by all of the branes of type IIA. A more conservative point of view is that the torus is merely a powerful bookkeeping device, and the type IIB theory is no more or less ten dimensional than it was before the advent of F-theory. This is perhaps supported in part by the fact that the only information about the torus which has physical meaning is its complex structure modulus τ. The Kähler structure, containing information about its size, is nowhere to be seen in the formulation. So the putative twelve dimensional dynamics would at best be purely (loosely speaking) topological, it would appear.

The spirit of string theory's history of advances is that one must keep one's mind and eyes open for new directions and often unexpected and fruitful changes of point of view. This is probably because we do not really know yet what the theory really is. So as long as a firm unambiguous computational advantage is obtained in exchange, most practitioners simply do not seem to care what explanatory words or terminology arises to decorate the new tools once they are found. It may well be that a formulation using dynamics in twelve dimensions does arise one day, and if it describes key pieces of physics in a manner more economical than current techniques, then it deserves a place alongside other important pieces of the puzzle of describing fundamental physics.

So no firm declaration is to be found in these pages concerning the twelve dimensional dynamical origins of F-theory. Instead, it is worth noting that there are also signs that many of the key pieces of F-theory – particularly the origin of the torus – can be seen directly to have more

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humble origins: It is simply a limit of the eleven dimensional picture of M-theory[133, 134].

Let us return to the duality between eleven dimensional supergravity on a circle of radius R_{10} and type IIA string theory. The type IIA string coupling is related to the circle radius by: $R_{10} = (g_s^A)^{2/3}\ell_p = g_s^A \ell_s$, since $\ell_p = (g_s^A)^{1/3}\ell_s$, recalling formulae from chapter 12. Once the circle is small enough, we are able to work with weakly coupled ten dimensional physics of the type IIA string to a good approximation. As we have described before, the D4-brane, the D2-brane, and the NS5-brane of type IIA arise from the M-branes reduced or wrapped on the circle, the D0-brane is a Kaluza–Klein momentum, and the D6-brane is a Kaluza–Klein monopole.

We can continue to compactify on another circle, this time of radius R_9, and shrink that one away as well. We know that this has a dual description in terms of the type IIB string theory, where now the theory is compactified on a circle of radius $R_9' = \ell_s^2/R_9$, and, crucially from equation (5.1), the type IIB string coupling is $g_s^B = g_s^A \ell_s/R_9$. We can go ahead and shrink away the second circle entirely as well, and use the ten dimensional type IIB description, which has no direct reference to the two circles we started with. However, we see that the type IIB string coupling can be expressed entirely in terms of the size of the two circles:

$$g_s^B = \frac{R_{10}}{R_9}. \tag{16.46}$$

So in fact, given the existence of M-theory, the type IIB string coupling can be interpreted entirely in terms of the ratio of the radii of two circles. These two circles make a torus, since they define a lattice upon which we can make an identification. Since equation (16.46) only refers to the ratio of the radii of the circles, we can rescale and write the lattice as of unit length in one direction (associated to x_{10}), and of length $1/g_s^B$ in the other (associated to x_9). See figure 16.8. Before making the identification on the lattice however, we are free to make a shift in the x_{10} direction before identifying to construct the torus. Different non-integer shifts give non-equivalent tori, while a shift by an integer gives the same torus. See figure 16.9. This shift is to be identified with the R–R periodic scalar $C_{(0)}$, a natural identification since it is correlated, by tracing backwards, with a familiar structure in the tenth direction. It is T-dual to the type IIA R–R potential $C_{(1)}$, which in turn is conjugate to momentum in the periodic direction x_{10} and so is directly related to a periodic shift.

What we have just described is our F-theory torus of the previous subsections, with complex structure $\tau = C_{(0)} + ie^{-\Phi}$. Notice that the fact that it seems to have no physical size is natural from this description. We arrived at it by sending the each circle to zero size, and so only the

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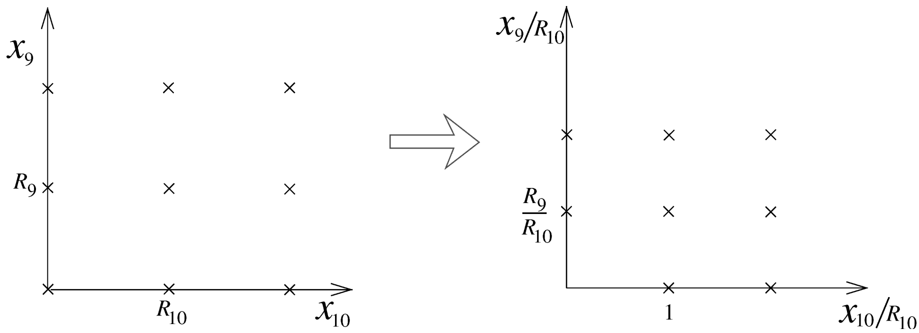

Fig. 16.8. The geometry of the compactification torus used to get type IIB string theory from M-theory.

