THE STABILITY THEOREMS FOR DISCRETE DYNAMICAL SYSTEMS ON TWO-DIMENSIONAL MANIFOLDS

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§1. Introduction

One of the basic problems in the theory of dynamical systems is the characterization of stable systems.

Let M be a closed (i.e. compact without boundary) connected smooth manifold with a smooth Riemannian metric and $\mathrm{Diff}^r(M)$ $(r \geq 1)$ denote the space of C^r diffeomorphisms on M with the uniform C^r topology. Let $f \in \mathrm{Diff}^s(M)$ with $s \geq r$. Then f is called C^r structurally stable if and only if there is a neighborhood $\mathscr{U}(f)$ of f in $\mathrm{Diff}^r(M)$ such that for any $g \in \mathscr{U}(f)$ there exists a homeomorphism $h: M \to M$ satisfying $gh = hf$.

Another important notion of stability is the Ω-stability. Recall that $x \in M$ is a non-wandering point of f if and only if for any neighborhood U of x, there is a nonzero integer m such that $f^m(U) \cap U \neq \phi$. The set $\Omega(f)$ of all the non-wandering points of f is a closed invariant set. f is called C^r Ω-stable if and only if there is a neighborhood $\mathscr{U}(f)$ of f in $\mathrm{Diff}^r(M)$ such that for any $g \in \mathscr{U}(f)$ there exists a homeomorphism $h: \Omega(f) \to \Omega(g)$ satisfying $gh = hf$ on $\Omega(f)$.

The essential condition to characterize these stabilities is "Axiom A" introduced by S. Smale in [17]. Namely, f satisfies Axiom A if and only if
  (a) $\Omega(f)$ is a hyperbolic set,
  (b) $\overline{\mathrm{Per}\,(f)} = \Omega(f)$,
where $\mathrm{Per}\,(f)$ denotes the set of all the periodic points of f. Recall that a compact f-invariant subset $\Lambda \subset M$ is a hyperbolic set if and only if there exist constants $c > 0$, $0 < \lambda < 1$ and a Tf-invariant splitting $TM|\Lambda = E^s \oplus E^u$ such that

$$\| Tf^n | E_p^s \| \leq c\lambda^n$$
$$\| Tf^{-n} | E_p^u \| \leq c\lambda^n$$

for all $p \in \Lambda$ and non-negative integers n.

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In [9], [18] and [8], the following is conjectured.

Structural stability conjecture. f is c^r structurally stable if and only if f satisfies Axiom A and Strong transversality condition.

Ω-stability conjecture. f is c^r Ω-stable if and only if f satisfies Axiom A and No cycle property.

For the definitions of Strong transversality condition and No cycle property, we refer to [9], [18].

The purpose of this paper is to give an affirmative answer to these conjectures for f of class C^2 in case of dim $M = 2$ and $r = 1$.

The sufficiency parts are known to be true for arbitrary dimension, that is; if f satisfies Axiom A and Strong transversality condition, then f is C^1 structurally stable (Robbin [13] for f of class C^2 and Robinson [15] for f of class C^1), and if f satisfies Axiom A and No cycle property, then f is C^1 Ω-stable (Smale [18]). Remark that if f is C^1 structurally (resp. Ω-) stable and of class C^s, then C^r structurally (resp. Ω-) stable for all $1 \leq r \leq s$.

Concerning the converses, it is known that C^r structural stability plus Axiom A implies Strong transversality condition [14], and C^r Ω-stability plus Axiom A implies No cycle property [8], for all $r \geq 1$. Thus, if we can verify that Ω-stability implies Axiom A, then the above conjectures are established.

In this paper, we investigate a certain class $F(M)$ of C^1 diffeomorphisms introduced by Mañé [5], [6] which contains all C^1 Ω-stable diffeomorphisms; namely, we put

$$F(M) = \text{int}_1 \{g \in \text{Diff}^1(M): \text{any periodic point of } g \text{ is hyperbolic}\},$$

where int_1 means "interior" with respect to C^1 topology of $\text{Diff}^1(M)$ and a periodic point x of g with period m is hyperbolic if and only if $T_x g^m: T_x M \to T_x M$ has no eigenvalue of absolute value one.

Our result is the following;

THEOREM. *Let* dim $M = 2$ *and* $f \in F(M)$. *If* f *is of class* C^2, *then* $\Omega(f)$ *is a hyperbolic set.*

By the theorem of Kupka-Smale [14], it can be seen that if f is C^1-Ω-stable, then $f \in F(M)$. Furthermore, by the "C^1 Closing lemma", we have $\overline{\text{Per}(f)} = \Omega(f)$, namely Axiom $A(b)$ holds for $f \in F(M)$, (see Lemma

3.1 in [6] for the proof).

Thus, as corollaries of our Theorem, we get;

Structural stability theorem. Let $\dim M = 2$ and $f \in \mathrm{Diff}^2(M)$. f is C^1 structurally stable if and only if f satisfies Axiom A and strong transversality condition.

Ω-stability theorem. Let $\dim M = 2$ and $f \in \mathrm{Diff}^2(M)$. f is C^1 Ω-stable if and only if f satisfies Axiom A and No cycle property.

In this paper, we investigate only C^1 stability, namely $F(M)$, because the "C^r Closing lemma" has not been established for $r \geq 2$. As we mentioned above, for the proof of Axiom $A(b)$ for $f \in F(M)$, we need the "C^1 Closing lemma", and this is again our main tool for the proof of our Theorem. As is pointed out by Robinson in [14], it seems to be an interesting problem whether C^2 structural (or Ω-) stability implies C^1 structural (or Ω-) stability or not.

This paper consists of 15 sections. In Sections $2 \sim 9$, we give technical preliminaries. The essential part of the proof of our Theorem is given in Sections $10 \sim 15$.

For the understanding of the basic ideas of the proof, the reader is recommended to look at the statements of (2.3), (4.1), (12.1), (12.2), (12.3) and all of Section 13; "Proof of the Theorem". Then it will be understood how easily the Theorem may be proved provided that (12.2) Lemma and (12.3) Lemma have been verified. The essence is in (12.2) Lemma and it is proved without much difficulty, once (11.1) Key lemma is established. Therefore, we will expend our almost all efforts in proving this Key lemma.

The present work is motivated by Mañé [5], Robinson [16] and Pliss [10], [11], especially, Pliss's excellent idea of "cutting off the suborbits" in [10]. We apply it to the proof of our (11.2) Main lemma in Section 15, but in his argument, there are some points which are not clear and we use this idea in a different way from his.

After I had finished this work, I was informed that in [Chin. Ann. of Math. 1(1980), 9–29.] S. D. Liao also asserted that he proved the stability conjectures for C^1 diffeomorphisms on 2-manifolds and for C^1 flows with isolated singularities on 3-manifolds. But his method is considerably different from ours.

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LIST OF SYMBOLS

[ ]: integer part of a number. r_1: (5.4).

$d(x, y)$: distance from x to y. r_2: (5.4).

Z_+: non-negative integers. r_3: (6.1).

a_p: §5. r_4: (6.2).

A_0: (11.1). r_5: §7.

$b_p(v, a_1, a_2)$: §7. r_6: (9.1).

$B(r)$, $B(r', r'')$: §6. r_7: (9.1).

$B_p(r)$, $B_p(r', r'')$: §6. r_8: (10.2).

c: (2.3), (2.4), (2.5). r_9: (14.4).

c_1: (5.4) (vi). $T(r, r')$: (14.1).

d_p: §5. $T_p(r, r')$: (14.3).

$E_p^u(f)$, $E_p^s(f)$: §2. $U(r)$: (5.4) (i).

E^1, E^2, E_p^1, E_p^2: (2.3) (ii). $\mathscr{U}_1$: (2.3) (i).

$F(M)$: §1. $\mathscr{U}_2$: (2.9).

G: (10.1). $\mathscr{U}(\varepsilon, f)$: §2.

G_1: (11.2). $V_p(r)$: §5.

m_0: (2.7), §5. $\beta = \lambda^{-1/2}$: §14.

m_1: proof of (10.3). ε_0: (10.1).

M_0: (5.4) (v). $\eta(p, n)$: §6.

M_1: §5. $\theta(p, n)$: §6.

N_*: (11.2). λ: (2.3), (2.4), (2.5).

N_0: (10.1). $\Lambda = \Lambda_1(f)$: §4.

N_1: (9.1). $\Lambda_i(f)$: (2.1).

N_2: (10.1). $\nu_0(N)$: (11.2).

N_3: (10.2). ψ_p: §5.

N_4: (12.2). $\omega(p, n)$: §6.

§2. Some preliminary results

From now on, we assume that M is a fixed closed 2-dimensional smooth manifold with a smooth Riemannian metric. For $f \in F(M)$, $p \in$ Per (f) is hyperbolic and we denote by $E_p^u(f)$ $(E_p^s(f))$ the unstable (stable) subspace of $T_p(M)$. We put

(2.1) $\Lambda_i(f) =$ closure $\{p \in \text{Per}\,(f) : \dim E_p^s(f) = i\}$ for $i = 0, 1, 2$.

Note that $\Lambda_0(f)$ and $\Lambda_2(f)$ are the sets of all the sources and the sinks of f respectively. In [11], Pliss proved the following,

(2.2) LEMMA. *For $f \in F(M)$, $\Lambda_0(f)$ and $\Lambda_2(f)$ are finite sets.*

From this lemma, we know that $\Lambda_0(f)$ and $\Lambda_2(f)$ are hyperbolic sets. The following lemma proved by Mañé [5] will play a key role in studying the precise properties of $f \in F(M)$.

(2.3) LEMMA. *For $f \in F(M)$, there exist $c > 0$ and $0 < \lambda < 1$ satisfying;*
(i) *there exists a C^1-neighborhood $\mathcal{U}_1$ of f such that*

$$\| Tg^{\pi(p)} \,|\, E_p^s(g) \| \leqq c\lambda^{\pi(p)}$$
$$\| Tg^{-\pi(p)} \,|\, E_p^u(g) \| \leqq c\lambda^{\pi(p)}$$

for all $g \in \mathcal{U}_1$ and $p \in \mathrm{Per}\,(g)$, where $\pi(p)$ denotes the period of p.
(ii) *there exists a Tf-invariant continuous splitting $TM \,|\, \Lambda_1(f) = E^1 \oplus E^2$ such that*

$$\| Tf^n \,|\, E_p^2 \| \cdot \| Tf^{-n} \,|\, E_{f^n(p)}^1 \| \leqq c\lambda^n$$

for all $n \in Z_+$ and $p \in \Lambda_1(f)$. Moreover if $p \in \Lambda_1(f) \cap \mathrm{Per}\,(f)$, then $E_p^1 = E_p^u(f)$ and $E_p^2 = E_p^s(f)$, where E_p^i $(i = 1, 2)$ denotes the fiber of E^i over p.

Note that $\dim E_p^1 = \dim E_p^2 = 1$ for all $p \in \Lambda_1(f)$.
By changing $0 < \lambda < 1$ if necessary, we may assume that the constant $c > 0$ of (2.3) (i) equals 1 on saddles, that is;

(2.4) LEMMA.

$$\| Tg^{\pi(p)} \,|\, E_p^s(g) \| \leqq \lambda^{\pi(p)}$$
$$\| Tg^{-\pi(p)} \,|\, E_p^u(g) \| \leqq \lambda^{\pi(p)}$$

for all $g \in \mathcal{U}_1$ and $p \in \Lambda_1(f) \cap \mathrm{Per}\,(f)$.

To simplify the notations and calculations in the succeeding sections, we also assume that,

(2.5) (i) $1/2 \leqq \lambda < 1$ and
(ii) $c \geqq 1$.

(i) will be used only to formulate (14.5). Throughout this paper, the symbols c and λ denote the constants given by (2.3), (2.4), and (2.5).
The next lemma is a modification of Lemma 2.1 in [10].

(2.6) LEMMA. *Let n be a positive integer and $h: \{0, 1, 2, \cdots, n\} \to \mathbf{R}$ be a map. Take $m \leqq \min_{0 \leqq i \leqq n-1}\{h(i+1) - h(i)\}$, and $\sigma > m$. If $h(n) - h(0) \leqq \ell$ for some $\ell \in \mathbf{R}$ with $n\sigma - \ell \geqq 0$, then there exist $\nu = [(n\sigma - \ell)/(\sigma - m)] + 1$ integers $0 \leq k_1 \leq k_2 < \cdots < k_\nu \leq n$ such that*

$$h(k + k_j) - h(k_j) \leq k\sigma$$

for all $0 \leq k \leq n - k_j$ and $1 \leq j \leq \nu$.

Proof. We shall select k_j inductively as follows. Let k_1 be the largest integer in $\{0, 1, \cdots, n\}$ satisfying $h(k_1) - h(0) \geq k_1\sigma$. Such an integer exists, because zero satisfies this condition. k_1 has the property that

$$h(k + k_1) - h(k_1) \leq k\sigma \qquad \text{for all } 0 \leq k \leq n - k_1,$$

because if $h(k + k_1) - h(k_1) > k\sigma$ for some $0 < k \leq n - k_1$, then

$$h(k + k_1) - h(0) = h(k + k_1) - h(k_1) + h(k_1) - h(0) > (k_1 + k)\sigma$$

which contradicts the definition of k_1.

Assume that k_j is defined and $k_j < n$. We select k_{j+1} as the largest integer in $\{k_j + 1, k_j + 2, \cdots, n\}$ satisfying

$$h(k_{j+1}) - h(k_j + 1) \geq (k_{j+1} - k_j - 1)\sigma .$$

Such an integer exists because $k_j + 1$ satisfies this condition. By the same argument as above, we have,

$$h(k + k_{j+1}) - h(k_{j+1}) \leq k\sigma \qquad \text{for all } 0 \leq k \leq n - k_{j+1} .$$

Now we shall estimate how many $\{k_j\}$ we can select. Since

$$\begin{aligned}
\ell \geq h(n) - h(0) &= (h(n) - h(k_j)) + (h(k_j) - h(k_{j-1} + 1)) \\
&\quad + (h(k_{j-1} + 1) - h(k_{j-1})) + \cdots + (h(k_1 + 1) - h(k_1)) \\
&\quad + (h(k_1) - h(0)) \\
&\geq (n - k_j)m + (k_j - k_{j-1} - 1)\sigma + m + \cdots + m + k_1\sigma \\
&= nm - (j - 1)(\sigma - m) + k_j(\sigma - m) ,
\end{aligned}$$

and $\sigma - m > 0$, we have

$$k_j \leq (j - 1) + (\ell - nm)/(\sigma - m) .$$

Namely, if $j \leq (n\sigma - \ell)/(\sigma - m)$, we can define k_{j+1} because

$$\begin{aligned}
k_j &\leq (n\sigma - \ell)/(\sigma - m) - 1 + (\ell - nm)/(\sigma - m) \\
&= n - 1 .
\end{aligned}$$

This completes the proof.

Applying (2.6) to our situation with $h(k) = \log \| Tf^k | E_p^1 \|$, we get the following,

(2.7) LEMMA. *Let $f \in F(M)$, $p \in \Lambda_1(f)$ and n be a positive integer. Put $m_0 = \inf_{p \in \Lambda_1(f)} \{ \| Tf | E_p^1 \|, \| Tf | E_p^2 \| \}$, and take $\rho > m_0$. If $\log \| Tf^n | E_p^1 \| \leq \ell$ for some $\ell \in \mathbf{R}$ with $n(\log \rho) \geq \ell$, then there are*

$$\nu = [(n(\log \rho) - \ell)/(\log \rho - \log m_0)] + 1$$

integers $0 \leq k_1 < k_2 < \cdots < k_\nu \leq n$ such that

$$\| Tf^k | E_{f^{k_j}(p)}^1 \| \leq \rho^k$$

for all $0 \leq k \leq n - k_j$ and $1 \leq j \leq \nu$.

Now we shall give here a C^1-distance on a neighborhood of $f \in \mathrm{Diff}^1(M)$. The definition is a usual one.

Let $\{ U_i, \alpha_i \}_{i \in I}$ and $\{ V_j, \beta_j \}_{j \in J}$ be local coordinate systems on M with finite index sets, and $\gamma : I \to J$ be a map such that $f(\overline{U}_i) \subset V_{\gamma(i)}$ for all $i \in I$. Let $g_1, g_2 \in \mathrm{Diff}^1(M)$ be sufficiently near f so that $g_k(\overline{U}_i) \subset V_{\gamma(i)}$ for all $i \in I$ ($k = 1, 2$). We define a C^1-distance on a neighborhood of f by

$$d_1(g_1, g_2) = \sup_{i \in I} \sup_{x \in \alpha_i(U_i)} \max \{ | \beta_{\gamma(i)} g_1 \alpha_i^{-1}(x) - \beta_{\gamma(i)} g_2 \alpha_i^{-1}(x) |, $$
$$\| T_x(\beta_{\gamma(i)} g_1 \alpha_i^{-1}) - T_x(\beta_{\gamma(i)} g_2 \alpha_i^{-1}) \| \} .$$

With this distance we can define an ε-neighborhood of f for small $\varepsilon > 0$, which is denoted by $\mathscr{U}(\varepsilon, f)$.

Clearly we have the next lemma which guarantees simultaneous perturbations with disjoint supports, that is;

(2.8) LEMMA. *Let $\varepsilon > 0$ be small and $g_1, g_2 \in \mathscr{U}(\varepsilon, f)$. If $\mathrm{supp}\,(g_1 f^{-1}) \cap \mathrm{supp}\,(g_2 f^{-1}) = \phi$, then we have;*

$$(g_1 f^{-1})(g_2 f^{-1})f \in \mathscr{U}(\varepsilon, f) .$$

Next lemma is an easy modification of (1.1) Lemma in [1], and will be used in Sections 3 and 8.

(2.9) LEMMA. *Let $f \in \mathrm{Diff}^1(M)$. There exists a C^1-neighborhood $\mathscr{U}_2$ of f satisfying the following: For any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $\{ p_1, \cdots, p_m \}$ is a finite set of points in M, $g \in \mathscr{U}_2$ and A_i ($i = 1, \cdots, m$) are linear automorphisms on $T_{g(p_i)}M$ satisfying $\| A_i - \mathrm{id} \| \leq \delta$ for all $1 \leq i \leq m$, then there exists a C^1 map g' such that*

(i) $d_1(g, g') < \varepsilon$

(ii) $(Tg')_{p_i} = A_i \circ (Tg)_{p_i}$ *for all* $1 \leq i \leq m$.

Moreover, for any neighborhood U of $\{p_1, \cdots, p_m\}$, we can require that $g = g'$ outside U.

§3. Stability of sinks and sources

Throughout the rest of this paper, f denotes a fixed C^2 diffeomorphism in $F(M)$. In this section, we shall show that the sinks and sources of f are stable under small perturbations.

(3.1) LEMMA. *Given $\varepsilon > 0$, there is a C^1-neighborhood $\mathscr{U}$ of f with the following property; for any $g \in \mathscr{U}$, there is a bijection $h \colon \Lambda_0(f) \cup \Lambda_2(f) \to \Lambda_0(g) \cup \Lambda_2(g)$ such that $d_0(h, \mathrm{id}) \leq \varepsilon$, where d_0 denotes the C^0 distance.*

Remark that if f is structurally stable, this lemma is obvious by the stability of hyperbolic sets [3]. But for $f \in F(M)$, it is not so easy, because we must guarantee that there appear neither new sinks nor sources under small perturbations. We may find an idea of our proof in [11]. Using (2.3) and (2.9), we can modify Theorem 3.1 in [11] as follows.

(3.2) LEMMA. *There exist a C^1-neighborhood $\mathscr{U}$ of f and $\theta_0 > 0$ satisfying the following; for any integer $\theta \geq \theta_0$, there is $m(\theta) \geq 0$ such that for $g \in \mathscr{U}$ and $p \in \Lambda_2(g)$ with $\pi(p) \geq m(\theta)$, we have*

$$\prod_{i=1}^{[\pi(p)/\theta]} \|(Tg^\theta)_{g^{(i-1)\theta}(p)}\| \leq (\lambda^{1/2})^{\pi(p)} ,$$

where $\pi(p)$ denotes the period of p.

