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A PROPERTY OF SERIES OF HOLOMORPHIC HOMOGENEOUS POLYNOMIALS WITH HADAMARD GAPS

JUN SOO CHOA

Recently J. Miao proved that if $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ is a holomorphic function with Hadamard gaps on the open unit disc $\mathbb{D}$ then $f \in X^p$ if and only if $f \in B^p$ if and only if $f \in B_0^p$ if and only if $\sum_{k=1}^{\infty} |a_k|^p < \infty$, where X^p, B^p and B_0^p denote respectively the class of holomorphic functions on $\mathbb{D}$ which satisfy $|f'(z)|^p \left(1-|z|^2\right)^{p-1} dxdy$ is a finite measure, a Carleson measure and a little Carleson measure on $\mathbb{D}$. In this paper we give a higher-dimensional version of Miao's result.

1. INTRODUCTION

Notation used in this note will be for the most part as in [8]. Let $\mathbb{B} = \mathbb{B}_n$ be the open unit ball of C^n $(n \geqslant 1)$ and V be the Lebesgue volume measure on $\mathbb{B}$ normalised so thet $V(\mathbb{B}) = 1$. We write S for the boundary of $\mathbb{B}$ and σ for the normalised surface measure on S. We shall set $\mathbb{D} = \mathbb{B}_1$. For z, $w \in C^n$, we let $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w_j}$ denote the complex inner product in C^n and $|z| = \langle z, z\rangle^{1/2}$. For a function f holomorphic on $\mathbb{B}$, the radial derivative $\mathcal{R}f$ of f is defined by

$$\mathcal{R}f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z) \quad (z \in \mathbb{B}).$$

Note that $\mathcal{R}f(z) = \sum_{k=0}^{\infty} k f_k(z)$ if f has the homogeneous polynomial expansion $f(z) = \sum_{k=0}^{\infty} f_k(z)$.

We say that a positive measure μ on $\mathbb{B}$ is a *Carleson measure* (respecitvely a *little Carleson measure*) if

$$\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \frac{\left(1-|a|^2\right)^n}{|1-\langle z, a\rangle|^{2n}} \, d\mu(z) < \infty \quad \left(\text{respectively } \lim_{|a| \nearrow 1} \int_{\mathbb{B}} \frac{\left(1-|a|^2\right)^n}{|1-\langle z, a\rangle|^{2n}} \, d\mu(z) = 0\right).$$

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For $0 < p < \infty$, a function f holomorphic on $\mathbb{B}$ is said to be a member of X^p, B^p or B_0^p, respectively, if $|\mathcal{R}f(z)|^p \left(1 - |z|^2\right)^{p-1} dV(z)$ is a finite measure, a Carleson measure or a little Carleson measure. It is clear that $B_0^p \subset B^p \subset X^p$ for each $p > 0$, and it is well-known that X^2 is the Hardy space H^2. (This follows from the Littlewood-Paley intergral inequalities [**2**, Lemma 3.2] and the trivial identity $\|f\|^2_{H^2(\mathbb{B})} = \int_S \|f_\zeta\|_{H^2(\mathbb{D})} \, d\sigma(\zeta)$). It is also known (see [4] or [9]) that $B^2 = BMOA$ and $B_0^2 = VMOA$.

The prototype of our work is the following result concerning Hadamard gap series on the open unit disc $\mathbb{D}$, which is due to Miao [**6**] and it was well-known (see, for example, [**3**, pp.44–45]) for the special case $p = 2$.

Theorem [Miao]. *If $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ is a holomorphic function on the open unit disc $\mathbb{D}$ that has Hadamard gaps, that is, $n_{k+1}/n_k \geqslant \lambda > 1$ for all k, then*

$$f \in X^p \iff f \in B^p \iff f \in B_0^p \iff \sum_{k=1}^{\infty} |a_k|^p < \infty.$$

In this paper we find a family of holomorphic functions on the open unit ball $\mathbb{B}$ that have the same phenomena as occurred in the theorem above. Our main result is stated in Section 3.

2. Definitions and Preliminary Results.

As usual, we write for $0 < p < \infty$

$$\|h\|_p = \left(\int_S |h(\zeta)|^p \, d\sigma\right)^{1/p}$$

and

$$\|h\|_\infty = \sup_{\zeta \in S} |h(\zeta)|$$

if h is a holomorphic homogeneous polynomial on C^n restricted to S. A holomorphic function f on $\mathbb{B}$ with the homogenous expansion $f(z) = \sum_{k=1}^{\infty} a_k P_{n_k}(z)$ (here, each P_{n_k} is a homogeneous polynomial of degree n_k) is said to have Hadamard gaps if $n_{k+1}/n_k \geqslant \lambda > 1$ for all $k = 1, 2, \cdots$.

Now we collect some material which will be used later.

The first Lemma below, which was proved by the author [1, Proposition 1], gives a criterion for a function f holomorphic on $\mathbb{B}$, with Hadamard gaps, to belong to the space X^p.