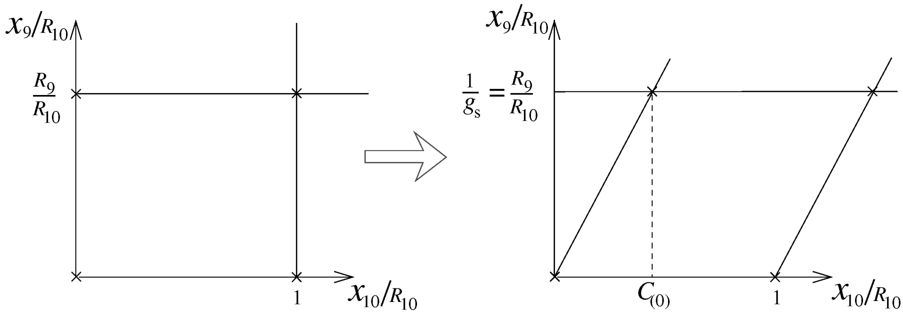

Fig. 16.9. Generalising the compactification lattice by including a shift. This is how the F-theory or type IIB theory torus arises from M-theory.

ratio of the circles has physical meaning in the resulting type IIB theory. Moreover, it is clear that the type IIB theory obtains its $SL(2,\mathbb{Z})$ structure in this way, and that it is truly and manifestly non-perturbative, given the construction.

So we see that at least locally, we can attribute the F-theory torus to the result of shrinking a physical torus in M-theory [133, 134]. Consequently, we should be able to make sense of, directly in M-theory, more complicated structures with varying type IIB couplings, like various branes, and even complete F-theory vacua.

16.2.1 M-branes and odd D-branes

The route of the previous subsection is just what we need to show the M-theory origin of type IIB's odd Dp-branes and NS5-brane. Of course, it is directly deducible from T-duality to the type IIA branes, but it is

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useful to recast things in terms of the M-theory reduction on the torus, following the steps above.

Imagine that we started in M-theory with an M2-brane, with one direction extended in x_{10}. It has tension $\tau_2^M = (2\pi)^{-2}\ell_p^{-3}$. Upon reduction, this becomes the type IIA string, with the correct tension $\tau_F \equiv \tau_{1,0} = \tau_2^M 2\pi R_{10} = (2\pi)^{-1}\ell_s^{-2}$, which becomes the type IIB string under the T_9-duality. We have used the fact that $\ell_p = (g_s^A)^{1/3}\ell_s$. Alternatively, the M2-brane could have been transverse to x_{10}, with one direction lying in x_9 instead. Then it would have become a D2-brane, with tension $\tau_2 = (2\pi)^{-2}\ell_s^{-3}(g_s^A)^{-1}$ and by T_9-duality a D1-brane in type IIB, with tension

$$\tau_1 \equiv \tau_{0,1} = \tau_2 2\pi R_9 = (2\pi)^{-1}\ell_s^{-2}(g_s^A)^{-1}R_9/\ell_s = (2\pi)^{-1}\ell_s^{-2}(g_s^B)^{-1},$$

where again we have used the fact that the type IIB string coupling is $g_s^B = g_s^A\ell_s/R_9$.

The two situations are related by a flip of the x_9 and x_{10} directions. This in turn is the S-transformation of the type IIB torus, and so we have correctly arrived at the S-duality action on the type IIB strings. It should be clear now how to get all of the (p,q) strings: we need to wrap the M2-brane p times on the x_{10} cycle and q times on the x_9 cycle. Let us check that we get the right tension formula. Wrapping as stated above, looking at figure 16.8 reveals that the length $2\pi R_{p,q}$ that the M2-brane is stretched is simply given by Pythagoras: $2\pi R_{p,q} = 2\pi\sqrt{(pR_{10})^2 + (qR_9)^2}$, and hence the resulting tension written in type IIB terms is:

$$\begin{aligned}
\tau_{p,q} &= \tau_2^M 2\pi R_{p,q} = (2\pi)^{-2}\ell_p^{-3}\cdot 2\pi R_{p,q} && (16.47)\\
&= (2\pi)^{-1}\ell_s^{-3}(g_s^A)^{-1}\sqrt{(pR_{10})^2 + (qR_9)^2} && (16.48)\\
&= (2\pi)^{-1}\ell_s^{-2}(g_s^B)^{-1}\sqrt{\frac{p^2R_{10}^2}{R_9^2} + q^2} && (16.49)\\
&= \sqrt{(p\tau_{1,0})^2 + (q\tau_{0,1})^2}, && (16.50)
\end{aligned}$$

where we have used the T-duality formula for the relation of the string couplings, the relation between ℓ_s and ℓ_p, etc. and we have recovered our earlier bound state formula (11.16). We can even go further and derive the more general formula for the case in which there is a background value, $c_{(0)}$, of the R–R scalar $C_{(0)}$ present. Recall that it is a shift in the x_{10} direction shown in figure 16.9. So in computing the length of the wrapped membrane, we ought to take into account this shift: looking at the diagram, it is elementary to see that every time ones goes around the x_9-cycle, one picks up a reduction of $c_{(0)}R_{10}$ in the total length stretched

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in the x_{10} direction. Therefore we should have, in this case, the more general expression $2\pi R_{p,q} = 2\pi\sqrt{([p - qc_{(0)}]R_{10})^2 + (qR_9)^2}$. Similar manipulations to the above give:

$$\tau_{p,q} = \sqrt{([p - qc_{(0)}]\tau_{1,0})^2 + (q\tau_{0,1})^2}, \tag{16.51}$$

which is a rewriting of equation (16.11).

Turning to D3-branes, it is immediately clear from this picture what its origins must be. We can take an M5-brane and wrap two of its directions on the torus as its shrinks away. Following the type IIA route, it becomes first a D4-brane from shrinking x_{10}, and then a D3-brane after shrinking x_9 and T-dualising. We can check that we get the right tension directly:

$$\begin{aligned}\tau_3 &= \tau_5^M \cdot 2\pi R_{10} \cdot 2\pi R_9 = (2\pi)^{-5}\ell_p^{-6}(2\pi)^2 R_{10}R_9 \\ &= (2\pi)^{-3}\ell_{\rm s}^{-6}(g_{\rm s}^{\rm A})^{-2}\ell_{\rm s} g_{\rm s}^{\rm A} R_9 = (2\pi)^{-3}\ell_{\rm s}^{-4}(g_{\rm s}^{\rm B})^{-1}.\end{aligned} \tag{16.52}$$

It is also clear that the D3-brane is invariant under $SL(2,\mathbb{Z})$ since it is wrapped entirely on both cycles of the torus.

For fivebranes, the story is similar to the case of the strings. There is a whole (p,q) family of them because there are two ways of getting a five dimensional extended object from the M5-brane: one either wraps it on the x_{10} cycle, in which case it becomes a D5-brane (which we ought to call $(1,0)$), or we wrap it on the x_9 cycle and so it becomes an NS5-brane $(0,1)$. It should be easy to see that the resulting tension of the (p,q) fivebrane made by wrapping the appropriate number of times on each cycle is (including the background $C_{(0)}$ field, and using the Pythagorean relation for $R_{p,q}$ above):

$$\tau^5_{p,q} = \sqrt{([p - qc_{(0)}]\tau^5_{1,0})^2 + (q\tau^5_{0,1})^2}, \tag{16.53}$$

which indeed gives the supergravity formula (16.25) given earlier. (In the computation, the above comes multiplied by $2\pi R_9'$, since that is what the resulting fivebrane is wrapped around on arrival in the type IIB theory.)

Finally, let us turn to the sevenbranes. In the stringy picture, these come from T-dualising transverse to D6-branes, but it is illuminating to think of it in the picture of reduction from M-theory. Recall that a D6-brane in M-theory comes from a clever twist of the geometry, making a Kaluza–Klein monopole. The metric is $(\mathbf{x} = (x_7, x_8, x_9))$:

$$\begin{aligned} ds^2_{11} &= -dt^2 + \sum_{i=1}^{6} dx_i^2 + V(r)(d\mathbf{x}\cdot d\mathbf{x}) + V(r)^{-1}(dx_{10} + \mathbf{A}\cdot d\mathbf{x})^2 \\ V(r) &= 1 + \frac{r_6}{r}, \qquad r^2 = \mathbf{x}\cdot\mathbf{x}, \qquad \nabla\times\mathbf{A} = \nabla V(r), \end{aligned} \tag{16.54}$$