Proof of (3.1). We only need to prove the lemma for Λ_2, that is; for any $\varepsilon > 0$ there is a C^1-neighborhood $\mathscr{U}$ of f such that for any $g \in \mathscr{U}$, there exists a bijection $h \colon \Lambda_2(f) \to \Lambda_2(g)$ with $d_0(h, \mathrm{id}) \leq \varepsilon$.

Suppose the contrary, i.e., there exist $\varepsilon > 0$ and sequences $g_n \to f$, $p_n \in \Lambda_2(g_n)$ such that $d(p_n, \Lambda_2(f)) > \varepsilon$.

If $\pi(p_n)$ are bounded as $n \to \infty$, by taking a subsequence we can assume that there is a $k \in Z_+$ with $\pi(p_n) = k$ for all n and $p_n \to p$ for some p. Clearly $f^k(p) = p$. Since $d(p, \Lambda_2(f)) \geq \varepsilon$, p is not a sink of f. As p is hyperbolic, there is a non-zero vector $v \in T_pM$ such that the norm of $Tf^i(v)$ increase exponentially as $i \to \infty$. This is a contradiction because p_n is a sink of g_n and $g_n \to f$.

Next, we see that the period of p_n are bounded as $n \to \infty$. We can apply the same technique which Pliss used to prove the finiteness of sinks and sources of $f \in F(M)$ in [11]. He showed that the period of sinks of the diffeomorphism in $F(M)$ cannot be arbitrarily large. Using (3.2), we can estimate the supremum of the period of the sinks uniformly in a C^1-neighborhood of f, and get the boundedness of $\{\pi(p_n)\}$. This completes the proof.

§4. Preliminary lemmas about the spectrum of f_*

In what follows, we put $\Lambda = \Lambda_1(f)$. Let

$$\Sigma^b(E^i) = \{\text{bounded sections } \Lambda \to E^i\} \qquad i = 1, 2,$$

and let $f_*: \Sigma^b(E^i) \to \Sigma^b(E^i)$ be defined by

$$f_*(\sigma) = Tf \circ \sigma \circ f^{-1} \qquad \sigma \in \Sigma^b(E^i).$$

$\Sigma^b(E^i)$ is a Banach space with the natural sup. norm and f_* is the automorphism of $\Sigma^b(E^i)$ canonically induced by f.

In this section we shall prove two lemmas that will be used in Sections 5 and 13.

The next lemma is a modification of an argument in [5].

(4.1) LEMMA. *Let* $\mu > 0$. *If*

$$\text{spec. rad. } f_*^{-1} | \Sigma^b(E^1) \geq \mu,$$

then there is a recurrent point $p_* \in \Lambda$ *such that*

$$\|Tf^n | E_{p_*}^1\| \leq \mu^{-n} \qquad \text{for all } n \in Z_+,$$

where spec. rad. *denotes the spectral radius.*

Proof. Let S be the family of compact f-invariant subsets of Λ such that

$$\text{spec. rad. } f_*^{-1} | \Sigma^b(E^1 | K) \geq \mu \qquad K \in S.$$

We define an order on S by inclusion. Then S is an inductively ordered set, and by Zorn's lemma, we may find a minimal element $K_0 \in S$. By an argument in [7], there is a sequence $\sigma_n \in \Sigma^b(E^1 | K_0)$ such that $\|\sigma_n\| = 1$ and

$$\|(f_* - rI)\sigma_n\| \leq 1/n,$$

where $r = (\text{spec. rad. } f_*^{-1} | \Sigma^b(E^1 | K_0))^{-1}$, and I denotes the identity. Note that $r \leq \mu^{-1}$.

Choose $p_n \in \Lambda$ such that $|\sigma_n(p_n)| \geq n/(n+1)$, $p_n \to p_*$ for some $p_* \in \Lambda$ and $\sigma_n(p_n) \to v \in E_{p_*}^1$. Clearly $|v| = 1$.

Now we shall prove

$$\| Tf^m | E_{p_*}^1 \| \leq r^m \qquad \text{for all } m \in Z_+ \,.$$

Put $L = f_* | \Sigma^b(E^1 | K_0)$ and $w = \max \{\sup_{0 \leq k \leq m} \|L^k\|, 1\}$, where m is now fixed. Then we have

$$\|(L^m - r^m I)\sigma_n\| = \|(L^m - rL^{m-1} + rL^{m-1} - r^m I)\sigma_n\|$$
$$\leq w/n + r \|(L^{m-1} - r^{m-1}I)\sigma_n\| \,,$$

and therefore

$$\|(L^m - r^m I)\sigma_n\| \leq (w/n)(1 + r + r^2 + \cdots + r^{m-2} + r^{m-1}$$
$$= (w/n)\cdot(1 - r^m)/(1 - r) \,.$$

Since $\dim E^1 = 1$ and $|v| = 1$, it suffices to show that

$$|Tf^m(v)| \leq r^m \,.$$

In fact,

$$|Tf^m \sigma_n(p_n)| = |Tf^m \sigma_n f^{-m}(f^m(p_n)) - r^m \sigma_n(f^m(p_n)) + r^m \sigma_n(f^m(p_n))|$$
$$\leq \|(L^m - r^m I)\sigma_n\| + r^m \,.$$

Since $\sigma_n(p_n) \to v$ and $\|(L^m - r^m I)\sigma_n\| \to 0$ as $n \to \infty$, we have $|Tf^m(v)| \leq r^m$ as claimed.

We can also see that $\omega(p_*) \in S$, where $\omega(p_*)$ denotes the ω-limit set of p_*. Since K_0 is a minimal element in S, $\omega(p_*) = K_0$. This implies that p_* is a recurrent point, and completes the proof.

The next lemma is necessary to prove that the pre-laminations $\{W_p^1\}_{p \in \Lambda}$ and $\{W_p^2\}_{p \in \Lambda}$ are of class C^2 in Section 5.

(4.2) LEMMA.

$$\text{spec. rad. } f_*^{-1} | \Sigma^b(E^1) \leq 1$$
$$\text{spec. rad. } f_* | \Sigma^b(E^2) \leq 1 \,.$$

Proof. We shall only prove the first case.

Suppose that there is a $\mu > 1$ such that

$$\text{spec. rad. } f_*^{-1} | \Sigma^b(E^1) \geq \mu \,.$$

Then from (4.1), there is a recurrent point $p_* \in \Lambda$ such that

$$\|Tf^n\,|\,E^1_{p_*}\| \leq \mu^{-n} \qquad \text{for all } n \in Z_+ \,.$$

From (2.3) (ii), we have

$$\|Tf^n\,|\,E^2_{p_*}\| \leq c(\lambda\mu^{-1})^n \qquad \text{for all } n \in Z_+ \,.$$

This implies that a neighborhood of p_* contracts exponentially under the iteration of f. Using the recurrence of p_* with precise but straightforward arguments, we can see that there is a sink sufficiently near p_*. This contradicts (2.2).

§5. Semi-invariant coordinate

In the succeeding sections, we study the behavior of the iterated images of a neighborhood of a point in Λ. A certain local coordinate system is indispensable for the precise estimates, and we shall define it in this section.

The definition is not a peculiar but a natural one. This local coordinate is defined on a neighborhood of each point $p \in \Lambda$, and has the laminae W^1_p, W^2_p of the locally f-invariant C^2 prelaminations $\{W^1_p\}_{p\in\Lambda}$, $\{W^2_p\}_{p\in\Lambda}$ as the x-axis and the y-axis of $\mathbf{R}^2$ respectively. Thus, with respect to this coordinate, f preserves x and y-axes.

The arguments in this section heavily depend on Section 5 (especially (5.1) Theorem and (5.5) Theorem) in [4], so refer to it for details.

Recall that, from (2.3) (ii), we have,

$$\|Tf^n\,|\,E^2_p\| \cdot \|Tf^{-n}\,|\,E^1_{f^n(p)}\| \leq c\lambda^n \qquad \text{for all } n \in Z_+ \text{ and } p \in \Lambda \,.$$

Moreover, we have the following,

(5.1) LEMMA. *There is an $n_0 \in Z_+$ such that if $n \geq n_0$,*

$$\|Tf^n\,|\,E^2_p\| \cdot \|Tf^{-n}\,|\,E^1_{f^n(p)}\|^k \leq c(\lambda^{1/2})^n$$

for all $p \in \Lambda$ and $k = 1, 2$.

Proof. For $k = 1$, this is just (2.3) (ii).

We shall show the case of $k = 2$. From (4.2), we have

$$1 \geq \text{spec. rad. } f_*^{-1}\,|\,\Sigma^b(E^1)$$
$$= \lim_{n\to\infty} \|f_*^{-n}\,|\,\Sigma^b(E^1)\|^{1/n} \,.$$

Hence, there exists $n_0 \in Z_+$ such that if $n \geq n_0$,

$$\| f_*^{-n} \,|\, \Sigma^b(E^1) \|^{1/n} \leq \lambda^{-(1/2)} \ .$$

Since

$$\| f_*^{-n} \,|\, \Sigma^b(E^1) \| \geq \| Tf^{-n} \,|\, E_p^1 \| \qquad \text{for all } p \in \Lambda \ ,$$

we have

$$\| Tf^{-n} \,|\, E_{f^n(p)}^1 \| \leq (\lambda^{-1})^{n/2} \qquad \text{for all } p \in \Lambda \ .$$

Combining with (2.3) (ii), we get the lemma.

Put $E = TM|\Lambda = E^1 \oplus E^2$ and we regard E as a vector bundle over E^1 by projecting each fiber onto E^1 along E^2. Let

$$\Sigma(E^1, E) = \{\text{continuous sections } E^1 \to E\} \ .$$

Define a metric on $\Sigma(E^1, E)$ by

$$\| \sigma - \sigma' \|_* = \sup_{p \in \Lambda} \sup_{\substack{x \in E^1 \\ x \neq 0}} \frac{|\sigma x - \sigma' x|}{|x|} \ .$$

Let

$$\Sigma_0 = \{\sigma \in \Sigma(E^1, E) \colon \| \sigma \|_* < \infty\}$$

and

$$\Sigma(1/2) = \{\sigma \in \Sigma_0 \colon \sigma_p(0) = 0, \ L(\sigma_p) \leq 1/2 \ \text{for all } p \in \Lambda\}$$

where $\sigma_p \colon E_p^1 \to E_p^2$ is defined by

$$(\sigma \,|\, E_p^1)(x) = (x, \sigma_p(x)) \in E_p^1 \oplus E_p^2 \qquad \text{for } x \in E_p^1 \ ,$$

and $L(\sigma_p)$ denotes the Lipschitz constant of σ_p. With the metric $\| \cdot \|_*$, Σ_0 is a Banach space and $\Sigma(1/2)$ is a closed subset (ref. (5.2) in [4]).

We define $\tilde{f} \colon TM|\Lambda \to TM|\Lambda$ by $\exp_{f(p)}^{-1} f \exp_p$ on a neighborhood of 0 in $T_p M$ for each $p \in \Lambda$, and extend it to all over $TM|\Lambda$ by combining with Tf by a smooth bump function. Note that $\sup_{p \in \Lambda} L((\tilde{f} - Tf)|T_p M)$ can be forced as small as desired by restricting $\tilde{f}$ to a smaller neighborhood of 0 before averaging with Tf.

Let $N \in Z_+$ satisfy $N \geq n_0$ and $c(\lambda^{1/2})^N < 1$, and put $g = f^N$. Then from (5.1), there is a $0 < \rho < 1$ such that

$$\| Tg \,|\, E_p^2 \| \cdot \| Tg^{-1} \,|\, E_{g(p)}^1 \|^k \leq \rho$$

for all $p \in \Lambda$ and $k = 1, 2$.

For any $\sigma \in \Sigma(1/2)$, we define

$$g_{\sharp}\sigma = \tilde{g}\sigma h , \qquad h = (\pi_1 \tilde{g}\sigma)^{-1} ,$$

where $\tilde{g} = f^N$ and $\pi_1 \colon E \to E^1$ is the canonical projection. By the almost same estimates as in [4], we can see that $g_{\sharp}$ is well-defined and a contraction on $\Sigma(1/2)$. This gives a unique invariant section σ^1 which satisfy

(5.2) For all $p \in \Lambda$,

  (i) $L(\sigma_p^1) \leq 1/2$

  (ii) $\sigma_p^1(0) = 0$

  (iii) $\sigma_p^1 \colon E_p^1 \to E_p^2$ is of class C^2

  (iv) $T_0\sigma_p^1 = 0$

  (v) $\tilde{g}\,(\text{graph}\,(\sigma_p^1)) = \text{graph}\,(\sigma_p^1)$

  (vi) $\sigma_p^1 \colon E_p^1 \to E_p^2$ depends continuously on $p \in \Lambda$ as C^2 map.

Put $W_p^1(r) = \exp_p\,(\text{graph}\,(\sigma_p^1)\,\cap\,V_p(r))$ for $r > 0$, where

$$V_p(r) = \{(v_1, v_2) \in E_p^1 \oplus E_p^2 \colon |V_1| \leq r,\ |v_2| \leq r\} .$$

$\{W_p^1(r)\}_{p \in \Lambda}$ is called locally g-invariant C^2 pre-limination for small $r > 0$ (ref. (5.5) in [4]).

Now we shall show that by taking $\sup_{p \in \Lambda} L((\tilde{f} - Tf)\,|\,T_pM)$ and $r > 0$ small, $\{W_p^1(r)\}_{p \in \Lambda}$ is f-invariant.

Recall that $g = f^N$, and $N \in \mathbf{Z}_{+}$ satisfy $N \geq n_0$ and $c(\lambda^{1/2})^N \leq \rho < 1$. Consider $g_1 = f^N$, $g_2 = f^{N+1}$ and $g_3 = f^{N(N+1)}$. As above, they satisfy

$$\|Tg_i\,|\,E_p^2\| \cdot \|Tg_i^{-1}\,|\,E_{g_i(p)}^1\|^k \leq \rho < 1$$

for all $p \in \Lambda$, $k = 1, 2$. Therefore,

$$g_{i\sharp} \colon \Sigma(1/2) \longrightarrow \Sigma(1/2)$$

has a unique fixed point σ_i for $i = 1, 2, 3$. Taking $\sup_{p \in \Lambda} L((\tilde{f} - Tf)\,|\,T_pM)$ sufficiently small, by induction, we can see that,

  (i) $(g_{1\sharp})^{N+1} = g_{3\sharp} = (g_{2\sharp})^N$ on $\Sigma(1/2)$

  (ii) $f_{\sharp} \colon \Sigma(1/2) \to \Sigma_0$ is well-defined by
      $f_{\sharp}\sigma = \tilde{f}\sigma h$, where $h = (\pi_1\tilde{f}\sigma)^{-1}$

and has the property that

$$g_{2\sharp} = f_{\sharp} \circ g_{1\sharp} \qquad \text{on } \Sigma(1/2) .$$

From (i), we get $\sigma_1 = \sigma_2 = \sigma_3$. Denoting this σ^1, from (ii), we have

$$\sigma^1 = g_{2\#}\sigma^1 = f_{\#}g_{1\#}\sigma^1 = f_{\#}\sigma^1 \ .$$

This implies that

$$\tilde{f}\,(\mathrm{graph}\,(\sigma^1)) = \mathrm{graph}\,(\sigma^1) \ .$$

Similarly, there is a $\tilde{f}$-invariant section

$$\sigma^2 \colon E^2 \longrightarrow E^1 \oplus E^2 = E = TM|\varLambda \ ,$$

satisfying the similar property to (5.2).

Now we are ready to give the precise definition of the "Semi-invariant coordinate for f". "Semi-invariant" means the invariance of x and y-axes by f.

For each $p \in \varLambda$, we define a C^2 diffeomorphism $a_p \colon T_pM \to T_pM$ by

$$a_p(v_1, v_2) = (v_1 - \sigma_p^2(v_2),\ v_2 - \sigma_p^1(v_1))$$

for $v_i \in E_p^i$.

By the properties of σ^i, we can see that a_p satisfy

(5.3)   For all $p \in \varLambda$,

    (i) a_p is a C^2 diffeomorphism

    (ii) $a_p(0) = 0$ and $T_0 a_p = \mathrm{id}$

    (iii) $a_p\,(\mathrm{graph}\,(\sigma_p^1)) = E_p^1 \oplus \{0\}$

           $a_p\,(\mathrm{graph}\,(\sigma_p^2)) = \{0\} \oplus E_p^2 .$

Remark. The properties $L(\sigma_p^1) \leq 1/2$, $L(\sigma_p^2) \leq 1/2$ are used to prove (i).

Let $p \in \varLambda$ and e_p^i be a unit vector in E_p^i. Define an isomorphism $d_p \colon T_pM = E_p^1 \oplus E_p^2 \to \boldsymbol{R}^2$ by

$$d_p(c_1 e_1 + c_2 e_2) = (c_1, c_2) \ , \qquad c_i \in \boldsymbol{R} \ .$$

Remark that this definition of d_p depends on the choice of the basis of $E_p^1 \oplus E_p^2 = T_pM$.

For small $r > 0$ and each $p \in \varLambda$, we define a C^2 local coordinate $\psi_p \colon (U_p(r), p) \to (\boldsymbol{R}^2, 0)$ by

$$\psi_p = d_p \circ a_p \circ \exp_p^{-1} \ ,$$

where $U_p(r)$ denotes the r-neighborhood of p.

ψ_p has the property that,

$$\psi_p(W_p^1(r')) \subset \{x\text{-axis}\}$$
$$\psi_p(W_p^2(r')) \subset \{y\text{-axis}\}$$

for small $r' > 0$ and $p \in \varLambda$.

Put $f_p = \psi_{f(p)} \circ f \circ \psi_p^{-1}$ for $p \in \Lambda$.

From the above argument, we can see that,

(5.4)   There exist $r_1 > 0$ and $r_2 > 0$ with the following properties .

(i) $\psi_p^{-1}: (U(r_1), 0) \to (M, p)$ is a well-defined C^2-diffeomorphism for all $p \in \Lambda$, where $U(r_1)$ is the r_1-neighborhood of the origin in $\mathbf{R}^2$.

(ii) $f_p: (U(r_1), 0) \to (\mathbf{R}^2, 0)$ is well-defined and preserves x and y-axes, for all $p \in \Lambda$.

(iii)   for $p, q \in \Lambda$, if $d(p, q) < r_2$ then $\psi_p \psi_q^{-1}: U(r_1) \to \mathbf{R}^2$ is a well-defined C^2 diffeomorphism.

(iv)   for any $\varepsilon > 0$, there exists $0 < \delta < r_2$ such that if $d(p, q) < \delta$ then $\|\psi_p \psi_q^{-1} - \mathrm{id}\|_2 < \varepsilon$ on $U(r_1)$, where ψ_p and ψ_q are defined with the bases which are near each other, and $\|\cdot\|_2$ denotes the C^2-norm for functions on $\mathbf{R}^2$.

(v)
$$\sup_{\substack{p \in \Lambda, x \in U(r_1) \\ i, j, k = 1, 2}} \left| \frac{\partial^2 f_p^k}{\partial x_i \partial x_j}(x) \right| = M_0 < \infty$$

where $f_p = (f_p^1, f_p^2)$.

(vi)   there is $c_1 > 0$ such that, for any measurable subset A of $U(r_1)$,

$$c_1 s(A) \le s(\psi_p^{-1}(A)) \le c_1^{-1} s(A)$$

for all $p \in \Lambda$, where $s(A)$ denotes the measure (area) of A.

From the definition of ψ_p, we get

(5.5) $\|Tf | E_p^i\| = \left| \dfrac{\partial f_p^i}{\partial x_i}(0, 0) \right|$ for all $p \in \Lambda$, and $i = 1, 2$.