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Lemma 1. *Let* $0 < p < \infty$ *and* $f(z) = \sum_{k=1}^{\infty} a_k P_{n_k}(z)$ *be a holomorphic function on* $\mathbb{B}$ *with Hadamard gaps. Then the following conditions are equivalent.*

(i) $f \in X^p$,

(ii) $\sum_{k=1}^{\infty} |a_k|^p \|P_{n_k}\|_p^p < \infty$.

The next lemma is taken from [**5**].

Lemma 2. *Let* $\alpha > 0$, $0 < p < \infty$, $a_k \geqslant 0$, $I_k = \{n : 2^k \leqslant n < 2^{k+1},\ k \in \mathbb{N}\}$ *and* $t_k = \sum_{n \in I_k} a_n$. *Then there exists a constant* $K(p, \alpha)$ *depending only on* p *and* α *such that*

$$K(p,\alpha)^{-1} \sum_{k=0}^{\infty} 2^{-k\alpha} t_k^p \leqslant \int_0^1 \left(\sum_{k=1}^{\infty} a_k x^k \right)^p (1-x)^{\alpha-1} dx \leqslant K(p,\alpha) \sum_{k=0}^{\infty} 2^{-k\alpha} t_k^p.$$

We use Lemma 2 to obtain the following result which gives a sufficient condition for a function f holomorphic on $\mathbb{B}$, with Hadamard gaps, to be a member of B_0^p. The proof below is a slight modification of that of [**6**, Theorem 2]. But we include it for the sake of completeness.

Lemma 3. *Let* $0 < p < \infty$ *and* $f(z) = \sum_{k=1}^{\infty} a_k P_{n_k}(z)$ *be a holomorphic function on* $\mathbb{B}$ *with Hadamard gaps. Then*

$$\sum_{k=1}^{\infty} |a_k|^p \|P_{n_k}\|_{\infty}^p < \infty \quad \textit{implies} \quad f \in B_0^p.$$

Proof: By Lemma 2, we have

$$\int_0^1 \left(\sum_{k=1}^{\infty} n_k |a_k| \|P_{n_k}\|_{\infty} r^{n_k} \right)^p (1-r)^{p-1} dr \leqslant K(p) \sum_{k=0}^{\infty} 2^{-kp} t_k^p,$$

where $t_k = \sum_{n_j \in I_k} n_j |a_j| \left\| P_{n_j} \right\|_{\infty} < 2^{k+1} \sum_{n_j \in I_k} |a_j| \left\| P_{n_j} \right\|_{\infty}$.

Let $n_{k+1}/n_k \geqslant \lambda > 1$ for all k. Then the number of coefficients a_j is at most $[\log_\lambda 2] + 1$ when $n_j \in I_k$, for $k = 1, 2, \cdots$. Thus by Hölder's inequality we have

$$\begin{aligned} \sum_{k=0}^{\infty} 2^{-kp} t_k^p &\leqslant 2^p \sum_{k=0}^{\infty} \left(\sum_{n_j \in I_k} |a_j| \left\| P_{n_j} \right\|_{\infty} \right)^p \\ &\leqslant 2^p \left([\log_\lambda 2] + 1 \right)^p \sum_{j=1}^{\infty} |a_j|^p \left\| P_{n_j} \right\|_{\infty}^p, \end{aligned}$$

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and so, by hypothesis, $\int_0^1 \Big(\sum_{k=1}^{\infty} n_k\,|a_k|\,\|P_{n_k}\|_\infty r^{n_k}\Big)^p (1-r)^{p-1} dr$ is finite. Hence for any $\varepsilon > 0$, there is a $\delta \in (0,1)$ such that

$$\int_\delta^1 \left(\sum_{k=1}^{\infty} n_k\,|a_k|\,\|P_{n_k}\|_\infty r^{n_k}\right)^p (1-r)^{p-1}\,dr < \varepsilon.$$

Then integration in polar coordinates and simple calculations give

$$\begin{aligned}
&\int_{\mathbb{B}} |\mathcal{R}f(z)|^p \left(1-|z|^2\right)^{p-1} \frac{\left(1-|a|^2\right)^n}{|1-\langle z,a\rangle|^{2n}}\,dV(z) \\
&= 2n \int_S \int_0^1 \left|\sum_{k=1}^{\infty} n_k a_k P_{n_k}(\zeta) r^{n_k}\right|^p (1-r^2)^{p-1} \frac{\left(1-|a|^2\right)^n}{|1-\langle r\zeta,a\rangle|^{2n}} r^{2n-1}\,dr\,d\sigma(\zeta) \\
&\leqslant C(n,p) \int_0^1 \left(\sum_{k=1}^{\infty} n_k\,|a_k|\,\|P_{n_k}\|_\infty r^{n_k}\right)^p (1-r)^{p-1} \left\{\int_S \frac{\left(1-|a|^2\right)^n}{|1-\langle r\zeta,a\rangle|^{2n}}\,d\sigma(\zeta)\right\} dr \\
&\leqslant C(n,p) \int_0^1 \left(\sum_{k=1}^{\infty} n_k\,|a_k|\,\|P_{n_k}\|_\infty r^{n_k}\right)^p (1-r)^{p-1} \frac{\left(1-|a|^2\right)^n}{\left(1-r^2|a|^2\right)^n}\,dr \\
&\leqslant C(n,p) \int_0^\delta \left(\sum_{k=1}^{\infty} n_k\,|a_k|\,\|P_{n_k}\|_\infty r^{n_k}\right)^p (1-r)^{p-1} \frac{\left(1-|a|^2\right)^n}{\left(1-r^2|a|^2\right)^n}\,dr + C(n,p)\varepsilon \\
&\leqslant C(n,p) \frac{\left(1-|a|^2\right)^n}{(1-\delta^2)^n} \int_0^1 \left(\sum_{k=1}^{\infty} n_k\,|a_k|\,\|P_{n_k}\|_\infty r^{n_k}\right)^p (1-r)^{p-1}\,dr + C(n,p)\varepsilon,
\end{aligned}$$