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for a single brane located at $r = 0$ in the (x_7, x_8, x_9) plane. The key point is that the x_{10} circle shrinks to zero at the location of the D6-brane, since the metric vanishes there. So we see that shrinking the x_9 circle as well to go to the type IIB theory (after T-dualising), gives us the x_9, x_{10} torus, which we discover has a cycle which degenerates over the (x_7, x_8) plane. This is just how we describe a D7-brane in F-theory language, and so we have recovered yet another key F-theory phenomenon as a limit of M-theory. To do better, and get (p, q) sevenbranes, we may consider placing x_9 on a circle (on the M-theory side), giving a physical torus after identification (with a shift to include $c_{(0)}$). We may then consider more general S^1 fibration geometries than those in equation (16.54). The analysis of monodromies in the non-compact directions is then identical to the F-theory one.

A key phenomenon which we discovered was a description of the enhancement of symmetry when two seven-branes coincide, described as the collision of singularities in the F-torus. Since this is described by fundamental strings going to zero length in the type IIB picture, we drew this suggestively as an S^1 fibration over the string making a $\mathbb{CP}^1$ cycle, as depicted in figure 16.7, and then identified the appearance of extra massless fields with the shrinking of the cycle. Since the F-torus has no dynamics associated with it, in the way it was described, that suggestion could not be honestly taken as anything more than a strongly plausible description. Now we see in the M-theory origins of the torus that this is exactly the correct description: on the M-theory side, an M2-brane can wrap both of its directions on the cycle stretching between two lifted D6-brane fibrations of the type in equation (16.54). We have already learned that a fundamental string comes from such a wrapped M2-brane, and after shrinking the torus, we recover precisely figure 16.7. So the sevenbrane enhanced gauge symmetries in F-theory come from wrapped M2-branes on collapsing cycles in M-theory.

In summary we now see how to connect type IIB theory, and indeed the F-theory description, to M-theory. We can do the reverse now, and take various F-theory vacua and turn them into M-theory vacua. Here is a simple rule: Place the theory on any circle. Shrink the circle away, and in the limit the F-theory torus acquires a physical size, returning us to eleven dimensional M-theory.

16.2.2 M-theory on K3 and heterotic on T^3

We now have enough information to construct the M-theory versions of some of the data which we obtained in F-theory in earlier sections. In particular, we discovered that F-theory on K3 is in fact dual to the heterotic string on T^2.

Starting with the F-theory configuration described in earlier sections, let us now compactify a harmless direction (any of $x_1, \ldots, x_7$) on a circle, and shrink it away. The result is M-theory on K3. Actually, on the dual side, we are simply placing the heterotic string theory on an additional circle, and so derive the non-trivial result that M-theory on a K3 is dual to the heterotic string on T^3. The fundamental heterotic string is that string which originated as a D3-brane wrapped on the $\mathbb{CP}^1$ base of the elliptic K3. We now see that this string is now an M5-brane wrapped on the entire K3 in the M-theory picture. The pattern of enhanced gauge symmetries is enlarged somewhat on both sides, and the moduli space is now locally:

$$\mathcal{M} = \frac{O(19,3)}{O(19) \times O(3)}. \tag{16.55}$$

16.2.3 Type IIA on K3 and heterotic on T^4

Finally, we can in fact compactify another of the harmless circles on the M-theory side, and the result is type IIA string theory on K3. Since we have done nothing non-trivial to the heterotic side either, we discover as a result that there is a duality between type IIA on K3 and the heterotic string on T^4. We have already mentioned this duality previously in insert 7.5 (p. 186) and in chapter 12. The F-theory moduli space is now locally:

$$\mathcal{M} = \frac{O(20,4)}{O(20) \times O(4)}. \tag{16.56}$$

16.3 Matrix theory

One of the most striking features of string duality is the discovery that eleven dimensions is dynamically relevant to string theory. It had always been thought of as a useful bookkeeping device to start with eleven dimensional supergravity and derive the structure of type IIA supergravity by dimensional reduction, but it was thought of as nothing more than that. However, once one takes the loop-protected BPS spectrum of D0-branes seriously, one is forced to try to interpret the tower of light states they supply at large string coupling, and a Kaluza–Klein story appears inevitable[149].

Further study showed that the dynamics of D0-branes implied that they clearly were sensitive to shorter scales[106] than just ℓ_s. In fact, now we know (see the discussion surrounding equation (12.15)) that the physics they were sensitive to was the scale $\ell_p = g_s^{1/3}\ell_s$, which at weak coupling a lot

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is shorter than the supposed minimum distance ℓ_s perturbative strings know about.