Regard $f_p^1(x, y)$ as a function of x. Applying the Taylor expansion up to 2nd order around $(0, y)$, we have,

$$f_p^1(x, y) = f_p^1(0, y) + x \cdot \frac{\partial f_p^1}{\partial x}(0, y) + \frac{x^2}{2} \cdot \frac{\partial^2 f_p^1}{\partial x^2}(\theta x, y) ,$$

for some $0 \le \theta \le 1$. By the Mean value theorem,

$$\frac{\partial f_p^1}{\partial x}(0, y) = \frac{\partial f_p^1}{\partial x}(0, 0) + y \cdot \frac{\partial^2 f_p^1}{\partial x \partial y}(0, \theta' y)$$

for some $0 \le \theta' \le 1$. Since f_p preserves y-axis, $f_p^1(0, y) = 0$. Thus, we get

$$|f_p^1(x, y)| \le |x| \cdot \left\{ \left| \frac{\partial f_p^1}{\partial x}(0, 0) \right| + M_0(|x| + |y|) \right\} .$$

Similarly,

$$|f_p^2(x, y)| \le |y| \cdot \left\{ \left| \frac{\partial f_p^2}{\partial y}(0, 0) \right| + M_0(|x| + |y|) \right\} .$$

Put

$$m_0 = \inf_{\substack{p \in \Lambda \\ i=1,2}} \| Tf | E_p^i \| \quad \text{and} \quad M_1 = M_0/m_0 .$$

Then, from (5.5), we have

(5.6)
$$|f_p^1(x, y)| \le |x| \cdot \| Tf | E_p^1 \| \cdot \{ 1 + M_1(|x| + |y|) \}$$
$$|f_p^2(x, y)| \le |y| \cdot \| Tf | E_p^1 \| \cdot \{ 1 + M_1(|x| + |y|) \}$$

for all $p \in \Lambda$ and $(x, y) \in U(r_1)$.

§6. Tracing boxes

In this section, we shall define the box neighborhoods for each $p \in \Lambda$ with respect to the semi-invariant coordinate, and see how the iterated images of these boxes behave.

Put

$$B(r', r'') = \{(x, y) \in R^2 : |x| \le r', \ |y| \le r'' \}$$

and

$$B(r) = B(r, r) .$$

For $p \in \Lambda$ and $0 < r, r', r'' < r_1/2$, we define (p, r', r'')-box $B_p(r', r'')$ by

$$B_p(r', r'') = \psi_p^{-1}(B(r', r''))$$

and (p, r)-box $B_p(r)$ by

$$B_p(r) = \psi_p^{-1}(B(r)) .$$

DEFINITION. Let $p \in \Lambda$, $n \in Z_+$ and $\delta > 0$. $B_p(r', r'')$ is (p, n, δ)-tracing box if and only if

$$f^k(B_p(r', r'')) \subset B_{f^k(p)}(\delta) \qquad \text{for all } 0 \le k \le n .$$

Remark. While the definition of ψ_p depends on the choice of the basis of $E_p^1 \oplus E_p^2$, the set $B_p(r', r'')$ does not, because $B(r', r'')$ is symmetric with respect to x and y-axes respectively.

Two near boxes are related in the following way.

(6.1) LEMMA. *There exists $r_3 > 0$ such that, if $p, q \in \Lambda$ and $B_p(r) \cap B_q(r') \ne \phi$ for $0 < r, \ r' < r_3$, then*

(i) $q \in B_p(r + 2r')$

(ii) $B_q(r') \subset B_p(r + 4r')$.

Proof.   Assume that $B_p(r) \cap B_q(r') \neq \phi$, that is;

$$B(r) \cap \psi_p\psi_q^{-1}B(r') \neq \phi .$$

From the above remark, $B_q(r')$ does not depend on the choice of the basis with respect to which ψ_q is defined. So, we can assume that ψ_p and ψ_q are defined by the bases which are near each other. Choose $0 < \delta < r_2$ as in (5.4) (iv) with $\varepsilon = 1/2$. Take $r_3 \leq \delta$. Then for $x \in B(r')$ with $r' < r_3$, we have from (5.4) (iv),

$$\begin{aligned}
|\psi_p\psi_q^{-1}(x) &- \psi_p\psi_q^{-1}(0) - x| \\
&= |(\psi_p\psi_q^{-1} - \mathrm{id})(x) - (\psi_p\psi_q^{-1} - \mathrm{id})(0)| \\
&\leq \{ \sup_{y \in B(r')} \|T_y(\psi_p\psi_q^{-1} - \mathrm{id})\|\} \cdot |x| \\
&\leq |x|/2 \leq \sqrt{2}\,(r'/2) .
\end{aligned}$$

So, for $i = 1, 2$,

$$|(\psi_p\psi_q^{-1}(x))_i - (\psi_p\psi_q^{-1}(0))_i| \leq \sqrt{2}\,(r'/2) + |x_i| < 2r' ,$$

where $x = (x_1, x_2)$ and $(\ )_i$ denotes the i-th coordinate.

Take a point $x_0 \in B(r) \cap \psi_p\psi_q^{-1}B(r')$ and put $x = (\psi_p\psi_q^{-1})^{-1}(x_0) \in B(r')$. Then, since $q = \psi_q^{-1}(0)$ and $\psi_p\psi_q^-(x) \in B(r)$,

$$\begin{aligned}
|(\psi_p(q))_i| &\leq |(\psi_p\psi_q^{-1}(x))_i - (\psi_p(q))_i| + |(\psi_p\psi_q^-(x))_i| \\
&< 2r' + r .
\end{aligned}$$

This implies (i). A similar argument gives (ii).

Let $p \in \Lambda$, $n \in Z_+$ and $N > 0$ (possibly not integer). For the sake of convenience, we put

$$\begin{aligned}
\theta(p, n) &= \|Tf^n | E_p^1\| \\
\omega(p, n) &= \|Tf^n | E_p^2\| \\
\eta(p, N) &= \max_{0 \leq k \leq [N]} \|Tf^k | E_p^1\| .
\end{aligned}$$

Remark that, since $\dim E^1 = \dim E^2 = 1$,

$$\begin{aligned}
\theta(p, n + m) &= \theta(p, n) \cdot \theta(f^n(p), m) \\
\omega(p, n + m) &= \omega(p, n) \cdot \omega(f^n(p), m)
\end{aligned}$$

for any $p \in \Lambda$ and $n, m \in Z_+$.

The iterated images of the box neighborhoods behave as follows.

(6.2) LEMMA. *Let $p \in \Lambda$, $N \geq 1$ (possibly not integer), $K \in Z_+$ and $r_4 = \min\{r_1/2c,\ 1/M_1(1+c)\}$. If $\rho \geq \eta(p, K)$ and $0 < r \leq r_4$, then*

$$f^i(B_p(r/\rho N)) \subset B_{f^i(p)}(r\theta(p, i)/\rho(N-i),\ r\omega(p, i)/\rho(N-i))$$

for all integers $0 \leq i \leq \min\{K, N-1\}$.

Proof. Since $\rho \geq \eta(p, K) \geq \theta(p, i)$ and $N - i \geq 1$ for all integers $0 \leq i \leq \min\{K, N-1\}$, we have $\theta(p, i)/\rho(N-i) \leq 1$. From (2.3) (ii), $\omega(p, i) \leq c\theta(p, i)$. Then since $r \leq r_4$, we have

$$r\theta(p, i)/\rho(N-i) \leq r_1/2c$$
$$r\omega(p, i)/\rho(N-i) \leq r_1/2\ .$$

This implies that $B_{f^i(p)}(r\theta(p, i)/\rho(N-i),\ r\omega(p, i)/\rho(N-i))$ is well-defined for all integers $0 \leq i \leq \min\{K, N-1\}$.

We shall prove the lemma by induction. For $i = 0$, it is trivial. To simplify notations, we put

$$B_j = B_{f^j(p)}(r\theta(p, j)/\rho(N-j),\ r\omega(p, j)/(N-j))$$

for all integers $0 \leq j \leq \min\{K, N-1\}$.

Suppose that the lemma is true for $i - 1$. Then

$$f^{i-1}(B_p(r/\rho N)) \subset B_{i-1}\ .$$

It suffices to show that $f(B_{i-1}) \subset B_i$. Namely we shall see that for $(x, y) = \psi_{f^{i-1}(p)}(z)$ and $z \in B_{i-1}$, we have

$$r\theta(p, i)/\rho(N-i) \geq |f_1(x, y)|$$
$$r\omega(p, i)/\rho(N-i) \geq |f_2(x, y)|$$

where $(f_1, f_2) = \psi_{f^i(p)} \circ f \circ \psi_{f^{i-1}(p)}^{-1}$. In fact, from (5.6),

$$
\begin{aligned}
& r\theta(p, i)/\rho(N-i) - |f_1(x, y)| \\
&\quad \geq \{r\theta(p, i)/\rho(N-i)\} - |x| \cdot \|Tf|E^1_{f^{i-1}(p)}\| \cdot \{1 + M_1(|x| + |y|)\} \\
&\quad \geq \{r\theta(p, i)/\rho(N-i)\} - \{r\theta(p, i)/\rho(N-i+1)\} \\
&\qquad \cdot \{1 + M_1 r(\theta(p, i-1) + \omega(p, i-1))/\rho(N-i+1)\} \\
&\quad = \{r\theta(p, i)/\rho\} \cdot \{1/\rho(N-i+1)^2(N-i)\} \\
&\qquad \cdot \{(N-i+1)(\rho - M_1 r(\theta(p, i-1) + \omega(p, i-1))) \\
&\qquad\qquad\qquad\qquad + M_1 r(\theta(p, i-1) + \omega(p, i-1))\}\ .
\end{aligned}
$$

Thus, we only need to prove $\rho - M_1 r(\theta(p, i-1) + \omega(p, i-1)) \geq 0$. In fact, we get

$$\begin{aligned}
\rho - M_1 r(\theta + \omega) &\geq \rho - M_1 r(\theta + c\theta) \\
&= \rho\{1 - M_1 r\theta(p, i-1)(1 + c)/\rho\} \\
&\geq \rho\{1 - M_1 r(1 + c)\} \geq 0 ,
\end{aligned}$$

because $\omega(p, i-1) \leq c\theta(p, i-1)$, $\rho \geq \theta(p, i-1)$ and $r \leq 1/M_1(1 + c)$.

Similarly $r\omega(p, i)/\rho(N - i) - |f_2(x, y)| \geq 0$. This completes the proof. By this lemma and (2.3) (ii), the next lemma is easily proved.

(6.3) LEMMA. *Under the same hypothesis as in* (6.2),

$$f^i(B_p(r/\rho N)) \subset B_{f^i(p)}(cr\theta(p, i)/\rho(N - i))$$

for all integers $0 \leq i \leq \min\{K, N - 1\}$.

This lemma says that the box $B_p(r/\rho N)$ is a $(p, \min\{K, N - 1\}, cr)$-tracing box. A more useful modification of (6.2) is the following,

(6.4) LEMMA. *Under the same hypothesis as in* (6.2),

$$f^j(B_{f^i(p)}(r\theta(p, i)/\rho(N - i))) \subset B_{f^{i+j}(p)}(cr\theta(p, i + j)/\rho(N - i - j))$$

for all integers i, j *satisfying* $0 \leq i + j \leq \min\{K, N - 1\}$.

Proof.

$$\begin{aligned}
\eta(f^i(p), K - i) &= \max_{0 \leq n \leq K - i} \|Tf^n | E^1_{f^i(p)}\| \\
&= \max_{0 \leq n \leq K - i} \{\|Tf^{n+i} | E^1_p\|/\|Tf^i | E^1_p\|\} \\
&\leq \eta(p, K)/\theta(p, i) \leq \rho/\theta(p, i) .
\end{aligned}$$

From (6.3), we have

$$\begin{aligned}
f^j(B_{f^i(p)}(r\theta(p, i)/\rho(N - i))) \\
= f^j(B_{f^i(p)}(r/(\rho/\theta(p, i))(N - i))) \\
\subset B_{f^{i+j}(p)}(cr\theta(f^i(p), j)/(\rho/\theta(p, i))(N - i - j)) \\
= B_{f^{i+j}(p)}(cr\theta(p, i + j)/\rho(N - i - j)) .
\end{aligned}$$

This implies the lemma.

§7. The fundamental lemma for the closing lemma

In this and the next sections we shall formulate "the closing lemma for $F(M)$" which plays an essential role in the proof of our theorem.

In the general case, "The C^1 closing lemma" has been established by C. Pugh and C. Robinson in [12], and here we just adapt their method to our situation. In our case, $\dim M = 2$ and we are given a Tf-invariant splitting $TM|\Lambda = E^1 \oplus E^2$, so we don't need the "Linear algebra" which complicates the argument of [12]. On the other hand, we will need delicate perturbations of f in the succeeding sections, so we must formulate the closing lemma precisely for our purpose.

In this section, we shall prove the fundamental lemma by which we can select a pair of points that are particularly well situated regarding other points considered. The proof is a fairly easy analogy of (4.2) Fundamental Lemma in [12] in terms of our semi-invariant coordinate.

To simplify computations, we use the following norm on $TM|\Lambda$.

$$|v| = (|v_1|^2 + |v_2|^2)^{1/2}$$

where $v \in T_p M$, $p \in \Lambda$ and $v = v_1 + v_2 \in E_p^1 \oplus E_p^2$. Clearly this norm is equivalent to the norm defined by the Riemannian metric.

Before formulating the Fundamental lemma, we need the following technical lemma.

(7.1) LEMMA. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sup_{p \in \Lambda, x \in V_p(\delta)} \{\|T_x(a_p - \mathrm{id})\|, \|T_x(a_p^{-1} - \mathrm{id})\|\} \leq \varepsilon .$$

The proof is straightforward by using (5.2) (vi), (5.3) (ii) and compactness of Λ.

Let $p \in M$, $v \in T_p M$ and $A: T_p M \to T_p M$ be linear. We define a map $b_p(v, A)$: {a neighborhood of 0 in $T_p M$} $\to M$ by

$$b_p(v, A)(x) = \exp_p(v + Ax) \qquad \text{for } x \in T_p M .$$

Especially, for $p \in \Lambda$ and positive numbers a_1, a_2, we define

$$b_p(v, a_1, a_2) = b_p(v, A(a_1, a_2))$$

where $A(a_1, a_2)(v_1 + v_2) = a_1 v_1 + a_2 v_2$ for $v_i \in E_p^i$.

Let $\delta_0 > 0$ be given in (7.1) with $\varepsilon = 1/8$. Define a number, $r_5 = \min\{r_1/60, \delta_0/60\}$.

(7.2) LEMMA (Fundamental lemma). *Let $\{p_n\}$ be a finite subset of M, $0 < r < r_5$ and $p \in \Lambda$. If there exist two points $p_i, p_j \in \{p_n\}$ satisfying $p_i, p_j \in B_p(r)$, then there exist two points $p_s, p_t \in \{p_n\}$, $v \in T_p M$ and positive numbers a_1, a_2 such that*

(i) $|v| + a_1 + a_2 \leq 30r$

(ii) $(2\sqrt{2})^{-1} \leq a_1/a_2 \leq 2\sqrt{2}$

(iii) $b_p(v, a_1, a_2)(V_p(1)) \subset B_p(30r)$

(iv) $p_s, p_t \in b_p(v, a_1, a_2)(V_p(\sqrt{3/4}))$

(v) $p_k \notin b_p(v, a_1, a_2)(V_p(1))$ for $p_k \in \{p_n\}$ with $p_k \neq p_s, p_t$.

Proof. Suppose that $p_i, p_j \in B_p(r)$. For p_n near p, we put $\tilde{p}_n = \exp_p^{-1} p_n$. Remark that we don't need to consider the points that are far from p. Put $x_0 = \tilde{p}_i$, $y_0 = \tilde{p}_j$ and we shall select a sequence (x_n, y_n) of the points of $\{\tilde{p}_n\}$ as follows. Define

$$\xi(x, y) = \{z \in T_pM \colon |x - z| \leq \sqrt{3/4}\,|x - y| \text{ or } |y - z| \leq \sqrt{3/4}\,|x - y|\}$$

for $x, y \in T_pM$. If some point $z \in \{\tilde{p}_n\}$ is contained in $\xi(x_0, y_0)$, let

$$z \text{ replace } y_0 \qquad \text{if } |z - x_0| \leq |z - y_0|$$
$$z \text{ replace } x_0 \qquad \text{if } |z - y_0| \leq |z - x_0|.$$

Let (x_1, y_1) be the pair so formed. Proceed as with (x_0, y_0), generating a sequence (x_n, y_n). This process ends at finite steps, because

(1) $$|x_n - y_n| \leq (\sqrt{3/4})^n |x_0 - y_0|$$

and $\{\tilde{p}_n\}$ is a finite set. Let $(x_k, y_k) = (\tilde{p}_s, \tilde{p}_t)$ be the final pair. This has the property that no other point of $\{\tilde{p}_n\}$ is contained in $\xi(x_k, y_k)$. Also we can see that

(2) $$|x_k - x_0| \leq \textstyle\sum_{n=1}^k |x_n - x_{n-1}| \leq \{\sum_{n=1}^k (3/4)^{n/2}\} |x_0 - y_0|$$

because either $x_n = x_{n-1}$ or $y_n = y_{n-1}$, and

$$|x_n - x_{n-1}| \leq \sqrt{3/4}\,|x_{n-1} - y_{n-1}|.$$

Now set

$$v = (1/2)(\tilde{p}_s + \tilde{p}_t)$$

and

$$a_i = ((1/3)|\tilde{p}_s - \tilde{p}_t|_i^2 + (1/16)|\tilde{p}_s - \tilde{p}_t|^2)^{1/2} \qquad \text{for } i = 1, 2$$

where $|u|_i = |u_i|$ for $u = u_1 + u_2 \in E_p^1 \oplus E_p^2 = T_pM$.

By easy calculations, we have

$$v + A(a_1, a_2)(V_p(1)) \subset \xi(\tilde{p}_s, \tilde{p}_t)$$
$$\tilde{p}_s, \tilde{p}_t \in v + A(a_1, a_2)(V_p(\sqrt{3/4}))$$

$$(2\sqrt{2})^{-1} \le a_1/a_2 \le 2\sqrt{2} \ .$$

It remains to show (i) and (iii). Since $\psi_p(p_i) \in B(r)$, we have $d_p^{-1}\psi_p(p_i) \in V_p(r)$. Therefore

$$\begin{aligned}
|\tilde{p}_i| = |\exp_p^{-1} p_i| &= |a_p^{-1}d_p^{-1}\psi_p p_i| \\
&\le |(a_p^{-1} - \mathrm{id})(d_p^{-1}\psi_p(p_i))| + |d_p^{-1}\psi_p(p_i)| \\
&\le \sqrt{2}\,r/8 + \sqrt{2}\,r = (9\sqrt{2}/8)r \ .
\end{aligned}$$

Similarly $|\tilde{p}_j| \le (9\sqrt{2}/8)r$. By using (1), (2) and (7.1), we can show that

$$|v| + a_1 + a_2 \le 30r$$

and

$$v + A(a_1, a_2)(V_p(1)) \subset a_p^{-1}d_p^{-1}B(30r) \ .$$

This completes the lemma.

To simplify notations in the succeeding sections, we introduce the following definition.

(7.3) DEFINITION. Let $p \in \Lambda$ and $r > 0$. Let $x, y \in M$, $v \in T_pM$ and a_1, a_2 be positive numbers. A quintuple (x, y, v, a_1, a_2) is (p, r)-connectable if and only if

(i) $|v| + a_1 + a_2 \le r$
(ii) $(2\sqrt{2})^{-1} \le a_1/a_2 \le 2\sqrt{2}$
(iii) $x, y \in b_p(v, a_1, a_2)(V_p(\sqrt{3/4}))$
(iv) $b_p(v, a_1, a_2)(V_p(1)) \subset B_p(r) \ .$

With this definition, we can formulate (7.2) as follows.

(7.4) LEMMA. *Let $\{p_n\}$ be a finite subset of M, $0 < r < r_5$ and $p \in \Lambda$. If there are two points $p_i, p_j \in B_p(r)$, then there exist $v \in T_pM$, positive numbers a_1, a_2 and two points p_s, p_t in $\{p_n\}$ such that (p_s, p_t, v, a_1, a_2) is $(p, 30r)$-connectable and moreover $p_k \ne b_p(v, a_1, a_2)(V_p(1))$ for $p_k \in \{p_n\}$ with $p_k \ne p_s, p_t$.*

§8. Closing lemma for $F(M)$

In this section, we shall give a precise formulation of the "Closing lemma for $F(M)$".