where the symbol $C(n,p)$ is used to denote positive constants, not necessarily the same at each occurrence, depending only on p and the dimension n. So

$$\limsup_{|a|\nearrow 1} \int_{\mathbb{B}} |\mathcal{R}f(z)|^p \left(1-|z|^2\right)^{p-1} \frac{\left(1-|a|^2\right)^n}{|1-\langle z,a\rangle|^{2n}}\,dV(z) \leqslant C(n,p)\varepsilon.$$

Therefore $f \in B_0^p$, since $\varepsilon > 0$ is arbitrary. This proves Lemma 2. ∎

The next result, which was also proved by the author [**1**, Proposition 5], shows that in general the converse of Lemma 3 above need not be true.

Proposition 4. *Let $0 < p < \infty$. Suppose $f(z) = \sum_{k=1}^{\infty} a_k P_{n_k}(z)$ is a holomorphic function on $\mathbb{B}$ with Hadamard gaps and $f(z)$ depends only on fewer variables than the dimension n. Then*

$$f \in B^p \quad \textit{if and only if} \quad \sup_k |a_k|\,\|P_{n_k}\|_\infty < \infty.$$

3. Main Theorem

In order to state our main result, we require the so-called Ryll-Wajtaszczyk polynomials. The sequence of such polynomials that we shall need here is slightly different from the original one [7] and was obtained by Ullrich [**10**, Corollary 1]. We state the existence of those polynomials as a lemma.

Lemma 5. *For each $p > 0$, there exist a constant $C(p,n) > 0$, depending only on p and n, and a sequence $(W_k)_{k=1}^{\infty}$ of polynomials in C^n homogenous of degree k such that*

(i) $\|W_k\|_\infty \leqslant 1$, *and*
(ii) $\|W_k\|_p \geqslant C(p,n)$

for every $k = 1, 2, \cdots$.

We are now ready to state and prove the main result of this paper

Main Theorem. *Let $0 < p < \infty$ and let $f(z) = \sum_{k=1}^{\infty} a_k W_{n_k}(z)$ be a holomorphic function on $\mathbb{B}$ with Hadamard gaps (in which each W_{n_k} is a Ryll-Wojtaszczyk polynomial that satisfies the two conditions (i) and (ii) of Lemma 5). Then the following conditions are equivalent.*

(a) $f \in X^p$,
(b) $f \in B^p$,
(c) $f \in B_0^p$,
(d) $\sum_{k=1}^{\infty} |a_k|^p < \infty$.

Proof: Since the implications $(c) \Longrightarrow (b) \Longrightarrow (a)$ are trivial, we have only to verify that $(d) \Longrightarrow (c)$ and $(a) \Longrightarrow (d)$. We first assume (d) holds. Then by Lemma 5 (i), we see that

$$\sum_{k=1}^{\infty} |a_k|^p \left\|W_{n_k}\right\|_\infty^p \leqslant \sum_{k=1}^{\infty} |a_k|^p$$

and thus $f \in B_0^p$ by Lemma 3, which shows the implication $(d) \Longrightarrow (c)$. Finally we assume (a) holds and show (d). From Lemma 2 and Lemma 5 (ii), we get

$$\infty > \sum_{k=1}^{\infty} |a_k| \left\|W_{n_k}\right\|_p^p \geqslant C(p,n)^p \sum_{k=1}^{\infty} |a_k|^p ,$$

which proves $(a) \Longrightarrow (d)$, and the proof is complete. ∎

The special interest of the Main Theorem is the case $p = 2$. So we record it as a corollary.

Corollary. *If* $f(z) = \sum_{k=1}^{\infty} a_k W_{n_k}(z)$ *is a holomorphic function on* $\mathbb{B}$ *with Hadamard gaps (in which each* W_{n_k} *is a Ryll-Wojtaszczyk polynomial), then*

$$f \in H^2 \iff f \in BMOA \iff f \in VMOA \iff \sum_{k=1}^{\infty} |a_k|^2 < \infty.$$

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Department of Mathematics Education
Sung Kyun Kwan University
Jongro-Gu, Seoul 110-745
Korea
e-mail: jschoa@yurim.skku.ac.kr