This might lead one to attempt to capture some of the eleven dimensional physics in terms of that of D0-branes, hoping that it might lead to an understanding of the formulation of M-theory in its own right. This is not really a fully accurate picture of the thought processes that led to the presentation of Matrix theory[157], but then this is not an attempt at a history[158]. It suffices for us here to uncover a little of what we can with the above motivating remarks, and leave the matter of the history of it to be explored in the literature or elsewhere.

16.3.1 Another look at D0-branes

For reasons that will be stated shortly, let us focus on the low energy effective Lagrangian for N D0-branes. This is simply a $0+1$ dimensional theory (a quantum mechanics) involving the nine spatial transverse coordinates X^i, $i=1,\dots,9$, and their superpartners. We start by considering the branes to be all in the same place, and so we have a $U(N)$ invariant system. We must remember to keep commutator terms which would normally vanish in the Abelian case.

The most efficient way of writing this action is in fact to start with ten dimensional maximally symmetric Yang–Mills theory and dimensionally reduce it all the way to $0+1$ dimensions. After a rescaling, the result is:

$$\mathcal{L} = \mathrm{Tr}\left[\frac{D_tX^iD_tX^i}{2g_s\ell_s} + \frac{[X^i,X^j]^2}{4g_s\ell_s(2\pi\ell_s^2)^2} - \frac{i}{2}\Theta D_t\Theta + \frac{1}{4\pi\ell_s^2}\Theta\Gamma^0[X^i,\Gamma^i\Theta]\right].$$

We have indeed thrown away any terms with higher powers of velocity than quadratic, the trace is over $U(N)$. The X^is all come from internal components of the gauge field, and so there is the usual factor of $2\pi\ell_s^2$ to convert a gauge field to a coordinate. There are no remaining appearances of gauge fields except for A_0, which is inside the covariant derivative only, having no kinetic term. It may therefore be thought of as simply a constraint field, enforcing $U(N)$ gauge invariance. Also, Θ is a rescaled version of the $SO(9)$ sixteen component fermion which would have appeared in ten dimensions.

From the Lagrangian above, we can write a Hamiltonian. The details are left as an exercise to the reader, and the result is remarkably simple:

$$\begin{aligned}\mathcal{H} &= \mathrm{Tr}\left[\frac{g_s\ell_s}{2}p_ip_i - \frac{[X^i,X^j]^2}{4g_s\ell_s(2\pi\ell_s^2)^2} - \frac{1}{4\pi\ell_s^2}\Theta\Gamma^0[X^i,\Gamma^i\Theta]\right]\\ &= R\mathrm{Tr}\left[\frac{1}{2}p_ip_i - \frac{[X^i,X^j]^2}{16\pi^2\ell_p^6} - \frac{1}{4\pi\ell_p^3}\Theta\Gamma^0[X^i,\Gamma^i\Theta]\right].\end{aligned} \tag{16.57}$$

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Possibly the most immediately striking thing about this Hamiltonian is the fact that everything naturally assembles itself into eleven dimensional quantities, as shown in the second line above. We have pulled out an overall factor of the inverse mass of the D0-brane, $(g_s\ell_s)^{-1}$, which is the inverse of the radius of the eleventh direction, which we have called R.

16.3.2 The infinite momentum frame

There is a striking proposal for an interpretation of the physics of the above Hamiltonian[157]. The idea is that the system captures the physics of states with momentum $p_{10} = N/R$ in the limit that N and R go to infinity. This is the *'infinite momentum frame'* (IMF), essentially a light cone frame. It uses the fact that D0-brane charge is momentum in the eleventh direction, and is quantised in units of $1/R$ if the direction is on a circle. We then take the limit in which the circle is large and the momentum in that direction is large, keeping the fraction fixed. This allows us to consider the decompactified limit where we are allowed to discuss a fully eleven dimensional choice like picking a boost direction.

To see that we have not neglected anything relevant in picking the original Lagrangian, notice that, if we separate momentum up into ten dimensional component, $\mathbf{p}$ and the eleven dimensional component $p_{10} = N/R$, we have:

$$E^2 = \frac{N^2}{R^2} + p^2 + m^2,$$

where m is the mass of the particle. In the limit that the eleven dimensional momentum is extremely large, we see that the dominant energy contribution is from states who have a finite fraction of the eleven dimensional momentum in the limit. In other words, since

$$E = \frac{N}{R} + \frac{1}{2}\frac{R}{N}(p^2 + m^2) + O\left(\frac{R}{N}\right)^2,$$

in the limit of $N/R \to \infty$, the energy a state with contribution mostly from the second terms will not be significant, and so it will not play a role in the dynamics.