First, by an entire analogy with [12], using $TM|\Lambda = E^1 \oplus E^2$ and (2.3) (ii) instead of $V^1 \oplus V^2$ and hyp $(Tf^n$: resp. $V^1 \oplus V^2)$ in [12], and perturbing f around the points $\{f^n(p_*)\}_{-N \le n \le N-1}$, we get the following;

(8.1) LEMMA. *For any C^1-neighborhood $\mathcal{U}$ of f, there exist $r > 0$ and $N \in Z_+$ with the following property. Let $p_* \in \Lambda$ and $p, q \in M$. If $v \in T_{p_*}M$ and positive numbers a_1, a_2 satisfy*

(I) $|v| + a_1 + a_2 \leq r$

(II) $(2\sqrt{2})^{-1} \leq a_1/a_2 \leq 2\sqrt{2}$

(III) $f^n(b_{p_*}(v, a_1, a_2)(V_{p_*}(1)))$ *are disjoint for* $-N \leq n \leq N$.

(IV) $p, q \in b_{p_*}(v, a_1, a_2)(V_{p_*}(\sqrt{3/4}))$,

then there exists $g \in \mathcal{U}$ such that

(i) $g^{2N}(f^{-N}(q)) = f^N(p)$

(ii) $\sup gf^{-1} \subset \bigcup_{n=-N+1}^{N} f^n(b_{p_*}(v, a_1, a_2)(V_{p_*}(1)))$.

When we apply the closing lemma in the succeeding sections, we require that the perturbation g should preserve the Tf-invariant subbundle E^1, E^2 and the norm of differentials restricted to them (for precise meaning, see (8.2) below). For that purpose, we must extend the splitting $TM|\Lambda = E^1 \oplus E^2$ to a neighborhood of Λ beforehand. Since both M and the 1-dimensional Grassmann bundle over M (whose fiber over $x \in M$ is the Grassmann manifold of all 1-dimensional subspaces of T_xM) are ANR, we can extend E^i continuously to a neighborhood U of Λ (see (4.4) Lemma in [2] for the proof). We still call them E^i. Although E^1 and E^2 are not necessarily Tf-invariant outside Λ, by restricting U to a neighborhood sufficiently near Λ, we can assume that $TM|U = E^1 \oplus E^2$.

The following is the version of the closing lemma which we shall use afterwards.

(8.2) LEMMA. *For any C^1-neighborhood $\mathcal{U}$ of f, there exist $r > 0$ and $N \in Z_+$ with the following property. Let $p_* \in \Lambda$ and $p, q \in M$. If $v \in T_{p_*}M$ and positive numbers a_1, a_2 satisfy*

(I) $|v| + a_1 + a_2 \leq r$

(II) $(2\sqrt{2})^{-1} \leq a_1/a_2 \leq 2\sqrt{2}$

(III) $f^n(b_{p_*}(v, a_1, a_2)(V_{p_*}(1)))$ *are disjoint for* $-N + 1 \leq n \leq N$

(IV) $p, q \in b_{p_*}(v, a_1, a_2)(V_{p_*}(\sqrt{3/4}))$,

then there exists $g \in \mathcal{U}$ such that

(i) $g^{2N}(f^{-N}(q)) = f^N(p)$

(ii) $\sup gf^{-1} \subset \bigcup_{n=-N+1}^{N} f^n(b_{p_*}(v, a_1, a_2)(V_{p_*}(1)))$

(iii) $Tg(E^i_{g^n(f^{-N}(q))}) = E^i_{g^{n+1}(f^{-N}(q))}$

for all $0 \leq n \leq 2N - 1$ *and* $i = 1, 2$

(iv) $$\| Tg \mid E^i_{g^n(f^{-N}(q))} \| = \| Tf \mid E^i_{f^{n-N}(q)} \|$$

for all $0 \leq n \leq N - 1$ *and* $i = 1, 2$

$$\| Tg \mid E^i_{g^n(f^{-N}(q))} \| = \| Tf \mid E^i_{f^{n-N}(p)} \|$$

for all $N \leq n \leq 2N - 1$ *and* $i = 1, 2$.

Proof. The basic idea of the proof is simple. We only have to perturb the differentials of g in (8.1) around the points $\{g^n(f^{-N}(q))\}_{0 \leq n \leq 2N}$ by using (2.9).

Take $\varepsilon' > 0$ so that $\mathcal{U}(\varepsilon', f) \subset \mathcal{U} \cap \mathcal{U}_2$, where $\mathcal{U}_2$ is given in (2.9). Let $r > 0$ and $N \in Z_+$ be given in (8.1) with $\mathcal{U}(\varepsilon'/4, f)$ as a C^1-neighborhood of f. Let $p, q \in M$, $v \in T_{p_*}M$ and $a_1 > 0$, $a_2 > 0$, with the properties (I) $\sim$ (IV). Then from (8.1), there exists $g \in \mathcal{U}(\varepsilon'/4, f)$ satisfying (i) and (ii). Take $r > 0$ so small that for all $-N \leq n \leq N$, $f^n(b_{p_*}(v, a_1, a_2)(V_{p_*}(1)))$ are contained in U on which extended E^1 and E^2 are defined. Define linear maps

$$G^q_j: T_{g^{N+j}(f^{-N}(q))}M \longrightarrow T_{f^j(q)}M \qquad \text{for } -N \leq j \leq 0$$

and

$$G^p_j: T_{g^{N+j}(f^{-N}(q))}M \longrightarrow T_{f^j(p)}M \qquad \text{for } 0 \leq j \leq N$$

so that they preserve E^1, E^2 and are isometry on them. Note that G^q_{-N} and G^p_N are identity maps on $T_{f^{-N}(q)}M$ and $T_{f^N(p)}M$ respectively.

Now we shall apply (2.9) to g with $\{g^n(f^{-N}(q))\}_{0 \leq n \leq 2N-1}$ as a finite set. As linear maps, we take

$$A_j = (G^q_{j+1})^{-1}(Tf)_{f^j(q)}G^q_j(Tg^{-1})_{g^{N+j+1}(f^{-N}(q))} \qquad \text{for } -N \leq j \leq -1$$

and

$$A_j = (G^p_{j+1})^{-1}(Tf)_{f^j(p)}G^p_j(Tg^{-1})_{g^{N+j+1}(f^{-N}(q))} \qquad \text{for } 0 \leq j \leq N - 1 .$$

By taking $r > 0$ smaller, the distance from $f^j(q)$, resp. $f^j(p)$, to $g^{N+j}(f^{-N}(q))$ for $-N \leq j \leq 0$, resp. $0 \leq j \leq N$, can be small enough to satisfy $\|A_j - \mathrm{id}\| < \delta'$ for all $-N \leq j \leq N - 1$, where $\delta' > 0$ is given in (2.9) with $\varepsilon = \varepsilon'/4$. Then from (2.9), we can find g' such that

(i) $d_1(g, g') < \varepsilon'/4$

(ii) $Tg' = A_j \circ Tg$ on $T_{g^{N+j}(f^{-N}(q))}M$ for all $-N \leq j \leq N - 1$.

Moreover, the support of this perturbations can be arbitrarily small. This proves the lemma.

To simplify notations in the succeeding sections, we introduce the following definition.

(8.3) DEFINITION. Let $p_* \in \Lambda$, $N \in Z_+$ and $r > 0$. Let (p, q, v, a_1, a_2) be (p_*, r)-connectable, where $p, q \in M$, $v \in T_{p_*}M$ and $a_1 > 0$, $a_2 > 0$. Assume that $f^n(b_{p_*}(v, a_1, a_2)(V_{p_*}(1)))$ are disjoint for $-N \leq n \leq N$ and contained in a neighborhood U of Λ on which extended E^1 and E^2 are defined. C^1 diffeomorphism g is $(p_*, N, p, q, v, a_1, a_2)$-connector if and only if g satisfies (i) $\sim$ (iv) in (8.2).

With this definition, we can formulate (8.2) as follows.

(8.4) LEMMA. *For any C^1-neighborhood $\mathscr{U}$ of f, there exist $r > 0$ and $N \in Z_+$ with the following property; let $p_* \in \Lambda$, $p, q \in M$, $v \in T_{p_*}M$ and $a_1 > 0$, $a_2 > 0$. If (p, q, v, a_1, a_2) is (p_*, r)-connectable and $f^n(b_{p_*}(v, a_1, a_2)(V_{p_*}(1)))$ are disjoint for $-N + 1 \leq n \leq N$, then there exists a $(p_*, N, p, q, v, a_1, a_2)$-connector g in $\mathscr{U}$.*

Remark that the definition of $(p_*, N, p, q, v, a_1, a_2)$-connector depends on the order of p and q, so generally $(p_*, N, p, q, v, a_1, a_2)$-connector and $(p_*, N, q, p, v, a_1, a_2)$-connector are different. This difference plays a remarkable role in the proof of our main lemma. Roughly speaking, if $q = f^n(p)$ for some positive integer n, $(p_*, N, p, q, v, a_1, a_2)$-connector closes up the orbit $\{f^k(p)\}_{0 \leq k \leq n}$ and $(p_*, N, q, p, v, a_1, a_2)$-connector cuts off it.

§9. Existence of a saddle

In this section, we shall give the following technical lemma which asserts that a sufficiently near recurrence with exponential expansion of the norm of Tf on E^1 and contraction on E^2 implies the existence of a saddle.

(9.1) LEMMA. *There exist $N_1 \in Z_+$, $r_6 > 0$ and $r_7 > 0$ such that if $p \in \Lambda$ and $m \in Z_+$ satisfy*

(i) $m \geq N_1$

(ii) $\|Tf^m | E_p^1\| \geq (\lambda^{-9/10})^m$
      $\|Tf^m | E_p^2\| \leq (\lambda^{9/10})^m$

(iii) $d(p, f^m(p)) \leq r_6$

(iv) *Putting* $(x, y) = \psi_p(f^m(p))$ *and* $r = \max\{|x|, |y|\}$, $B_p(4r)$ *is a (p, m, r_7)-tracing box,*

then, there is a saddle in $B_p(4r)$. Moreover, the period of this saddle is a divisor of m.

This kind of argument is used sometimes in the theory of hyperbolic sets, but in our case, the condition on the norm of differentials of f^k at p are not given for $0 < k < m$. Instead, we have a condition on the size of the tracing box $B_p(4r)$. While the basic idea of the proof is the same as in the case of hyperbolic sets, the precise estimation is rather messy.

Firstly, for a vector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbf{R}^2$ with $v_1 \neq 0$, we define

$$\text{slope}(v) = s(v) = |v_2|/|v_1| \ .$$

For $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in GL(2, R)$ with $\alpha \neq 0$, $\delta \neq 0$, define

$$h(A) = |\delta|/|\alpha| \ .$$

Now we consider for A the following condition,

(1) $|\beta|, |\gamma| \leq \varepsilon$, $a \leq |\alpha|, |\delta|$ for positive constants ε, a .

Then, we have easily,

(9.2) LEMMA. *If A satisfies (1) and $1 \geq \varepsilon a^{-1} s(v)$, then*

$$s(Av) \leq h(A)(1 - \varepsilon a^{-1} s(v))^{-1}(\varepsilon a^{-1} + s(v)) \ .$$

Next, let us consider a sequence $\left\{ A_i = \begin{bmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{bmatrix} \right\}_{i \in Z_+}$ *satisfying (1) and furthermore*

(2) $\prod_{n=i}^{j} h(A_n) \leq c(\lambda^{9/10})^{j-i+1}$ *for all $i, j \in Z_+$ with $i \leq j$.*

We shall estimate the slope of $A_n \cdots A_1 v$. For the sake of simplicity, we put

$$\rho = (1 - \lambda^{8/10})^{-1} \ , \quad \mu = \lambda^{-1/10} \ , \quad \tau = \varepsilon a^{-1} \ .$$

(9.3) LEMMA. *Assume that $s(v) \leq 8\rho c\tau$ and let $\varepsilon > 0$ be so small that $(1 - c\rho(1 + 8c)\tau^2)^{-1} \leq \mu$. Then we have*

(3) $s(A_n \cdots A_1 v) \leq \{ \sum_{i=1}^{n} (\prod_{j=i}^{n} \mu h(A_j)) + 8\rho c \prod_{j=1}^{n} \mu h(A_j) \} \tau$
    for all $n \in Z_+$.

Proof. We shall prove the lemma by induction.
For $n = 1$, by using $s(v) \leq 8\rho c\tau$ and (9.2), we get

$$s(A_1v) \le h(A_1)(1 - \tau 8\rho c\tau)^{-1}(\tau + 8\rho c\tau)$$
$$\le h(A_1)\mu(1 + 8\rho c)\tau ,$$

because, $(1 - \tau^2 8\rho c)^{-1} \le (1 - c\rho(1 + 8c)\tau^2)^{-1} \le \mu.$

Suppose the lemma is true for $n - 1$. Then from (2), we have

$$s(A_{n-1} \cdots A_1v) \le \{\sum_{i=1}^{n-1}(\prod_{j=i}^{n-1}\mu h(A_j)) + 8\rho c \prod_{j=1}^{n-1}\mu h(A_j)\}\tau$$
$$\le \{\sum_{i=1}^{n-1}\mu^{n-i}c(\lambda^{9/10})^{n-i} + 8\rho c\mu^{n-1}c(\lambda^{9/10})^{n-1}\}\tau$$
$$\le \{\sum_{i=1}^{\infty}(\lambda^{8/10})^i + 8\rho c(\lambda^{8/10})^{n-1}\}c\tau$$
$$\le \{1 + 8c(\lambda^{8/10})^{n-1}\}\rho c\tau .$$

Namely,

$$(4) \qquad s(A_{n-1} \cdots A_1v) \le \{1 + 8c(\lambda^{8/10})^{n-1}\}\rho c\tau .$$

This implies that

$$(1 - \tau s(A_{n-1} \cdots A_1v))^{-1} \le (1 - \tau^2(1 + 8c)\rho c)^{-1} \le \mu ,$$

and

$$\tau s(A_{n-1} \cdots A_1v) \le \tau^2(1 + 8c)\rho c$$
$$< 1 - \mu^{-1} < 1 .$$

so, by (9.2), we get

$$s(A_n \cdots A_1v) \le h(A_n)\mu\tau\{1 + \sum_{i=1}^{n-1}(\prod_{j=i}^{n-1}\mu h(A_j)) + 8\rho c \prod_{j=1}^{n-1}\mu h(A_j)\}$$

which implies (3) for n.

By the same calculation proving the above (4), we have

(9.4) LEMMA. *Under the same hypothesis as* (9.3),

$$s(A_n \cdots A_1v) \le \{1 + 8c(\lambda^{8/10})^n\}\rho c\tau \qquad \text{*for all* } n \in Z_+ .$$

Proof of (9.1). *For* $q \in \Lambda$ *and* $x \in B(r_7)$, *we put*

$$T_xf_q = \begin{bmatrix} \alpha_q & \beta_q \\ \gamma_q & \delta_q \end{bmatrix} \quad \text{and} \quad T_0f_q = \begin{bmatrix} \alpha_q^0 & 0 \\ 0 & \delta_q^0 \end{bmatrix}$$

where $f_q = \psi_{f(q)} \circ f \circ \psi_q^{-1}$.

By taking r_7 small, we can assume that

(5) (i) there is a constant $a > 0$ such that

$$a \le |\alpha_q|, |\delta_q| \qquad \text{for any } q \in \Lambda$$

(ii) there is $\varepsilon > 0$ such that $|\beta_q|, |\gamma_q| \le \varepsilon$ for any $q \in \Lambda$ and

$$(1 - c\rho(1 + 8c)(\varepsilon a^{-1})^2)^{-1} \le \mu$$

(iii) $\lambda^{1/20} \leq |\alpha_q|/|\alpha_q^0| \leq \lambda^{-1/20}$

$\lambda^{1/20} \leq |\delta_q|/|\delta_q^0| \leq \lambda^{-1/20}$ for any $q \in \Lambda$.

We put $\tau = \varepsilon a^{-1}$ as before, and $\psi_p(f^m(p)) = (x, y)$, $r = \max\{|x|, |y|\}$ as in the assumption of the lemma.

Put $q = f^m(p)$ and let z be an arbitraly point in $B(4r)$. Take $v \in T_z\mathbf{R}^2$ such that $s(v) \leq 8\rho c\tau$. Let us consider $T_z(\psi_p \circ f^m \circ \psi_p^{-1})(v)$. To simplify notation, put $f_p^n = \psi_{f^n(p)} \circ f^n \circ \psi_p^{-1}$, $z_n = f_p^n(z)$ and $p_n = f^n(p)$ for $0 \leq n \leq m$.

Noting that $T_z(\psi_p \circ f^m \circ \psi_p^{-1}) = T_{z_m}(\psi_p\psi_q^{-1}) \circ T_zf_p^m$, first, we estimate the slope and the norm of $T_zf_p^m(v)$.

From (5) (i) $\sim$ (iii), (9.1) (iv) and (2.3) (ii), a sequence $\{T_{z_n}f_{p_n}\}_{n=0,\cdots,m-1}$ satisfies the hypothesis of (9.3). Therefore, if we take m large enough so that $8c(\lambda^{8/10})^m \leq 1$, then by (9.4), $s(T_zf_p^m(v)) \leq 2\rho c\tau$. If $d(p, f^m(p))$ is sufficiently small (i.e. we take r_6 small enough), then by (5.4) (iv), we get $s(T_z(\psi_p \circ f^m \circ \psi_p^{-1})(v)) \leq 4\rho c\tau$.

Thus, $T_z(\psi_p \circ f^m \circ \psi_p^{-1})$ preserves the sector

$$S_{4\rho c\tau} = \{v \in T_x\mathbf{R}^2 : s(v) \leq 4\rho c\tau, \ x \in B(r_7)\}$$

for $z \in B(4r)$.

Next, we investigate the norm of $T_z(\psi_p \circ f^m \circ \psi_p^{-1})(v)$. Assume that $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $|v_1| = 1$. Generally, if a matrix $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ satisfy the assumption of (9.2), then

(6)
$$\begin{aligned} |\pi_1 Av| = |\alpha v_1 + \beta v_2| &\geq |\alpha||v_1| - |\beta||v_2| \\ &\geq |\alpha||v_1|(1 - \varepsilon a^{-1}s(v)) , \end{aligned}$$

where π_1 denotes the canonical projection to the first coordinate. From this (6), (9.4), (5) (iii) and (9.1) (ii), we have

$$\begin{aligned} |\pi_1 T_zf_p^m(v)| &\geq \left|\frac{\partial f_{p_{m-1}}^1}{\partial x_1}(z_{m-1})\right| \cdot |\pi_1 T_zf_p^{m-1}(v)|(1 - \tau s(T_zf_p^{m-1}(v))) \\ &\geq \left|\frac{\partial f_{p_{m-1}}^1}{\partial x_1}(z_{m-1})\right| \cdot |\pi_1 T_zf_p^{m-1}(v)|(1 - \tau^2(1 + 8c)\rho c) \\ &\geq \left|\frac{\partial f_{p_{m-1}}^1}{\partial x_1}(z_{m-1})\right| \cdot |\pi_1 T_zf_p^{m-1}(v)| \cdot \mu^{-1} \\ &\cdots \\ &\geq \left\{\prod_{i=0}^{m-1}\left|\frac{\partial f_{p_i}^1}{\partial x_1}(z_i)\right|\right\} \cdot \lambda^{m/10} \\ &\geq (\lambda^{-8/10})^m \lambda^{m/10} = (\lambda^{-7/10})^m . \end{aligned}$$

Namely, if we take m large enough to satisfy $(\lambda^{-7/10})^m \geq 5$, then we have $|\pi_1 T_z f_p^m(v)| \geq 5$. If we take $d(p, f^m(p))$ sufficiently small (i.e. by lessening r_6), using (5.4) (iv), we have $|\pi_1 T_z(\psi_p \circ f^m \circ \psi_p^{-1})(v)| \geq 4$.

Applying the same argument to f^{-1}, we have that;

(i) for $z \in \psi_p f^m(B_p(4r))$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in T_z \mathbf{R}^2$ with $|v_2| = 1$ and $\tilde{s}(v) = |v_1|/|v_2| \leq 8\rho c\tau$, $\psi_p \circ f^{-m} \circ \psi_p^{-1}$ preserves the sector

$$\tilde{S}_{4\rho c\tau} = \{v \in T_x \mathbf{R}^2 : \tilde{s}(v) \leq 4\rho c\tau, \ x \in B(r_7)\}$$

on $\psi_p f^m(B_p(4r))$.

(ii) $|\pi_2 T_z(\psi_p \circ f^{-m} \circ \psi_p^{-1})(v)| \geq 4$, where π_2 denotes the canonical projection to the second coordinate.

Thus, we know that $\psi_p \circ f^m \circ \psi_p^{-1}$ expands $B(4r)$ with respect to the first coordinate and contracts with respect to the second coordinate. Then, by using the above properties, it can be seen that $\bigcap_{n \in \mathbf{Z}} f^{mn}(B_p(4r))$ consists of only one point which must be a saddle.

§10. Disjointness of tracing boxes

The purpose of this section is to prove the following (10.2) Lemma which gives a precise estimate about the length of the positive orbit on which the iterated box neighborhoods are pairwise disjoint. This (10.2) plays two important roles in the arguments in Sections 12 and 15. One is to guarantee the disjointness of the supports of perturbations when we apply the closing lemma, and another is to guarantee that the length of suborbits are not smaller than a given number. Refer to (15.1) and (15.2) for details.

We need the following constants to formulate (10.2).

(10.1) CONSTANTS. $N_1 \in \mathbf{Z}_+$ is given in (9.1). C^1 neighborhood $\mathscr{U}_1$ of f is given in (2.3) (i). Let C^1 neighborhood $\mathscr{U}_3$ of f be given in (3.1) with $\varepsilon = (1/4)d(\varLambda_1, \varLambda_0 \cup \varLambda_2)$. Let $r_0 > 0$ and $N_0 \in \mathbf{Z}_+$ be given in (8.2) with $\mathscr{U} = \mathscr{U}(\varepsilon_0, f)$, where $\varepsilon_0 > 0$ is chosen so that $\mathscr{U}(\varepsilon_0, f) \subset \mathscr{U}_1 \cap \mathscr{U}_3$. Moreover, we assume that N_0 is so large that $c\lambda^{N_0} \leq 1$ and r_0 is small enough to satisfy $B_p(r_0) \subset U_p((1/8)d(\varLambda_1, \varLambda_0 \cup \varLambda_2))$ for any $p \in \varLambda$, where $U_p(r)$ denotes the r-neighborhood of p. Define $N_2 = \max\{2N_1, 4N_0\}$ and $G = 12\,(\log m_0^{-1})/(\log \lambda^{-1})$.

(10.2) LEMMA. There exist $r_8 > 0$ and $N_3 \in \mathbf{Z}_+$ with the following property. Let $p \in \varLambda$ and $0 < r \leq r_8$. If integer $N \geq N_3$ satisfy $\log \eta(p, GN) \leq (GN/10)\,(\log \lambda^{-1})$, then $B_i(r) \cap B_j(r) = \phi$ for any integers $0 \leq i < j \leq 2N$,

where $B_i(r) = B_{f^i(p)}(r\theta(p, i)/\eta(p, GN)(GN - i))$.

Main tools for the proof of (10.2) are (8.2) and (9.1). To apply them, we need that length of the suborbit of the positive orbit of $p \in \Lambda$ is not so small. So we must manage the case of short suborbits (i.e. $j - i$ is small) before proving (10.2), namely;

(10.3) LEMMA. *There exist $r_8' > 0$ and $N_3' \in Z_+$ with the following property. Let $p \in \Lambda$ and $0 < r \le r_8'$. If integer $N \ge N_3'$ satisfy $\log \eta(p, GN) \le (GN/10)(\log \lambda^{-1})$, then $B_i(r) \cap B_j(r) = \phi$ for any integers $0 \le i < j \le 3N$ with $j - i \le N_2$.*

(10.4) *Remark.* Since $G \ge 12$ and $\theta(p, i)/\eta(p, GN) \le 1$ for $i \le 12N$, we have $\theta(p, i)/\eta(p, GN)(GN - i) \le 1$ for $i \le 11N$. Therefore $B_i(r)$ is well-defined if $i \le 11N$ and $r \le r_1/2$ (ref. (5.4)).

For the proof of (10.3), we need the following;

(10.5) LEMMA. *Let $n_* \in Z_+$ be given. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for $x \in M$ and a positive integer $k \le n_*$ with $d(x, f^k(x)) < \delta$, there is a periodic point of f in the ε-neighborhood of x. Moreover, the period of this periodic point is a divisor of k.*

The proof is straightforward and left to the reader.

Proof of (10.3). Suppose that $B_i(r) \cap B_j(r) \ne \phi$ for some integers $0 \le i < j \le 3N$ with $j - i \le N_2$. From (10.4), $\theta(p, i)/\eta(p, GN)(GN - i) \le 1$ for $i \le 11N$. So, by taking $r \le r_3$, we can apply (6.1) and have;

(i) if $\theta(p, i)/(GN - i) \ge \theta(p, j)/(GN - j)$, then $f^j(p) \in B_i(3r)$

(ii) if $\theta(p, i)/(GN - i) \le \theta(p, j)/(GN - j)$, then $f^i(p) \in B_j(3r)$.

Namely, we have

(1)   there exist $q \in \Lambda$ and integers $0 \le t \le 3N$, $0 < m \le N_2$ such that $q, f^m(q) \in B_t(3r)$.

In fact, take $m = j - i$, $q = f^i(p)$ and $t = i$ in case of (i), $t = j$ in case of (ii) accordingly.

Now we claim that;

(2)   Let $K \ge 3$. If $Kr \le \min \{r_3, r_4\}$, then

$$f^m(B_t(Kr)) \subset B_t(10cm_1^m Kr) \qquad \text{where } m_1 = \sup_{p \in \Lambda} \{\| Tf \,|\, E_p^1 \|, \; \| Tf \,|\, E_p^2 \|\} \, .$$

In fact, from (6.4), we have $f^m(B_t(Kr)) \subset B_{t+m}(cKr)$, and from (1), $f^m(q) \in B_t(Kr) \cap B_{t+m}(cKr)$. Putting $\theta_i = \theta(p, i)$ and $\eta = \eta(p, GN)$, we get from (6.1) that;

$$B_{t+m}(cKr) \subset B_{f^t(p)}(Kr\theta_t/\eta(GN - t) + 4cKr\theta_{t+m}/\eta(GN - t - m)) \ .$$

By taking N with $9N \geq 2N_2$, we have $1/(GN - t - m) \leq 2/(GN - t)$ and

$$B_{f^t(p)}(Kr\theta_t/\eta(GN - t) + 4cKr\theta_{t+m}/\eta(GN - t - m))$$
$$\subset B_{f^t(p)}(Kr\theta_t(1 + 8cm_1^m)/\eta(GN - t)) \ ,$$

because $\theta_{t+m} = \theta_t \cdot \theta(f^t(p), m)$, $m \leq N_2$ and $\theta(f^t(p), m) \leq m_1^m$. Since $c \geq 1$ and $m_1 \geq 1$, this proves (2).

From (1) and (2), we have that;

$$q, f^m(q), f^{2m}(q) \in B_t(10cm_1^m 3r) \ .$$

Taking r so that $3r(10cm_1^{N_2})^{N_1 - 1} \leq \min\{r_3, r_4\}$ and applying (2) $N_1 - 1$ times, we get

$$(3) \qquad\qquad q, f^m(q), \cdots, f^{N_1 m}(q) \in B_t((10cm_1^m)^{N_1 - 1} 3r) \ .$$

Choose $\varepsilon_1 > 0$ so that, for $x, y \in \Lambda$ with $d(x, y) < \varepsilon_1$ we have $\lambda^{1/10} \leq \|Tf \mid E_x^i\|/\|Tf \mid E_y^i\| \leq \lambda^{-1/10}$ for $i = 1, 2$, and $\varepsilon_2 > 0$ so that, for $x, y \in M$ with $d(x, y) < \varepsilon_2$, $d(f^n(x), f^n(y)) < \varepsilon_1$ holds for all $0 \leq n \leq N_2 N_1$.

Now let δ_1 be a positive number given in (10.5) with $\varepsilon = \varepsilon_2$ and $n_* = N_2 N_1$.

Next, take r so small that $B_t(K_0 r)$ is contained in $\delta_1/2$-neighborhood of $f^t(p)$, where we put $K_0 = 3(10cm_1^m)^{N_1 - 1}$. Then, since $q, f^{m N_1}(q) \in B_t(K_0 r)$ and $N_1 m \leq N_2 N_1$, there exists a periodic point z whose period is a divisor of $N_1 m$ in ε_2-neighborhood of q by (10.5). From the definition of ε_2,

$$(4) \qquad\qquad d(f^n(q), f^n(z)) < \varepsilon_1 \qquad \text{for all } 0 \leq n \leq N_1 m \ .$$

By taking ε_2 smaller than $(1/4)d(\Lambda_1, \Lambda_0 \cup \Lambda_2)$, we know that z is a saddle of f. Therefore, from (2.4),

$$(5) \qquad\qquad \begin{aligned} \|Tf^{\pi(z)} \mid E_z^1\| &\geq \lambda^{-\pi(z)} \\ \|Tf^{\pi(z)} \mid E_z^2\| &\leq \lambda^{\pi(z)} \end{aligned}$$

where $\pi(z)$ denotes the period.

Since $N_1 m$ is a multiple of $\pi(z)$, from (4), (5) and the definition of ε_1, we have

$$(6) \qquad \begin{aligned} \| Tf^{N_1 m} \,|\, E_q^1 \| &\geq (\lambda^{-9/10})^{N_1 m} \\ \| Tf^{N_1 m} \,|\, E_q^2 \| &\leq (\lambda^{9/10})^{N_1 m} . \end{aligned}$$

From (3), $q, f^{N_1 m}(q) \in B_t(K_0 r)$. Applying (6.1), we have $B_t(K_0 r) \subset B_q(4K_0 r \theta_t / \eta(GN - t))$. Therefore,

$$(7) \qquad f^{N_1 m}(q) \in B_q(4K_0 r \theta_t / \eta(GN - t)) .$$

Since $f^t(p) \in B_q(4K_0 r \theta_t / \eta(GN - t))$, from (6.1), we have

$$(8) \qquad B_q(16K_0 r \theta_t / \eta(GN - t)) \subset B_t(64K_0 r) .$$

Take r so small that $64K_0 r \leq r_4$. Then from (6.4),

$$(9) \qquad B_t(64K_0 r) \text{ is a } (f^t(p), GN - t - 1, 64cK_0 r)\text{-tracing box} .$$

Since $G \geq 12$, $t \leq 3N$ and $m \leq N_2$, taking N large, we can assume that $N_1 m \leq GN - t - 1$. So, from (6), (7), (8), q and m satisfy the hypothesis of (9.1). Applying (9.1), we know that there exists a saddle z' in $B_q(16K_0 r \theta_t / \eta(GN - t))$ and the period of z' is a divisor of $N_1 m$. So, from (8), we get;

(10)   there exists a saddle z' whose period is a divisor of $N_1 m$ in $B_t(64K_0 r)$.

One can easily see that if $N \geq N_1 m$ then;

(11)   there exists $k' \in Z_+$ such that
$$5N (\log m_0^{-1})/(\log \lambda^{-1}) \leq k' N_1 m \leq GN - t - 1 .$$

Take r so small that $64cK_0 r$-boxes are contained in ε_1-neighborhood for all points in Λ. Then from (9), (10), (11), we have that $d(f^n(z'), f^{t+n}(p)) < \varepsilon_1$ for all $0 \leq n \leq k' N_1 m$. From (2.4) and the definition of ε_1,

$$(12) \qquad \| Tf^{k' N_1 m} \,|\, E_{f^t(p)}^1 \| \geq (\lambda^{-9/10})^{k' N_1 m} .$$

Now it remains only easy calculations to finish the proof. Using $\dim E^1 = 1$ and (11), (12), we have,

$$\begin{aligned} \log \| Tf^{t + k' N_1 m} \,|\, E_p^1 \| &= \log \| Tf^t \,|\, E_p^1 \| + \log \| Tf^{k' N_1 m} \,|\, E_{f^t(p)}^1 \| \\ &\geq -t (\log m_0^{-1}) + (9/10) k' N_1 m (\log \lambda^{-1}) \\ &\geq (9N/2) (\log m_0^{-1}) - 3N (\log m_0^{-1}) \\ &= (3N/2) (\log m_0^{-1}) . \end{aligned}$$

From (11), $t + k' N_1 m \leq GN - 1$. Therefore,

$$\log \| Tf^{t+k'N_1 m} | E_p^1 \| \le \log \eta(p, GN) \ .$$

On the other hand, from the hypothesis of the lemma,

$$\log \eta(p, GN) \le (GN/10) (\log \lambda^{-1})$$
$$= (6N/5) (\log m_0^{-1}) \ ,$$

which is a contradiction. This completes the proof of (10.3).

Now we shall prove (10.2).

Assume that $B_i(r) \cap B_j(r) \ne \phi$ for $0 \le i < j \le 2N$. By the same argument as in the proof of (10.3), we have

(13) there exist $q \in \Lambda$ and integers $0 \le t \le 2N$, $0 \le m \le 2N$ such that $q, f^m(q) \in B_t(3r)$.

Take r so that $3r \le r_5$. Applying (7.4) with $\{f^n(q)\}_{0 \le n \le m}$ as a finite subset of M, we have from (13) that there exist $v \in T_{f^t(p)} M$, positive numbers a_1, a_2 and two points $f^{k_1}(q), f^{k_2}(q)$ with $0 \le k_1 < k_2 \le m$, such that $(f^{k_1}(q), f^{k_2}(q), v, a_1, a_2)$ is $(f^t(p), 90r\theta_t/\eta(GN - t))$-connectable and moreover,

(14) $f^n(q) \notin b_{f^t(p)}(v, a_1, a_2)(V_{f^t(p)}(1))$ for $0 \le n \le m$ with $n \ne k_1, k_2$.

We claim that;

(15) $\{f^n(B_t(90r))\}_{-N_0 \le n \le N_0}$ are disjoint .

In fact, suppose that $f^{n_1}(B_t(90r)) \cap f^{n_2}(B_t(90r)) \ne \phi$ for some $-N_0 \le n_1 < n_2 \le N_0$, namely,

$$B_t(90r) \cap f^{n_2-n_1}(B_t(90r)) \ne \phi \quad \text{and} \quad 0 \le n_2 - n_1 \le 2N_0 \ .$$

From (6.4), we have that

$$f^{n_2-n_1}(B_t(90r)) \subset B_{t+n_2-n_1}(90rc) \ .$$

Since $c \ge 1$,

(16) $B_t(90cr) \cap B_{t+n_2-n_1}(90cr) \ne \phi$.

If we take $N \ge 2N_0$, then $t + n_2 - n_1 \le 2N + 2N_0 \le 3N$. By taking r with $90cr \le r_8'$, (16) contradicts (10.3) and this proves (15).

With (15), we can apply the closing lemma to $(f^{k_1}(q), f^{k_2}(q), v, a_1, a_2)$. Take r so that $90r \le r_0$. From (8.4) and (10.1), we have that there exists a $(f^t(p), N_0, f^{k_1}(q), f^{k_2}(q), v, a_1, a_2)$-connector g in $\mathcal{U}(\varepsilon_0, f)$.

From (14), $f^{k_2-N_0}(q)$ is a periodic point of g whose period is $k_2 - k_1$. Put $q_0 = f^{k_2-N_0}(q)$. From the definition of r_0 (ref. (10.1)), q_0 is a saddle of

g. Since g preserves E^1 and E^2, by using (2.3) (ii), we see that $E_{q_0}^1 = E_{q_0}^u(g)$ and $E_{q_0}^2 = E_{q_0}^s(g)$. Therefore, from (2.4),

$$\| Tg^{k_2-k_1} \,|\, E_{q_0}^1 \| \geq (\lambda^{-1})^{k_2-k_1}$$
$$\| Tg^{k_2-k_1} \,|\, E_{q_0}^2 \| \leq \lambda^{k_2-k_1} \,.$$

So, from the properties of g (ref. (8.3) and (8.2) (iv)), we get;

$$(17) \qquad \begin{aligned} &\| Tf^{k_2-k_1} \,|\, E_{f^{k_1(q)}}^1 \| \geq (\lambda^{-1})^{k_2-k_1} \\ &\| Tf^{k_2-k_1} \,|\, E_{f^{k_1(q)}}^1 \| \leq \lambda^{k_2-k_1} \,. \end{aligned}$$

Recall that $f^{k_1}(q), f^{k_2}(q) \in B_t(90r)$. From (6.1), we have

$$(18) \qquad f^{k_2}(q) \in B_t(90r) \subset B_{f^{k_1(q)}}(360r\theta_t/\eta(GN-t)) \,,$$

$$(19) \qquad B_{f^{k_1(q)}}(16 \cdot 90r\theta_t/\eta(GN-t)) \subset B_t(64 \cdot 90r) \,.$$

By taking r with $64 \cdot 90r \leq r_4$, we have from (6.4) that

$$(20) \qquad B_t(64 \cdot 90r) \text{ is a } (f^t(p), GN-t-1, 64 \cdot 90cr)\text{-tracing box} \,.$$

By the same argument as in the proof of (15), it can be seen that $k_2 - k_1 \geq 2N_1$. Since $k_2 - k_1 \leq 2N$ and $GN - t \geq 10N$, $GN - t - 1 > 9N > k_2 - k_1$. Thus, from (17), (18), (19) and (20), taking r with $64 \cdot 90cr \leq r_7$, we can apply (9.1) and have a periodic point z in $B_{f^{k_1(q)}}(16 \cdot 90r\theta_t/\eta(GN-t))$ whose period is a divisor of $k_2 - k_1$. Therefore from (19), we get;

(21) there exists a saddle z in $B_t(64 \cdot 90r)$ whose period is a divisor of $k_2 - k_1$.

The rest of the proof is the same as that of (10.3).

§11. Main lemma and Key lemma

In this section, we state (11.2) Main lemma and by using it, prove (11.1) Key lemma which is literally the key to the proof of our theorem. The proof of (11.2) will be given in Section 15.

(11.1) KEY LEMMA. *There exists a constant $A_0 > 0$ such that if for $m \in \mathbf{Z}_+$ and $\ell > 0$,*

$$\log \| Tf^m \,|\, E_p^1 \| \leq -\ell$$

holds at some $p \in \Lambda$, then $m \geq A_0 \cdot \ell^4$.

For the statement of our Main lemma, we put

$$G_1 = (\log \lambda^{-1})/8 \, (\log m_0^{-1}) \, ,$$

and

$$\nu_0(N) = [G_1 N / 4N_0] \qquad \text{for an integer } N \, .$$

In what follows, we use a terminology "Z_+-interval" which means a interval in Z_+, and denote it by $I = [u, v]$ for $u, v \in Z_+$ with $u \le v$, namely,

$$I = [u, v] = \{n \in Z_+ : u \le n \le v\} \, .$$

We define length $(I) = v - u$.

(11.2) MAIN LEMMA. *There exists an integer $N_* \ge 8N_0/G_1$ with the following property: Let $p \in \Lambda$. If for an integer $N \ge N_*$ and $m \in Z_+$,*

$$\log \| Tf^m \, | \, E_p^1 \| \le (N^{10/9}/2) \, (\log \lambda)$$

holds, then there exist integers $0 \le m' < m$, $0 < \nu \le \nu_0(N)$, and ν disjoint Z_+-intervals $\{I_i = [u_i, v_i]\}_{1 \le i \le \nu}$ in $[0, m - m']$ with the following properties;

- (i) $\| Tf^n \, | \, E_{f^{m'}(p)}^1 \| \le 1$ *for all* $0 \le n \le m - m'$,
- (ii) $\sum_{i=1}^{\nu} (v_i - u_i) > \nu_0(N) \cdot N$,
- (iii) $\| Tf^{v_i - u_i} \, | \, E_{f^{u_i + m'}(p)}^1 \| \ge (\lambda^{-1})^{v_i - u_i}$ *for all* $i = 1, \cdots, \nu$.