In fact, this justifies our dropping of higher order terms in the basic Lagrangian, since those corrections (subleading in *ten dimensional* momentum) will not have a chance to contribute to the limit. Actually, the only sector which has a chance of contributing (from the ten dimensional perspective) are the D0-branes, together with the *lightest open strings* connecting them. These are precisely the sectors which appear in the Hamiltonian in equation (16.57).

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The Hamiltonian above may therefore be studied in the light of this proposal in purely eleven dimensional terms. Apparently, we are to somehow recover all of the physics of M-theory this way, since eleven dimensional Lorentz invariance (*assumed* to be preserved) would suggest that we can always boost any situation into this frame. Of course, we can only do this is we can understand how to extract the physics appropriate to questions we might ask. Now we see, for example, why the bound state questions of chapter 11 were pertinent. A graviton of momentum n is in fact a bound state of n D0-branes, and so we must establish that a normalisable wavefunction for such a system exists. This is not a solved problem for arbitrary n, as already stated in chapter 11.

The scattering of gravitons with no exchange of longitudinal (eleventh direction) momentum is nicely described in terms of matrices in this language. The αth graviton of momentum $p_\alpha = n_\alpha/R$ is represented by a $n_\alpha \times n_\alpha$ block of the X^i (each X^i representing matrix position in the ith transverse coordinate). The trace of the $n_\alpha \times n_\alpha$ block of the matrix is the centre-of-mass position of the graviton. Interaction between the block diagonal parts can be determined by integrating out off-diagonal degrees of freedom, which correspond to integrating out the massive open strings stretching between the widely separated clumps and and determining the effective interactions between the clumps in that way. It has been shown that this reproduces rather nicely the expected results for graviton-graviton scattering.

In fact, a lot more can be done along those lines, including recovering the basic lightcone world-volume M2-brane description by a change of variables, making contact with the much earlier work[255] on the M2-brane Lagrangian done back when it was thought to be a viable fundamental object[157].

Another striking feature of the description is that there is a natural statement about the importance of the onset of non-commutativity of the description of spacetime at high energy. Recall that the X^i are supposed to be related to spacetime coordinates as well. They are naturally (and essentially) presented as $N \times N$ matrices here. It is only when the X^i are large that we recover the usual picture of them as commuting spacetime coordinates, for only in that limit is is favourable for the Hamiltonian to select sectors for which $[X^i, X^j]$ vanishes. Then, the X^i can all be simultaneously diagonalised into their eigenvalues x^i, the nine transverse spacetime positions[26, 157].

Note that an interpretation of the model at *finite* N has also been proposed[280]. It is simply a *discrete light cone quantisation* (DLCQ) of the theory. In other words, at finite N, the fact that the theory is on a circle of radius R is taken seriously. The theory is taken as being in the light cone

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frame, with a compact null direction. Such techniques have been used successfully elsewhere in order to supply the non-perturbative definition of field theories such as QCD.[281]

Note also that there is another matrix model proposal for capturing important degrees of freedom. It is based on structures in the type IIB string and D-instantons in particular[342].

16.3.3 Matrix string theory

Of course, one thing which we ought to be able to recover is the fact that we get the type IIA superstring upon compactification of a dimension on a circle. In fact, we should be able to do this on *any* spatial circle. How are we to see this here?

What we would like to do is compactify one of the directions X^i. There are a number of ways of working out just what that means for our model, but there is a particularly simple way[282], given all that we have studied so far: by T-duality, working with D0-branes in the presence of one of the X^i compact is equivalent to working with D1-branes extended in that compact direction. It must be that the model we need is a large N model built from D1-branes wound on a circle. As the size of the circle shrinks to smaller and smaller size, this picture is increasingly the more useful one to use. In fact, an extremely important sector to include is the family of light strings stretching between D0-branes after winding around the circle some number of times.

We know how to write the just the model that we want. It is 1+1 dimensional Yang–Mills on a circle. We can write it down by starting from the beginning again, or we can simply obtain it from the present matrix model. To do so, if X^9 is to be our compact direction, of radius R_9, we need only replace X^9 by $R_9\mathcal{D}_\sigma$, where $0 \le \sigma \le 2\pi$ and $\mathcal{D}_\sigma$ is the covariant derivative.