(11.3) *Remark.* Since $G_1 \le 1/8$, $N_* \ge 64N_0 \ge 64$.

Now we prove (11.1) assuming (11.2). For that purpose, we consider the following proposition depending on two numbers $\theta \ge 1$ and $A > 0$.

PROP. $[\theta, A]$. *If for $m \in Z_+$ and $\ell > 0$,*

$$\log \| Tf^m \, | \, E_p^1 \| \le -\ell \qquad \text{holds at some } p \in \Lambda \, ,$$

then $m \ge A \cdot \ell^\theta$.

If Prop. $[\theta, A]$ is true for some $\theta \ge 4$ and for some constant $A > 0$, then clearly we get (11.1). In what follows, we shall investigate for what values of θ and A, Prop. $[\theta, A]$ holds.

As the first step to see this, we give;

(11.4) LEMMA. *Prop. $[1, (\log m_0^{-1})^{-1}]$ holds.*

Recalling that $m_0 = \inf_{p \in \Lambda} \{\| Tf \, | \, E_p^1 \|, \, \| Tf \, | \, E_p^2 \|\}$, this lemma is obvious.

Now our strategy is to increase the value of θ by applying (11.2) inductively, namely;

(11.5) LEMMA. *Assume that Prop. $[\theta, A]$ is true and for $m \in Z_+$ and $\ell > 0$, we have;*

(i) $\ell \geq (N_*^{10/9}/2)(\log \lambda^{-1})$

(ii) $\log \| Tf^m | E_p^1 \| \leq -\ell$ *for some* $p \in \Lambda$, *then* $m \geq \tilde{k}(\theta) \cdot A \cdot \ell^{9(1+\theta)/10}$,

where $k(\theta) = 2^{-(\theta+11)/10} \cdot (\log \lambda^{-1})^{(\theta-9)/10} \cdot G_1/N_0$.

Proof.   We put

(1) $$N = [(2\ell/(\log \lambda^{-1}))^{9/10}] + 1 .$$

Then, from the assumption on m, ℓ and N_*, we have

(2) $$N \geq N_* \geq 8N_0/G_1 \quad \text{and} \quad \log \| Tf^m | E_p^1 \| \leq (N^{10/9}/2)(\log \lambda) .$$

Therefore we can apply (11.2) and have that; there are integers $0 \leq m'$ $< m_0$, $0 < \nu \leq \nu_0(N)$ and ν disjoint Z_+-intervals $\{I_i\}_{1 \leq i \leq \nu}$ in $[0, m - m']$ with the properties (i) $\sim$ (iii) in (11.2). We put $I_i = [u_i, v_i]$. Without loss of generality, we can assume that $I_1 < I_2 < \cdots < I_\nu$ (i.e. $v_{i-1} < u_i$ for $2 \leq i \leq \nu$).

Define disjoint Z_+-intervals $\{\tilde{J}_i\}_{1 \leq i \leq \nu}$ in $[0, u_\nu]$ as follows.

$$\tilde{J}_1 = [0, u_1] \quad \text{and} \quad \tilde{J}_i = [v_{i-1}, u_i] \qquad \text{for } 2 \leq i \leq \nu .$$

We put $q = f^{m'}(p)$ and

(3) $$\begin{aligned} -\tilde{a}_1 &= \log \| Tf^{u_1} | E_q^1 \| \\ -\tilde{a}_i &= \log \| Tf^{u_i - v_{i-1}} | E_{f^{v_{i-1}}(q)}^1 \| \qquad \text{for } 2 \leq i \leq \nu . \end{aligned}$$

From (11.2) (iii), (3) and $\dim E^1 = 1$, we have

$$\begin{aligned} \log \| Tf^{v_\nu} | E_q^1 \| &= \log \| Tf^{u_1} | E_q^1 \| + \log \| Tf^{v_1 - u_1} | E_{f^{u_1}(q)}^1 \| \\ &\quad + \log \| Tf^{u_2 - v_1} | E_{f^{v_1}(q)}^1 \| + \cdots + \log \| Tf^{v_\nu - u_\nu} | E_{f^{u_\nu}(q)}^1 \| \\ &\geq -\tilde{a}_1 + \log (\lambda^{-1})^{v_1 - u_1} + (-\tilde{a}_2) + \cdots + (-\tilde{a}_\nu) + \log (\lambda^{-1})^{v_\nu - u_\nu} . \end{aligned}$$

On the other hand, from (11.2) (i),

$$\log \| Tf^{v_\nu} | E_q^1 \| \leq 0 .$$

So, we have

(4) $$\left(-\sum_{i=1}^{\nu} \tilde{a}_i \right) + (\log \lambda^{-1}) \sum_{i=1}^{\nu} (v_i - u_i) \leq 0 .$$

We select intervals in $\{\tilde{J}_i\}_{1 \leq i \leq \nu}$ with the property that $\tilde{a}_i > 0$ and denote them by $J_1 < J_2 < \cdots < J_{\nu'}$. Put $J_i = [s_i, t_i]$ and

(5) $$-a_i = \log \| Tf^{t_i - s_i} | E_{f^{s_i}(q)}^1 \| \qquad \text{for } 1 \leq i \leq \nu' .$$

Note that $a_i > 0$ for all $1 \leq i \leq \nu'$. Clearly

(6) $$\nu' \leq \nu ,$$

(7) $$\sum_{i=1}^{\nu} \tilde{a}_i \le \sum_{i=1}^{\nu'} a_1 .$$

From (11.2) (ii) and (2),

$$\sum_{i=1}^{\nu} (v_i - u_i) > \nu_0(N) \cdot N = [G_1 N / 4N_0] \cdot N$$
$$\ge (G_1 / 4N_0 - 1/N) \cdot N^2 \ge (G_1 / 8N_0) \cdot N^2 ,$$

namely;

(8) $$\sum_{i=1}^{\nu} (v_i - u_i) > (G_1 / 8N_0) \cdot N^2 .$$

From (4), (7) and (8),

(9) $$\sum_{i=1}^{\nu'} a_i \ge (G_1 (\log \lambda^{-1})/8N_0) \cdot N^2 .$$

From hypothesis, we can apply Prop. $[\theta, A]$ to (5) and have

(10) $$t_i - s_i = \text{length} (J_i) \ge A \cdot a_i^{\theta} \qquad \text{for all } 1 \le i \le \nu' .$$

First, we consider the case of $\theta = 1$. From (9) and (10) with $\theta = 1$, one can easily see that

(11) $$\sum_{i=1}^{\nu'} \text{length} (J_i) \ge (AG_1 (\log \lambda^{-1})/8N_0) \cdot N^2 .$$

Next, we assume that $\theta > 1$. From (10),

(12) $$A^{-1} \cdot \sum_{i=1}^{\nu'} \text{length} (J_i) \ge \sum_{i=1}^{\nu'} a_i^{\theta} .$$

By the Hölder's inequality, we have

(13) $$\{\sum_{i=1}^{\nu'} a_i^{\theta}\}^{1/\theta} \cdot (\nu')^{1-(1/\theta)} \ge \sum_{i=1}^{\nu'} a_i .$$

From (6), $\nu' \le G_1 N / 4N_0$. Since $1 - (1/\theta) > 0$,

(14) $$(\nu')^{1-(1/\theta)} \le (G_1 N / 4N_0)^{1-(1/\theta)} .$$

From (12), (13) and (14), we have

(15) $$\sum_{i=1}^{\nu'} a_i \le \{A^{-1} \cdot \sum_{i=1}^{\nu'} \text{length} (J_i)\}^{1/\theta} \cdot (G_1 N / 4N_0)^{1-(1/\theta)} .$$

From this (15) and (9),

(16) $$\{A^{-1} \cdot \sum_{i=1}^{\nu'} \text{length} (J_i)\}^{1/\theta} \cdot (G_1 N / 4N_0)^{1-(1/\theta)}$$
$$\ge (G_1 (\log \lambda^{-1})/8N_0) \cdot N^2 .$$

By an easy calculation, we have

(17) $$\sum_{i=1}^{\nu'} \text{length} (J_i) \ge ((\log \lambda^{-1})/2)^{\theta} \cdot (AG_1 / 4N_0) \cdot N^{1+\theta} .$$

Although we proved (17) for $\theta > 1$, if we put $\theta = 1$ in (17), this coincides with (11). Namely, (17) is valid for $\theta \ge 1$.

From (1), we have

$$N \geq \{2\ell/(\log \lambda^{-1})\}^{9/10} .$$

By substituting it in (17), we have

$$\textstyle\sum_{i=1}^{\nu'} \text{length}\,(J_i) \geq 2^{-(\theta+11)/10} \cdot (\log \lambda^{-1})^{(\theta-9)/10} \cdot (AG_1/N_0) \cdot \ell^{9(1+\theta)/10} .$$

Since $\{J_i\}$ are disjoint intervals in $[0, m-m']$,

$$m > \textstyle\sum_{i=1}^{\nu'} \text{length}\,(J_i) .$$

This completes the proof.

Now we remark that we can take away formally the hypothesis (i) in (11.5). In fact, put $\ell_0 = N_*^{10/9} (\log \lambda^{-1})/2$, and we shall consider the case of $0 < \ell < \ell_0$. Suppose that for $m \in Z_+$ and $0 < \ell < \ell_0$,

$$\log \|Tf^m | E_p^1\| \leq -\ell$$

holds at some $p \in \Lambda$. Then $m \geq 1$, because $\ell > 0$. Putting $k_0(\theta) = \ell_0^{-9(1+\theta)/10}$, we get $k_0(\theta) \cdot \ell^{9(1+\theta)/10} < 1$, namely,

$$m \geq k_0(\theta) \cdot \ell^{9(1+\theta)/10} .$$

Now define $k_0(\theta, A) = \min \{k(\theta) \cdot A, k_0(\theta)\}$. Then, from (11.5), we have;

(11.6) LEMMA. *If* Prop. $[\theta, A]$ *holds, then* Prop. $[9(1+\theta)/10, k_0(\theta, A)]$ *also holds.*

Recall that Prop. $[1, (\log m_0^{-1})^{-1}]$ is true by (11.4). Starting from this, by applying (11.6) repeatedly more than 5 times, we can get Prop. $[\theta, A_0]$ for $\theta > 4$ and for some constant $A_0 > 0$. This implies (11.1) Key lemma.

Remark that applying (11.6) much more times, we can have Prop. $[\theta, A]$ for θ arbitrarily close to 9. But we don't need such large θ.

§ 12. Expansive intervals

(12.1) DEFINITION. Let $p \in \Lambda$ and $N \in Z_+$. A Z_+-interval $I = [u, v]$ is called a (p, N)-expansive interval if and only if I satisfies

 (i)   length $(I) = v - u \geq N$

 (ii) $\|Tf^{v-u} | E_{f^u(p)}^1\| \geq (\lambda^{-1/2})^{v-u} .$

(12.2) LEMMA. *There exists $N_4 \in Z_+$ such that if*

$$\log \|Tf^m | E_p^1\| \leq (N/2)\,(\log \lambda) \qquad \text{for } p \in \Lambda,\ m \in Z_+$$

and integer $N \geq N_4$, then there exists a (p, N)-expansive interval in $[0, m]$.

(12.3) LEMMA. *For any $p \in \Lambda$ and $N \in Z_+$, there exists a (p, N)-expansive interval.*

The essence of the proof of our Theorem is in (12.2). (12.3) merely asserts the existence of the expansive intervals and doesn't claim where such intervals exist in Z_+. By a straightforward argument with rather rough estimates, one can prove (12.3) using (10.5) and (8.2), so, the proof is left to the reader. In the rest of this section, we shall prove (12.2).

Suppose for $p \in \Lambda$, and $m, N \in Z_+$, it holds that;

$$(1) \qquad \log \| Tf^m \,|\, E_p^1 \| \leq (N/2) (\log \lambda) .$$

Let $0 \leq k < m$ be the largest integer such that

$$\log \| Tf^k \,|\, E_p^1 \| \geq (N/4) (\log \lambda) + (\log m_0) .$$

Then, we can see that;

$$(2) \qquad \log \| Tf^k \,|\, E_p^1 \| < (N/4) (\log \lambda) ,$$

$$(3) \qquad \| Tf^n \,|\, E_{f^k(p)}^1 \| \leq 1 \qquad \text{for all } 0 \leq n \leq m - k ,$$

$$(4) \qquad \log \| Tf^{m-k} \,|\, E_{f^k(p)}^1 \| \leq (N/4) (\log \lambda) - (\log m_0) .$$

From (2), $\log \| Tf^k \,|\, E_p^1 \| < 0$. Therefore, by applying (2.7) with $\rho = \exp (N^{-4/3})$, we have $\nu = [kN^{-4/3}/(N^{-4/3} - (\log m_0))] + 1$ integers $0 \leq k_1 < k_2 \cdots < k_\nu \leq k$ such that

$$(5) \qquad \begin{aligned} & \log \| Tf^n \,|\, E_{f^{k_i(p)}}^1 \| \leq nN^{-4/3} \\ & \qquad \text{for all } 0 \leq n \leq k - k_i \text{ and } i = 1, 2, \cdots, \nu . \end{aligned}$$

From (3), we can easily see that

$$(6) \qquad \begin{aligned} & \log \| Tf^n \,|\, E_{f^{k_i(p)}}^1 \| \leq nN^{-4/3} \\ & \qquad \text{for all } 0 \leq n \leq m - k_i \text{ and } i = 1, 2, \cdots, \nu . \end{aligned}$$

From (4),

$$\log \| Tf^{m-k} \,|\, E_{f^k(p)}^1 \| \leq -\{(N/4) (\log \lambda^{-1}) - (\log m_0^{-1})\} .$$

Taking N so large that $(N/4) (\log \lambda^{-1}) > \log m_0^{-1}$, and applying (11.1), we have

$$\begin{aligned} m - k &\geq A_0\{(N/4) (\log \lambda^{-1}) - (\log m_0^{-1})\}^4 \\ &= A_0\{(1/4) (\log \lambda^{-1}) - (1/N) (\log m_0^{-1})\}^4 \cdot N^4 . \end{aligned}$$

If we choose N so that

$$(1/4) \, (\log \lambda^{-1}) - (1/N) \, (\log m_0^{-1}) \geq (1/8) \, (\log \lambda^{-1}) \,,$$

then we have

(7) $\qquad m - k \geq A_1 N^4 \,,$ $\qquad$ where $A_1 = A_0 \{ (1/8) \, (\log \lambda^{-1}) \}^4 \,.$

Now we fix a constant $r > 0$ such that

(8) $\qquad 270 cr \leq r_8$ and $90r < r_0 \,,$ $\qquad$ where r_8 is given in (10.2) .

We consider boxes $B_{f^{k_1}(p)}(r/GN)$ for each $f^{k_i}(p)$. From (5.4) (vi),

(9) $\qquad \begin{aligned} s(B_{f^{k_i}(p)}(r/GN)) &\geq c_1 s(B(r/GN)) \\ &= c_1 (2r/GN)^2 \,, \qquad \text{where } s(\cdot) \text{ denotes the area} \,. \end{aligned}$

Now let us estimate ν. From (2),

$$\log \| Tf^k \, | \, E_p^1 \| \leq - \{ (N/4) \, (\log \lambda^{-1}) \} \,.$$

From (11.1), $k \geq A_0 \{ (N/4) \, (\log \lambda^{-1}) \}^4$. Therefore,

(10) $\qquad\qquad\qquad\qquad k \geq A_1 N^4 \,.$

From this (10),

$$kN^{-4/3}/(N^{-4/3} - \log m_0) \geq A_1 N^4 N^{-4/3}/(N^{-4/3} + \log m_0^{-1}) \,.$$

Taking N with $N^{-4/3} \leq \log m_0^{-1}$, we have

$$\begin{aligned} \nu &> kN^{-4/3}/(N^{-4/3} - \log m_0) \\ &\geq \{ A_1/2 \, (\log m_0^{-1}) \} \cdot N^{8/3} \,. \end{aligned}$$

Namely, putting $A_2 = A_1/2 \, (\log m_0^{-1})$, we get;

(11) $\qquad\qquad\qquad\qquad \nu > A_2 N^{8/3} \,.$

We consider the sum of the area of $B_{f^{k_i}(p)}(r/GN)$ $(i = 1, \cdots, \nu)$. By (9) and (11),

$$\begin{aligned} \sum_{i=1}^{\nu} s(B_{f^{k_i}(p)}(r/GN)) &\geq \sum_{i=1}^{\nu} c_1 (2r/GN)^2 \\ &> A_2 c_1 (2r/G)^2 \cdot N^{2/3} \,, \end{aligned}$$

which exceeds the surface area of M by taking sufficiently large N. Therefore,

(12) $\qquad$ there exist at least two points $f^{k_i}(p), f^{k_j}(p)$ such that

$$B_{f^{k_i}(p)}(r/GN) \cap B_{f^{k_j}(p)}(r/GN) \neq \phi \,.$$

Assume that $k_i < k_j$ and put $f^{k_i}(p) = p_0$, $f^{k_j}(p) = q_0$ and $k_j - k_i = k_0$. By taking N with $r/GN \leq r_3$, where r_3 is from (6.1), and applying (6.1), we have;

$$(13) \qquad\qquad q_0 \in B_{p_0}(3r/GN) \,.$$

Then applying (7.4) with $\{f^n(p_0)\}_{0 \leq n \leq k_0}$ as a finite set, we have from (13) that;

(14)   there exist $v \in T_{p_0}M$, positive numbers a_1, a_2 and two points $f^{n_1}(p_0)$, $f^{n_2}(p_0)$ with $0 \leq n_1 < n_2 \leq k_0$, such that $(f^{n_1}(p_0), f^{n_2}(p_0), v, a_1, a_2)$ is $(p_0, 90r/GN)$-connectable and $f^n(p_0) \notin b_{p_0}(v, a_1, a_2)(V_{p_0}(1))$ for all $0 \leq n \leq k_0$ with $n \neq n_1, n_2$.

We want to close up this suborbit from $f^{n_1}(p_0)$ to $f^{n_2}(p_0)$ using closing lemma, but we need some preparations beforehand.

By taking N so large that $A_1 N^4 \geq 2GN$ and $2GN^{-1/3} \leq 1$, we have from (6), (7);

$$(15) \qquad \log \| Tf^n \,|\, E_{p_0}^1 \| \leq nN^{-4/3} \leq 1 \qquad \text{for all } 0 \leq n \leq 2GN \,.$$

Thus, we have;

$$(16) \qquad\qquad \log \eta(p_0, GN) \leq 1 \,.$$

Next, take N with $N \geq N_3$ and $(GN/10)(\log \lambda^{-1}) \geq 1$, then by (8) and (10.2), we have;

$$(17) \quad B_i(270cr) \cap B_j(270cr) = \phi \qquad \text{for any integer } 0 \leq i < j \leq 2N \,,$$

where

$$B_i(270cr) = B_{f^i(p_0)}(270cr\theta(p_0, i)/\eta(p_0, GN)(GN - i)) \,.$$

We claim that;

$$(18) \qquad f^n(B_{p_0}(90r/GN)) \qquad \text{are disjoint for } -N_0 \leq n \leq N_0 \,.$$

In fact, assume that

$$f^n(B_{p_0}(90r/GN)) \cap f^{n'}(B_{p_0}(90r/GN)) \neq \phi \,,$$

for some $-N_0 \leq n < n' \leq N_0$, namely,

$$B_{p_0}(90r/GN) \cap f^{n'-n}(B_{p_0}(90r/GN)) \neq \phi \,.$$

By (6.4) and (16),

$$f^{n'-n}(B_{p_0}(90r/GN)) \subset f^{n'-n}(B_{p_0}(270r/\eta(p_0, GN)GN))$$
$$\subset B_{f^{n'-n}(p_0)}(270cr\theta(p_0, n'-n)/\eta(p_0, GN)(GN-n'+n)),$$

which contradicts (17).

By the same argument, we can see that;

$$(19) \qquad\qquad n_2 - n_1 > N .$$

From (8), (14), (18) and (8.4), there exists a $(p_0, N_0, f^{n_1}(p_0), f^{n_2}(p_0), v, a_1, a_2)$-connector g in $\mathscr{U}(\varepsilon_0, f)$. Therefore, $f^{n_2-N_0}(p_0)$ is a periodic point of g with period $n_2 - n_1$. By the definition of $\mathscr{U}(\varepsilon_0, f)$ (ref. (10.1)), it can be seen that $f^{n_2-N_0}(p_0)$ must be a saddle of g. From the properties of g (ref. (8.3) and (8.2) (iii) (iv)) and (2.3) (ii), we can see that;

$$E^u_{f^{n_2-N_0(p_0)}}(g) = E^1_{f^{n_2-N_0(p_0)}} .$$

Therefore, from (2.4),

$$(20) \qquad\qquad \| Tg^{n_2-n_1} | E^1_{f^{n_2-N_0(p_0)}} \| \geq (\lambda^{-1})^{n_2-n_1} .$$

By the property of g (ref. (8.2) (iv)), we have;

$$(21) \qquad\qquad \| Tf^{n_2-n_1} | E^1_{f^{n_1(p_0)}} \| \geq (\lambda^{-1})^{n_2-n_1} .$$

(19) and this (21) imply $[n_1, n_2]$ is a (p_0, N)-expansive interval in $[0, k_0]$. Recall that $p_0 = f^{k_i}(p)$, $q_0 = f^{k_j}(p)$ and $k_j - k_j = k_0$. So, $[k_i + n_1, k_i + n_2]$ is a (p, N)-expansive interval in $[0, m]$. This completes the proof.

§13. Proof of the Theorem

In this section, we shall prove our Theorem assuming (11.2) Main lemma whose proof will be given in Section 15. Since we have already finished the essential part, the proof of the Theorem itself is now easy.

Proof. From Lemma 3.1 in [6], we have

$$\Omega(f) = \Lambda_0(f) \cup \Lambda_1(f) \cup \Lambda_2(f) .$$

From (2.2), $\Lambda_0(f)$ and $\Lambda_2(f)$ are finite sets and consequently they are hyperbolic sets. Therefore we have only to show that $\Lambda_1(f)$ is a hyperbolic set.

Let $TM | \Lambda_1(f) = E^1 \oplus E^2$ be the Tf-invariant splitting given in (2.3) (ii). Put $\Lambda = \Lambda_1(f)$ as before. It is enough to show that

(1)   there exist $c' > 0$ and $0 < \lambda' < 1$ such that

(i) $\|Tf^{-n}|E_p^1\| \le c'\lambda'^n$,

(ii) $\|Tf^n|E_p^2\| \le c'\lambda'^n$,   for all $n \in Z_+$ and $p \in \Lambda$.

We shall only prove (i). The same argument as (i) with f^{-1} on E^2 gives (ii).

Suppose that (i) does not hold, then it follows that

$$\text{spec. rad. } f_*^{-1}|\Sigma^b(E^1) \ge 1 .$$

From (4.1), we have that

(2)   there is a recurrent point $p_* \in \Lambda$ such that

$$\|Tf^k|E_{p*}^1\| \le 1 \quad \text{for all } k \in Z_+ .$$

We fix a positive integer $N \ge N_4$. From (12.3), there exists a (p_*, N)-expansive interval. Let $I = [m, n]$ be the minimum element of the set of all (p_*, N)-expansive intervals, where the order for intervals is given lexicographically, that is; for $I_1 = [m_1, n_1]$ and $I_2 = [m_2, n_2]$, $I_1 \le I_2$ iff $m_1 < m_2$ or, $m_1 = m_2$ and $n_1 \le n_2$.

From the definition of the expansive interval, we have

$$\|Tf^n|E_{p*}^1\| = \|Tf^{n-m}|E_{f^m(p*)}^1\| \cdot \|Tf^m|E_{p*}^1\|$$
$$\ge (\lambda^{-1/2})^N \cdot \|Tf^m|E_{p*}^1\| .$$

From (2), we have

$$\log \|Tf^m|E_{p*}^1\| \le (N/2)(\log \lambda) .$$

Then, by (12.2), it follows that there exists a (p_*, N)-expansive interval in $[0, m]$. This contradicts the fact that $I = [m, n]$ in the minimum among all (p_*, N)-expansive intervals. This proves the Theorem.

§ 14. Taper neighborhoods

In this section, we shall introduce for each $p \in \Lambda$ a certain class of neighborhoods with special shape which plays an essential role in the proof of the Main lemma in Section 15.

For notational convenience, we put

$$\alpha = \lambda^{1/20} , \quad\quad \beta = \lambda^{-1/2} .$$

(14.1) DEFINITION. Let $r' > r > 0$. We define a subset of R^2 by

$$T(r, r') = \bigcup_{0 \le t \le (\log r'/r)/(\log \beta)} B(r\alpha^t, r\beta^t) .$$

In other words, $(x, y) \in T(r, r')$ iff there exists $0 \leq t \leq (\log r'/r)/(\log \beta)$ such that $|x| \leq r\alpha^t$, $|y| \leq r\beta^t$.

By elementary computations, we can see that

$$s(T(r, r')) = (4/9)r^{11/10}(10r'^{9/10} - r^{9/10}) ,$$

where $s(\ )$ denotes the area. Since $r' > r > 0$, we have

(14.2) LEMMA.

$$s(T(r, r')) > 4r^{11/10} \cdot r'^{9/10} .$$

It is very important that $s(T(r, r'))$ has order $11/10$ with respect to r (ref. (11) in Section 15).

(14.3) DEFINITION. Let $r_1/2 > r' > r > 0$ and $p \in \Lambda$. We define $T_p(r, r') = \psi_p^{-1}(T(r, r'))$.

We call such a neighborhood $T_p(r, r')$, *taper neighborhood*. By drawing the picture, one can see the reason of the name.

Remark that, since $T(r, r')$ is symmetric with respect to x and y-axes in $\mathbf{R}^2$, $T_p(r, r')$ is independent of the choice of the basis of $E_p^1 \oplus E_p^2$ with which ψ is defined.

The iterated images of $T_p(r, r')$ under f are given as follows.

(14.4) LEMMA. *Let* $r_9 = \min \{r_1/2, (\lambda^{-1/20} - 1)/2M_1c\}$, $r_9 > r' > r > 0$ *and* $p \in \Lambda$. *If*

$$\|Tf^n | E_p^1\| \leq 1 \quad \text{for all } 0 \leq n \leq [(\log r'/r)/(\log \beta)] + 1 ,$$

then

$$f^i(B_p(r\alpha^t, r\beta^t)) \subset B_{f^i(p)}(\theta(p, i)r\alpha^{t-i}, \omega(p, i)r\beta^{t-i}(\lambda^{11/20})^i)$$

for all $0 \leq t \leq (\log r'/r)/(\log \beta)$ *and integers* $0 \leq i \leq [t + 1]$.

Proof. We fix $0 \leq t \leq (\log r'/r)/(\log \beta)$ and prove this lemma by induction on $i = 0, 1, \cdots, [t + 1]$.

When $i = 0$, it is trivial. Assuming that the lemma is true for $i - 1$, we show that,

$$f(B_{f^{i-1}(p)}(\theta(p, i - 1)r\alpha^{t-i+1}, \omega(p, i - 1)r\beta^{t-i+1}(\lambda^{-11/20})^{i-1})$$
$$\subset B_{f^i(p)}(\theta(p, i)r\alpha^{t-i}, \omega(p, i)r\beta^{t-i}(\lambda^{-11/20})^i) .$$

Let $(f_1, f_2) = \psi_{f^i(p)} \circ f \circ \psi_{f^{i-1}(p)}^{-1}$, and $(x, y) \in \psi_{f^{i-1}(p)}(B_{f^{i-1}(p)}(\text{———}))$. It is enough to show that

$$|f_1(x, y)| \leq \theta(p, i) r \alpha^{t-i}$$
$$|f_2(x, y)| \leq \omega(p, i) r \beta^{t-i} (\lambda^{-11/20})^i .$$

By using the fact that $\theta(p, i-1) \leq 1$, $\omega(p, i-1) \leq c\lambda^{i-1}\theta(p, i-1)$ and $i \leq [t+1]$ i.e. $t-i+1 \geq 0$, we have

$$\begin{aligned}
|x| + |y| &\leq \theta(p, i-1) r \alpha^{t-i+1} + \omega(p, i-1) r \beta^{t-i+1}(\lambda^{-11/20})^{i-1} \\
&\leq r(\alpha^{t-i+1} + \beta^{t-i+1} c(\lambda^{9/20})^{i-1}) \\
&\leq r(1 + c\beta^{t-i+1}) \leq r + cr\beta^t \\
&\leq r + cr' \leq 2cr' .
\end{aligned}$$

So, by the hypothesis on r', we have

$$1 + M_1(|x| + |y|) \leq \lambda^{-1/20} .$$

By (5.6),

$$\begin{aligned}
|f_1(x, y)| &\leq |x| \cdot \| Tf | E^1_{f^{i-1}(p)} \| \{1 + M_1(|x| + |y|)\} \\
&\leq \theta(p, i-1) r \alpha^{t-i+1} \| Tf | E^1_{f^{i-1}(p)} \| \cdot \lambda^{-1/20} \\
&= \theta(p, i) r \alpha^{t-i} , \\
|f_2(x, y)| &\leq |y| \cdot \| Tf | E^2_{f^{i-1}(p)} \| \{1 + M_1(|x| + |y|)\} \\
&\leq \omega(p, i-1) r \beta^{t-i+1}(\lambda^{-11/20})^{i-1} \| Tf | E^2_{f^{i-1}(p)} \| \lambda^{-1/20} \\
&= \omega(p, i) r \beta^{t-i} (\lambda^{-11/20})^i .
\end{aligned}$$

This completes the proof.

Let $0 \leq t \leq (\log r'/r)/(\log \beta)$ and $i = [t+1]$. Then, since $-1 \leq t-i \leq 0$, we have $\alpha^{t-i} \leq \alpha^{-1} = \lambda^{-1/20} \leq \lambda^{-1}$ and $\beta^{t-i} \leq 1$, and since $\omega(p, i) \leq c\lambda^i\theta(p, i)$, $\omega(p, i)(\lambda^{-11/20})^i \leq c\theta(p, i)(\lambda^{9/20})^i \leq c\theta(p, i)$. So, from (14.4), we have

$$f^i(B_p(r\alpha^t, r\beta^t)) \subset B_{f^i(p)}(\theta(p, i)r\lambda^{-1}, \theta(p, i)rc) .$$

Thus, recalling that $\lambda^{-1} \leq 2$ from (2.5), we get;

(14.5) LEMMA. *Let $r_9 > r' > r > 0$ and $p \in \Lambda$. If*

$$\| Tf^n | E^1_p \| \leq 1 \quad \text{for all } 0 \leq n \leq [(\log r'/r)/(\log \beta)] + 1 ,$$

then for any $z \in T_p(r, r')$, there is an integer $0 \leq i \leq [(\log r'/r)/(\log \beta)] + 1$ such that

$$f^i(z) \in B_{f^i(p)}(2cr\theta(p, i)) .$$

Remark. The assumption $\lambda \geq 1/2$ (ref. (2.5)) is only used to simplify the notation. The argument in Section 15 works similarly without it.

§15. Proof of (11.2) Main lemma

In this section, we shall give a proof of (11.2) Main lemma. It is rather hard but completes our whole work.

Let $p \in \Lambda$, $m \in \mathbf{Z}_+$ and $N \in \mathbf{Z}_+$. Assume that

$$(1) \qquad \log \| Tf^m \,|\, E_p^1 \| \leq (N^{10/9}/2)\,(\log \lambda) \;.$$

We want to show that, if N is sufficiently large, then we can select a positive integer $m' < m$ such that

$$\| Tf^n \,|\, E_{f^{m'}(p)}^1 \| \leq 1 \qquad \text{for all } 0 \leq n \leq m - m'$$

and construct pairwise disjoint intervals in $[0, m - m']$ with the properties given in the statement of (11.2).

As the first step, we choose a suborbit of the positive orbit of $p \in \Lambda$ which is nearly recurrent and satisfies a good condition for norms of differential of f restricted to E^1.

Let $0 < n_1 \leq m$ be the largest integer with the property;

$$(2) \qquad \log \| Tf^{n_1} \,|\, E_p^1 \| \geq (N^{10/9}/4)\,(\log \lambda) + (\log m_0) \;.$$

If we take N large enough to satisfy

$$(3) \qquad N^{10/9} > G_1^{-1} \;,$$

then by easy computations, it can be seen that

$$(4) \qquad \log \| Tf^{n_1} \,|\, E_p^1 \| < (N^{10/9}/4)\,(\log \lambda) \;,$$

$$(5) \qquad \log \| Tf^{m - n_1} \,|\, E_{f^{n_1}(p)}^1 \| < (N^{10/9}/8)\,(\log \lambda) \;.$$

Recalling (11.4), we have

(6) If $q \in \Lambda$, $n \in \mathbf{Z}_+$ and $\ell > 0$ satisfy

$$\log \| Tf^n \,|\, E_q^1 \| \leq -\ell \;, \qquad \text{then } n \geq (\log m_0^{-1})^{-1} \cdot \ell \;.$$

Then, from (5) and (6),

$$(7) \qquad m - n_1 \geq G_1 N^{10/9} \;.$$

By applying (2.7) to (4) with $\rho = 1$, we find $\nu_1 = [N^{10/9}\,(\log \lambda^{-1})/4\,(\log m_0^{-1})] + 1$ integers $0 \leq k_1 < k_2 < \cdots < k_{\nu_1} \leq n_1$ with the property that;

$$(8) \qquad \| Tf^n \,|\, E_{f^{k_i}(p)}^1 \| \leq 1 \qquad \text{for all } 0 \leq n \leq n_1 - k_i \text{ and } 1 \leq i \leq \nu_1 \;.$$

From the definition of n_1 and (8).

(9) $\| Tf^n | E^1_{f^{k_i(p)}} \| \leq 1$ for all $0 \leq n \leq m - k_i$.

We fix $r > 0$ such that

(10) $180 c^4 r < \min \{ r_0, r_3, r_4, r_9 \}$,

and consider the taper neighborhoods $T_{f^{k_i(p)}}(r/GN, r)$ for each $1 \leq i \leq \nu_1$. Let us estimate the sum of the area of these taper neighborhoods. From (14.2) and (5.4) (vi),

$$\sum_{i=1}^{\nu_1} s(T_{f^{k_i(p)}}(r/GN, r)) \geq \sum_{i=1}^{\nu_1} c_1 \cdot s(T(r/GN, r)) > \nu_1 \cdot c_1 \cdot 4 \cdot (r/GN)^{11/10} \cdot r^{9/10}$$
$$= \{ (\log \lambda^{-1})/(\log m_0^{-1}) \} \cdot c_1 \cdot G^{-11/10} \cdot r^2 \cdot N^{1/90} \ ,$$

which exceeds the surface area of M by taking N large enough. So, we have

(11)   there are at least two points $f^{k_i}(p), f^{k_j}(p)$ such that;

$$T_{f^{k_i(p)}}(r/GN, r) \cap T_{f^{k_j(p)}}(r/GN, r) \neq \phi \ .$$

We assume that $k_j > k_i$, and put;

(12) $f^{k_i}(p) = p_0,\ f^{k_j}(p) = q_0$ and $k_j - k_i = \tau$, so $f^\tau(p_0) = q_0$.

From (7), (8), (9),

(13) $\| Tf^n | E^1_{p_0} \| \leq 1$ for all $0 \leq n \leq G_1 N^{10/9} + \tau$,

               $\| Tf^n | E^1_{q_0} \| \leq 1$ for all $0 \leq n \leq G_1 N^{10/9}$.

From (11), we may take a point $z \in T_{p_0}(r/GN, r) \cap T_{q_0}(r/GN, r)$. We take N large enough to satisfy $(\log \beta)^{-1} (\log GN) + 1 \leq G_1 N^{10/9}$, then from (14.5), we have

(14)   (i)   there is an integer $0 \leq i \leq (\log \beta)^{.}$ $(\log GN) + 1$ such that
               $f^i(z) \in B_{f^i(p_0)}(2cr\theta(p_0, i)/GN)$,
        (ii)   there is an integer $0 \leq j \leq (\log \beta)^{.}$ $(\log GN) + 1$ such that
               $f^j(z) \in B_{f^j(q_0)}(2cr\theta(q_0, j)/GN)$.

Take large N such that

(15) $(\log \beta)^{-1} (\log GN) + 1 < [G_1 N] < [GN] < G_1 N^{10/9}$.

We put $\tau_1 = [G_1 N]$, then from (13) and (6.4),

$$f^{\tau_1 - i}(B_{f^i(p_0)}(2cr\theta(p_0, i)/GN)) \subset B_{f^{\tau_1}(p_0)}(2c^2 r\theta(p_0, \tau_1)/(GN - \tau_1 + i))$$
$$\subset B_{f^{\tau_1}(p_0)}(2c^2 r\theta(p_0, \tau_1)/(GN - \tau_1)) \ .$$

So, from (14), we have

(16) $f^{\tau_1}(z) \in B_{f^{\tau_1(p_0)}}(2c^2r\theta(p_0, \tau_1)/(GN - \tau_1))$.

Similarly,

(17) $f^{\tau_1}(z) \in B_{f^{\tau_1(q_0)}}(2c^2r\theta(q_0, \tau_1)/(GN - \tau_1))$.

Let $N_0 \in Z_+$ be given in (10.1), and take N so large that,

(18) $N \geq 8N_0 (\log m_0^{-1})/(\log \lambda^{-1}) = G_1^{-1}N_0$.

Put $\nu_0 = \nu_0(N) = [G_1N/4N_0]$ and for each integer $1 \leq k \leq \nu_0$, put

$$\tau_k = \tau_1 + (k - 1)4N_0 \quad \text{and} \quad f^{\tau_k}(p_0) = p_k, \; f^{\tau_k}(q_0) = q_k \; .$$

Note that, from (12), $f^{\tau}(p_k) = q_k$ for all $1 \leq k \leq \nu_0$. Also clearly,

(19) $\tau_k < 2[G_1N] \leq 2G_1N \quad$ for all $1 \leq k \leq \nu_0$.

From (16), (17),

$$B_{p_1}(2c^2r\theta(p_0, \tau_1)/(GN - \tau_1)) \cap B_{q_1}(2c^2r\theta(q_0, \tau_1)/(GN - \tau_1)) \neq \phi \; .$$

By (6.1),

(i)   if $\theta(p_0, \tau_1) \geq \theta(q_0, \tau_1)$, then $q_1 \in B_{p_1}(6c^2r\theta(p_0, \tau_1)/(GN - \tau_1))$,

(ii)   if $\theta(p_0, \tau_1) \leq \theta(q_0, \tau_1)$, then $p_1 \in B_{q_1}(6c^2r\theta(q_0, \tau_1)/(GN - \tau_1))$.

So, from (9), in either case, we have

(20)   there exist $p_* \in \Lambda$ such that

$$\|Tf^n \,|\, E_{p_*}^1\| \leq 1 \quad \text{for all } 0 \leq n \leq G_1N^{10/9} \text{ and}$$
$$p_1, q_1 \in B_{f^{\tau_1(p_*)}}(6c^2r\theta(p_*, \tau_1)/(GN - \tau_1)) \; .$$

In fact, we can take $p_0 = p_*$ (resp. $q_0 = p_*$) in case (i) (resp. (ii)). From this (20) and (6.4), we get

(21) $p_k, q_k \in B_{f^{\tau_k(p_*)}}(6c^3r\theta(p_*, \tau_k)/(GN - \tau_k)) \quad$ for all $1 \leq k \leq \nu_0$.