The model which results is:

$$\begin{aligned}\mathcal{H} = R\int_0^{2\pi} d\sigma\mathrm{Tr}\Bigg[&\frac{1}{2}p_i p_i - \frac{[X^i,X^j]^2}{16\pi^2\ell_p^6} - \frac{1}{4\pi\ell_p^3}\Theta\Gamma^0[X^i,\Gamma^i\Theta]\\ &-\frac{E^2}{16\pi^2\ell_p^6} - \frac{R_9^2(\mathcal{D}_\sigma X^j)^2}{16\pi^2\ell_p^6} - \frac{R_9}{4\pi\ell_p^3}\Theta\Gamma^0\mathcal{D}_\sigma\Theta\Bigg]\\ = R\int_0^{2\pi} d\sigma\mathrm{Tr}\Bigg[&\frac{1}{2}p_i p_i - \frac{[X^i,X^j]^2}{16\pi^2 g_s^2\ell_s^6} - \frac{1}{4\pi g_s\ell_s^3}\Theta\Gamma^0[X^i,\Gamma^i\Theta]\\ &-\frac{F^2}{16\pi^2 g_s^2\ell_s^6} - \frac{(\mathcal{D}_\sigma X^j)^2}{16\pi^2\ell_s^3} - \frac{1}{4\pi\ell_s^2}\Theta\Gamma^0\mathcal{D}_\sigma\Theta\Bigg]. \end{aligned} \quad (16.58)$$

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Notice that at the end, we made the substitution

$$R_9 = g_s \ell_s, \quad \text{and } \ell_p = g_s^{1/3} \ell_s,$$

as appropriate to the case of the type IIA model we expect to arrive at in the limit. Indeed, we see that the model naturally cleans itself up into the string variables. The electric field $F_{01} = \dot{A}_\sigma$ is the non-trivial gauge field strength of the model, an electric flux, in fact. The 16 component field Θ has naturally split into an $\mathbf{8}_c \oplus \mathbf{8}_s$ under the natural $SO(8)$ which acts here. One is left moving on the string and the other is right moving. The X^i transform as the $\mathbf{8}_v$, of course. This model therefore has the manifest supersymmetry we expect for the type IIA model and is in 'Green–Schwarz' form[108]. It is also the model we arrived at (but for a single D1-brane) in section 12.1 within the type IIB string theory. There, it represented the type IIB soliton string and the opposite chiralites of the left and right movers was appropriate to the expected zero modes on the soliton.

N.B. It is amusing to note that to describe compactification of spacetime dimensions, one has to work with a *higher dimensional* matrix model. This exchanges the role of dimensional reduction and the inverse procedure, dimensional 'oxidation'.

Now this model, with $U(N)$ gauge symmetry, is to be interpreted not as a soliton, but as a matrix definition of the type IIA string theory[283, 284]. The limits we are taking to get the free string are two-fold: we must take $R \to \infty$ and $N \to \infty$, as before, and we must also take $g_s \to 0$, which is of course the same as $R_9 \to 0$.

To study the model, let us consider the supersymmetric vacua, i.e. the moduli space $[X^i, X^j] = 0$. The $X^i(\sigma)$ can be chosen as diagonal matrices:

$$X^i(\sigma) = \begin{pmatrix} x_1^i(\sigma) & 0 & 0 & \cdots & \cdots \\ 0 & x_2^i(\sigma) & 0 & \cdots & \cdots \\ 0 & 0 & x_3^i(\sigma) & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \cdots & x_N^i(\sigma) \end{pmatrix}. \tag{16.59}$$

Naively the moduli space is just the space of eigenvalues, $(\mathbb{R}^8)^N$. Notice, however, that a discrete subgroup of the gauge symmetry still acts. It is the permutations of the eigenvalues, which we shall denote as $\mathcal{S}_N$. Since we must divide by this, the vacuum moduli space is therefore the orbifold $(\mathbb{R}^8)^N/\mathcal{S}_N$.

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The strings we've defined are lying in the direction parametrised by σ, but we must study this a bit more carefully. A configuration representing strings which are of the same length of the σ circle satisfies

$$X^i(\sigma + 2\pi) = X^i(\sigma).$$

One way to think of this configuration is as representing N closed strings. The x^i_n may be thought of as the x^i coordinate of the nth string, parameterised by σ. The $x^i(\sigma)$ are otherwise arbitrary functions (subject to the equations of motion) of τ and σ, and so can truly represent arbitrary strings in various shapes. (See figure 16.10.) In fact, one of these strings has energy of order $1/N$ that required to contribute to the physics in the limit, since it is T-dual to a single D0-brane among the very large N of the whole model. What we need is a method of making a string with a larger share of the longitudinal momentum.

The matrix model naturally contains such strings too. First, note that there is a natural symmetry group which we shall denote $\mathcal{S}_N$, which acts on the strings by permuting the N eigenvalues of the matrices. The strings are all identical, and so this is a very natural model. We can use this permutation symmetry to make long strings, by making configurations which satisfy:

$$X^i(\sigma + 2\pi) = s_2 X^i(\sigma),$$

where s_n is the element of $\mathcal{S}_N$ representing the permutation of n objects. The following configuration is an example:

$$s_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix}.$$

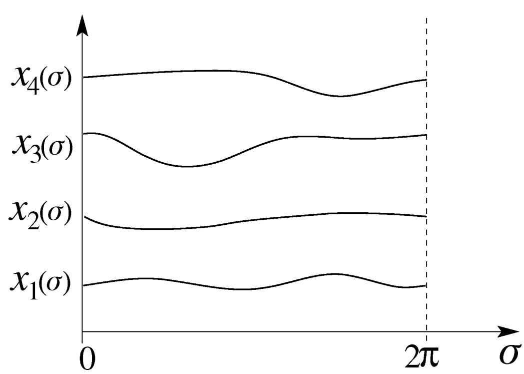

Fig. 16.10. Four minimum length strings in the matrix.

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This should remind the reader of a twisted sector from our orbifold techniques in various previous chapters, such as in section 4.8. This matrix implements a permutation of the two eigenvalues x_2 and x_3 one goes around the σ circle. So, in fact, since $s_2^2 = 1$, in order to make a closed string with eigenvalues in the 2 and 3 position, one must go around the σ circle twice. So we have made a configuration representing a string of twice the length of the basic strings. See figure 16.11. In this way, we see that the model contains closed strings which possess a large enough fraction of their energy in momentum in the eleventh direction in order to survive the limit.

To see that we get the right sort of theory, note that the limit $g_s \to 0$ actually defines a flow of the 1+1 dimensional Yang–Mills theory to the IR. There, the theory is expected to become a fully conformally invariant fixed point, representing the free type IIA matrix string. Notice that this is in fact a new way of constructing a *string field theory* of the strings, in the infinite momentum or light cone frame. It is a field theory in the sense that there are fields which create and destroy complete string configurations, the matrices $X^i(\tau, \sigma)$ themselves. The interactions between strings can be studied as well, and the splitting/joining operation is implemented by the addition of a special 'irrelevant' operator to the conformal field theory, deforming it away from the fixed point[284] towards the UV (see insert 3.1, p. 84).

It should be noted that the matrix string model at finite N has also been given an interpretation in its own right as a DLCQ definition of the theory. Also, matrix string theories (either DLCQ or IMF) for all of the other ten dimensional can be defined by similar methods. In fact, the technique has been used to supply a definition of theories (such as the special six dimensional non-gravitational string theories and their low

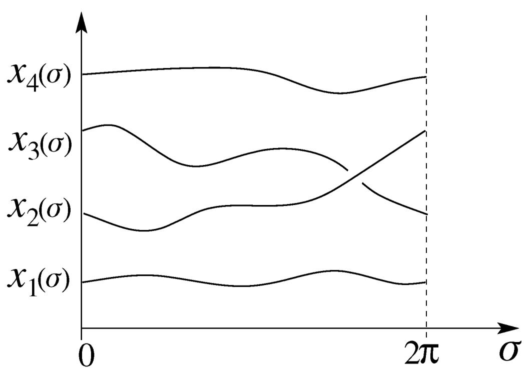

Fig. 16.11. A twisted sector representing two minimum length strings and one of twice the length.

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energy field theory limits mentioned at the end of section 12.3.2) which the usual Lagrangian techniques seem to fail to define even perturbatively[285].

As already noted, describing further compactified spacetime dimensions leads us to study higher dimensional Yang–Mills field theories in various limits, implicitly related to world-volume theories of D-branes. Unfortunately, once one gets to the study of six uncompactified directions, progress seems to stop. This is because the matrix theory is now a $5+1$ dimensional Yang–Mills field theory, which in the required matrix theory limit does not seem to make sense[158].

For this and other reasons, it seems at the time of writing that Matrix theory, while apparently a tantalising glimpse into the correct direction which will lead to a definition of M-theory, is incomplete. In retrospect, this is perhaps not surprising, since it is still rather closely wedded to D-brane techniques, being largely a reinterpretation of the physics of open strings and D-branes in various limits, albeit a very instructive and useful one. The search for a definition of M-theory must continue.