Now, since k_i in (11) and (12) satisfies the property (i) of the statement of (11.2), we can take $m' = k_i$ in (11.2). In what follows, we shall construct the disjoint intervals in $[0, \tau + \tau_{\nu_0}]$ with the properties (ii), (iii) in the statement of (11.2). Our strategy is the following. We identify the Z_+-interval $[0, \tau + \tau_{\nu_0}]$ with the positive orbit of $p_0 = f^{k_i}(p)$, that is $\{f^n(p_0)\}_{0 \leq n \leq \tau + \tau_{\nu_0}} = \{p_0 \cdots p_1 \cdots p\nu_0 \cdots q_0 \cdots q_1 \cdots q\nu_0\}$. First, we select connectable subintervals inductively by using (21) and (7.4). These intervals

are not necessarily disjoint. We regard the union of these intervals as a disjoint union of w_{ν_0} intervals, so w_{ν_0} is possibly less than $\nu_0(N)$. We will prove that these disjoint intervals are exactly the intervals wanted. Concerning the above positive orbit of p_0, remark that q_0 can not be contained in $\{f^n(p_0)\}_{0 \le n \le \tau_{\nu_0}}$, namely, $\tau > \tau_{\nu_0}$. In fact, as we will see later, $\tau > N$ and $\tau_{\nu_0} < N/4$.

Before proceedings to the main step of the proof, we shall make some preparations.

From (10), (15), (20) and (10.2), we get

(22)
$$B_{f^i(p_*)}(180c^4r\theta(p_*, i)/(GN - i))$$
$$\text{are disjoint with each other for } 0 \le i \le 2N .$$

By this, we have the following two lemmas.

(15.1) LEMMA. For $\tau_1 \le i \le \tau_{\nu_0}$ and $-N_0 \le n \le N_0$, any two of

$$f^n(B_{f^i(p_*)}(180c^3r\theta(p_*, i)/(GN - i)))$$

are disjoint when they have different $i + n$.

(15.2) LEMMA. Let $x \in M$, $s \in Z_+$ and $0 \le n_1 \le n_2 \le 2\tau_1$. If

$$x \in B_{f^{n_2}(p_*)}(180c^3r\theta(p_*, n_2)/(GN - n_2))$$

and

$$f^s(x) \in B_{f^{n_1}(p_*)}(180c^3r\theta(p_*, n_1)/(GN - n_1)) ,$$

then $s > N$.

The proofs of (15.1) and (15.2) are straightforward by using (6.4), (18) and (22). (15.1) will guarantee the disjointness of the supports of the perturbations when we apply the closing lemma on several places simultaneously, and (15.2) guarantees the length of the suborbit with such recurrence.

To complete the proof of the Main lemma, we first fix the following notation.

For a set of pairs of integers $\{s_k < t_k\}_{1 \le k \le \nu_0}$ in $[\tau_1, \tau + \tau_{\nu_0}]$, we put $J_k = \bigcup_{j=1}^{k} [s_j, t_j]$ and $J_0 = \phi$, where $[s_j, t_j]$ denotes the Z_+-interval. Regard J_k as a disjoint union of Z_+-intervals and denote it by $J_k = \bigcup_{j=1}^{w_k} I_j^k$ and $I_j^k = [u_j^k, v_j^k]$, where $I_1^k < I_2^k < \cdots < I_{w_k}^k$. Note that $w_k \le k$. Next we put $p_*^k = f^{\tau_k}(p_*)$ for $1 \le k \le \nu_0$.

Then we give;

(15.3) LEMMA. *There exists a set of pairs of integers $\{s_k < t_k\}_{1 \le k \le \nu_0}$ in $[\tau_1, \tau + \tau_{\nu_0}]$ with the following properties for all $1 \le k \le \nu_0$;*

(i) *There exist $v_k \in T_{p_*^k}M$ and positive numbers a_1^k, a_2^k such that $(f^{s_k}(p_0), f^{t_k}(p_0), v_k, a_1^k, a_2^k)$ is $(p_*^k, 180c^3r\theta(p_*, \tau_k)/(GN - \tau_k))$-connectable, namely;*

   (ia) $|v_k| + a_1^k + a_2^k \le 180c^3r\theta(p_*, \tau_k)/(GN - \tau_k)$

   (ib) $(2\sqrt{2})^{-1} \le a_1^k/a_2^k \le 2\sqrt{2}$

   (ic) $f^{s_k}(p_0), f^{t_k}(p_0) \in b_{p_*^k}(v_k, a_1^k, a_2^k)(V_{p_*^k}(\sqrt{3/4}))$

   (id) $b_{p_*^k}(v_k, a_1^k, a_2^k)(V_{p_*^k}(1)) \subset B_{p_*^k}(180c^3r\theta(p_*, \tau_k)/(GN - \tau_k))$.

  (ii) $s_k, t_k \notin J_{k-1}$.

  (iii)   *If $n \in [\tau_1, \tau + \tau_{\nu_0}] - J_{k-1}$ and $n \ne s_k, t_k$, then*
$$f^n(p_0) \notin b_{p_*^k}(v_k, a_1^k, a_2^k)(V_{p_*^k}(1)) .$$

  (iv) $t_k \le \tau + \tau_k$.

  (v) $s_k > 4[G_1N] = 4\tau_1$.

  (vi)   length $(I_j^k) = v_j^k - u_j^k > N$ for all $1 \le j \le w_k$.

  (vii) $\sum_{k=1}^{w_k}$ length $(I_j^k) > kN$.

  (viii) $\|Tf^{v_j^k - u_j^k} | E_j^1 u_j^k(p_0)\| \ge (\lambda^{-1})^{v_j^k - u_j^k}$ *for all $1 \le j \le w_k$.*

If this (15.3) is verified and such integers s_k, t_k exist for all $1 \le k \le \nu_0$, then the Main lemma will be established, because from (vii) and (viii), $\{I_j^{\nu_0}\}_{1 \le j \le w_{\nu_0}}$ are exactly the intervals we want.

Proof of (15.3). We shall prove (15.3) by induction on $k = 1, \cdots, \nu_0$.

First, let us select $s_1, t_1 \in [\tau_1, \tau + \tau_{\nu_0}]$. From (21), $p_1 = f^{\tau_1}(p_0)$ and $q_1 = f^{\tau_1}(q_0) = f^{\tau + \tau_1}(p_0)$ are contained in $B_{p_*^1}(6c^3r\theta(p_*, \tau_1)/(GN - \tau_1))$. So, applying (7.4) with $\{f^n(p_0): n \in [\tau_1, \tau + \tau_{\nu_0}]\}$ as a finite set, we have integers s_1, t_1 $(s_1 < t_1)$ in $[\tau_1, \tau + \tau_{\nu_0}]$, $v_1 \in T_{p_*^1}M$ and positive numbers a_1^1, a_2^1 satisfying (i) and (iii) in the statement of (15.3). For $k = 1$, (ii) is not necessary. So, let us prove that s_1 and t_1 satisfy (iv) $\sim$ (viii).

From (ic) and (id),

$$f^{s_1}(p_0), f^{t_1}(p_0) \in B_{p_*^1}(180c^3r\theta(p_*, \tau_1)/(GN - \tau_1)) .$$

Applying (15.2) with $x = f^{s_1}(p_0)$, $s = t_1 - s_1$ and $n_1 = n_2 = \tau_1$, we get;

(23) $t_1 - s_1 > N$.

So, (vi) and (vii) are satisfied.

Let us prove (iv), i.e. $t_1 \le \tau + \tau_1$. Suppose $t_1 > \tau + \tau_1$. As we observed above,

$$f^{t_1}(p_0) \in B_{p_*^1}(180c^3r\theta(p_*, \tau_1)/(GN - \tau_1)) .$$

On the other hand, from (21) and (6.4), we have

$$f^{t_1}(p_0) = f^{t_1 - \tau - \tau_1}(q_1) \in B_{f^{t_1 - \tau}(p_*)}(6c^4 r \theta(p_*, t_1 - \tau)/(GN - t_1 + \tau)) \,,$$

which contradicts (22), because from (19),

$$t_1 - \tau < (\tau + \tau_{\nu_0}) - \tau = \tau_{\nu_0} < 2G_1 N \le N/4 \,.$$

Next, let us prove (viii). From (i), $(f^{s_1}(p_0), f^{t_1}(p_0), v_1, a_1^1, a_2^1)$ is $(p_*^1, 180c^3 r \theta(p_*, \tau_1)/(GN - \tau_1))$-connectable. By (10), (10.1), (15.1) and (8.4), there exists a $(p_*^1, N_0, f^{s_1}(p_0), f^{t_1}(p_0), v_1, a_1^1, a_2^1)$-connector g in $\mathscr{U}(\varepsilon_0, f)$. From the definition of the connector g, $f^{t_1 - N_0}(p_0)$ is a periodic point of g. Moreover, its period is $t_1 - s_1$ and from (10) and (3.1), this periodic point must be a saddle of g. From (8.3), (8.2) (iii) (iv) and (2.3) (ii), we can see that

$$E_{f^{t_1 - N_0(p_0)}}^u(g) = E_{f^{t_1 - N_0(p_0)}}^1 \,.$$

Therefore, from (2.4),

$$\| Tg^{t_1 - s_1} | E_{f^{t_1 - N_0(p_0)}}^1 \| \ge (\lambda^{-1})^{t_1 - s_1} \,.$$

From (8.2) (iv), we get,

$$\| Tf^{t_1 - s_1} | E_{f^{s_1}(p_0)}^1 \| \ge (\lambda^{-1})^{t_1 - s_1} \,,$$

which implies (viii).

Finally, we shall prove (v). Suppose $s_1 \le 4\tau_1 = 4[G_1 N]$. From (vi) and (viii),

$$\| Tf^{t_1} | E_{p_0}^1 \| = \| Tf^{t_1 - s_1} | E_{f^{s_1}(p_0)}^1 \| \cdot \| Tf^{s_1} | E_{p_0}^1 \|$$
$$\ge (\lambda^{-1})^{t_1 - s_1} \cdot m_0^{s_1} > (\lambda^{-1})^N \cdot m_0^{s_1} \,.$$

Namely, we have,

$$\log \| Tf^{t_1} | E_{p_0}^1 \| > N (\log \lambda^{-1}) - 4G_1 N (\log m_0^{-1}) = (N/2) (\log \lambda^{-1}) \,,$$

which contradicts (13), because from (15) and (19),

$$t_1 \le \tau + \tau_1 < \tau + G_1 N^{10/9} \,,$$

Next, assuming that (15.3) is valid for all $1 \le i \le k - 1$, we prove that there exist integers s_k and t_k in $[\tau_1, \tau + \tau_{\nu_0}]$ with the properties (i) ~ (viii).

Let $Q = \{f^n(p_0): n \in [\tau_1, \tau + \tau_{\nu_0}] - J_{k-1}\}$. Since $s_i > 4\tau_1$ and $t_i \le \tau + \tau_i$ for all $1 \le i \le k - 1$, $p_k = f^{\tau_k}(p_0)$ and $q_k = f^{\tau_k + \tau}(p_0)$ are contained in Q. Moreover, from (21),

$$p_k, q_k \in B_{p_*^k}(6c^3 r\theta(p_*, \tau_k)/(GN - \tau_k)) .$$

So, applying (7.4) with Q as a finite set, we get $v_k \in T_{p_*^k}M$, positive numbers a_1^k, a_2^k and integers $s_k, t_k \in [\tau_1, \tau + \tau_{\nu_0}] - J_{k-1}$ ($s_k < t_k$) satisfying (i), (ii) and (iii).

By the same argument as in case of $k = 1$, we can see that $t_k \leq \tau + \tau_k$, i.e. (iv). Let us prove that s_k and t_k satisfy (v) $\sim$ (viii).

Consider $\{I_j^{k-1}\}_{1 \leq j \leq w_{k-1}}$. Since (ii) holds for all $1 \leq i \leq k$, we have;

(24)   Each I_j^{k-1} satisfies $I_j^{k-1} \subset [s_k, t_k]$ or $I_j^{k-1} \cap [s_k, t_k] = \phi$.

(25)   Each I_j^{k-1} coincides with some of $\{[s_i, t_i]\}_{1 \leq i \leq k-1}$.

Let $L_1 < L_2 < \cdots < L_w$ be intervals in $\{I_j^{k-1}\}_{1 \leq j \leq w_{k-1}}$ satisfying $I_j^{k-1} \subset [s_k, t_k]$. If there are no such intervals, that is; $I_j^{k-1} \cap [s_k, t_k] = \phi$ for all $1 \leq j \leq w_{k-1}$, then by the same argument as in case of $k = 1$, it can be seen that $\{s_i, t_i\}_{1 \leq i \leq k}$ satisfy the properties (i) $\sim$ (viii). So, we assume that $w \geq 1$.

From (25), for any $1 \leq n \leq w$, we can put $L_n = [s_{i_n}, t_{i_n}]$ with some $1 \leq i_n \leq k - 1$. From the definition of $\{L_n\}_{1 \leq n \leq w}$, we have;

$$s_k < s_{i_1} .$$

Since (ic) and (id) hold for all $1 \leq i \leq k$,

$$f^{s_{i_1}}(p_0) \in B_{p_*^{i_1}}(180c^3 r\theta(p_*, \tau_{i_1})/(GN - \tau_{i_1}))$$

and

$$f^{s_k}(p_0) \in B_{p_*^k}(180c^3 r\theta(p_*, \tau_k)/(GN - \tau_k)) .$$

So, applying (15.2) with $x = f^{s_k}(p_0)$, $s = s_{i_1} - s_k$, $n_1 = \tau_{i_1}$ and $n_2 = \tau_k$, we get

$$s_{i_1} - s_k > N .$$

From this fact and the hypothesis of induction, one can easily see that $\{s_i, t_i\}_{1 \leq i \leq k}$ satisfy (vi) and (vii).

If (viii) is proved, by the same argument as in case of $k = 1$ using (vi) and (viii), we can verify (v). Thus, we shall finish the proof of (15.3) (and so the Main lemma) by proving (viii).

Since $1 \leq i_n \leq k - 1$, from (i), $(f^{s_{i_n}}(p_0), f^{t_{i_n}}(p_0), v_{i_n}, a_1^{i_n}, a_2^{i_n})$ is $(p_*^{i_n}, 180c^3 r\theta(p_*, \tau_{i_n})/(GN - \tau_{i_n}))$-connectable. By (id), (10.1), (15.1) and (8.4),

(26)   there exists a $(p_*^{i_n}, N_0, f^{t_{i_n}}(p_0), f^{s_{i_n}}(p_0), v_{i_n}, a_1^{i_n}, a_2^{i_n})$-connector g_n
        in $\mathcal{U}(\varepsilon_0, f)$.

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And since (i) holds for k, $(f^{s_k}(p_0), f^{t_k}(p_0), v_k, a_1^k, a_2^k)$ is $(p_*^k, 180c^3 r\theta(p_*, \tau_k)/(GN - \tau_k))$-connectable, and similarly;

(27)   there exists a $(p_*^k, N_0, f^{s_k}(p_0), f^{t_k}(p_0), v_k, a_1^k, a_2^k)$-connector g_0 in $\mathscr{U}(\varepsilon_0, f)$.

It is important to remark that $f^{t_{i_n}}(p_0)$ appears before $f^{s_{i_n}}(p_0)$ in (26) and $f^{s_k}(p_0)$ appears before $f^{t_k}(p_0)$ in (27). From (8.3) and (8.2), g_n and g_0 have the property that

$$g_n^{2N_0}(f^{s_{i_n} - N_0}(p_0)) = f^{t_{i_n} + N_0}(p_0)$$
$$g_0^{2N_0}(f^{t_k - N_0}(p_0)) = f^{s_k + N_0}(p_0) .$$

It means that g_n cuts off the suborbit $\{f^n(p_0) : s_{i_n} \leq n \leq t_{i_n}\}$. It seems that g_0 closes up the suborbit $\{f^n(p_0) : s_k \leq n \leq t_k\}$. But, $f^{t_k - N_0}(p_0)$ is not necessarily a periodic point of g_0, because from (iii), the points $\{f^n(p_0) : n \in \bigcup_{j=1}^{w} L_j\}$ may be contained in the support of $g_0 f^{-1}$. We construct a periodic point by cutting off these intervals $\{L_n\}_{1 \leq n \leq w}$ by g_n.

We put,

(28) $h_0 = g_0 f^{-1}$ and $h_n = g_n f^{-1}$ for $1 \leq n \leq w$,

and

(29) $g = h_w \circ h_{w-1} \circ \cdots \circ h_1 \circ h_0 \circ f .$

From (26), (27), (id), (8.2) (ii) and (15.1),

(30) $\{\text{supp } h_n\}$ are disjoint for $0 \leq n \leq w$.

Since $g_n \in \mathscr{U}(\varepsilon_0, f)$ for all $0 \leq n \leq w$, from (2.8), we have

(31) $g \in \mathscr{U}(\varepsilon_0, f) .$

Let us consider the positive orbit of $f^{t_k - N_0}(p_0)$ for g. We put

(32)
$$g^{N_0}(f^{t_k - N_0}(p_0)) = x_0 ,$$
$$g^{N_0}(f^{s_{i_n} - N_0}(p_0)) = x_n \quad \text{for } 1 \leq n \leq w .$$

Then, from (26), (27), (30), (ic), (id) and (iii), we get;

(33)
$$g^{s_{i_1} - s_k}(x_0) = x_1 ,$$
$$g^{s_{i_n} - t_{i_{n-1}}}(x_{n-1}) = x_n \quad \text{for all } 2 \leq n \leq w ,$$
$$g^{t_k - t_w}(x_w) = x_0 .$$

Thus, x_0 is a periodic point of g with period

$$(t_k - s_k) - \sum_{n=1}^{w} (t_{i_n} - s_{i_n}) .$$

Since $g \in \mathcal{U}(\varepsilon, f)$, from (10.1) and (3.1), x_0 must be a saddle of g.

Now, it holds that $E_{x_0}^u(g) = E_{x_0}^1$ and $E_{x_0}^s(g) = E_{x_0}^2$. In fact, as g preserves E^1 and E^2 on the orbit of the saddle x_0, $E_{x_0}^1$ and $E_{x_0}^2$ must be either $E_{x_0}^u(g)$ or $E_{x_0}^s(g)$. From (33) and the properties of the connector (8.3) and (8.2) (iii), (iv), we have;

(34)   for $j = 1, 2$,

$$\| Tg^{s_{i_1} - s_k} | E_{x_0}^j \| = \| Tf^{s_{i_1} - s_k} | E_{f^{s_k}(p_0)}^j \|$$
$$\| Tg^{s_{i_n} - t_{i_{n-1}}} | E_{x_{n-1}}^j \| = \| Tf^{s_{i_n} - t_{i_{n-1}}} | E_{f^{t_{i_{n-1}}}(p_0)}^j \|$$

for all $2 \leq n \leq w$, and

$$\| Tg^{t_k - t_{i_w}} | E_{x_w}^j \| = \| Tf^{t_k - t_{i_w}} | E_{f^{t_{i_w}}(p_0)}^j \| .$$

From the construction of s_i, t_i, it is clear that

$$s_{i_n} - t_{i_n} \geq 4N_0 \qquad \text{for all } 2 \leq n \leq w ,$$

and

$$t_k - t_{i_w} \geq 4N_0 .$$

And as we observed above, $s_{i_1} - s_k > N$. If we take N large so that $c\lambda^N < 1$, then from (2.3) (ii), (34) and the hypothesis of N_0 in (10.1) (i.e. $c\lambda^{N_0} \leq 1$), we have,

$$\| Tg^{\pi(x_0)} | E_{x_0}^2 \| \cdot \| Tg^{\pi(x_0)} | E_{x_0}^1 \|^{-1} < 1 ,$$

where $\pi(x_0)$ denotes the period. So, from (2.4), we have that

$$E_{x_0}^u(g) = E_{x_0}^1 , \qquad E_{x_0}^s(g) = E_{x_0}^2 ,$$

and moreover,

(35) $$\| Tg^{\pi(x_0)} | E_{x_0}^1 \| \geq (\lambda^{-1})^{\pi(x_0)} .$$

From the hypothesis of induction,

$$\| Tf^{t_{i_n} - s_{i_n}} | E_{f^{s_{i_n}}(p_0)}^1 \| \geq (\lambda^{-1})^{t_{i_n} - s_{i_n}}$$

for all $1 \leq n \leq w$. Therefore, from (34), (35) and the fact that dim $E^1 = 1$, we get (viii) for k.

This completes the proof of (15.3) and so (11.2) Main lemma.

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