A SUMMATION FORMULA INVOLVING $\sigma_k(n)$, $k > 1$

C. NASIM

1. Introduction. The existence of certain formulae analogous to Poisson's summation formula (**9**, pp. 60–64),

$$\beta^{1/2}\left\{\tfrac{1}{2}F_c(0) + \sum_{n=0}^{\infty} F_c(n\beta)\right\} = \alpha^{1/2}\left\{\tfrac{1}{2}f(0) + \sum_{n=0}^{\infty} f(n\alpha)\right\},$$

where $\alpha\beta = 2\pi$, $\alpha > 0$, and $F_c(x)$ is the Fourier cosine transform of $f(x)$, but involving number-theoretic functions as coefficients, was first demonstrated by Voronoï (**10**) in 1904. He proved that

$$\sum_{n>a}^{n\leq b} \tau(n)f(n) = \int_a^b f(u)R(u)\,du + \tfrac{1}{2}\tau(b)f(b) - \tfrac{1}{2}\tau(a)f(a) + \sum_{n=1}^{\infty} \tau(n) \int_a^b f(u)\alpha(nu)\,du,$$

where $\tau(n)$ is an arithmetic function, $f(x)$ is continuous in (a, b) and $\alpha(x)$ and $R(x)$ are analytic functions dependent on $\tau(n)$. Later, numerous papers were published by various authors giving formulae of this type involving $d(n)$, the number of divisors of n (**3**), and $r_p(n)$, the number of ways of expressing n as the sum of p squares of integers (**8**).

In 1937, Ferrar (**4**) developed a general theory of summation formulae, using complex analysis. Around that time, Guinand (**5**) also published papers where he developed the general theory from a different point of view. He applied the theory of mean convergence for the transforms of class $L^2(0, \infty)$. Later in 1950, Bochner (**1**) gave a general summation formula. However, these theories failed to give a satisfactory form of the summation formula with coefficients $\sigma_k(n)$, $k > 1$, where $\sigma_k(n)$ is the sum of kth powers of the divisors of n, although in 1939, Guinand (**5**), gave a formula involving $\sigma_k(n)$, when $0 < |k| < 1$. The difficulties arose from the divergence of certain integrals at the origin. In the present paper, methods are developed to overcome these difficulties and a summation formula with $\sigma_k(n)$ as coefficients is proved for $k > 1$. The main result gives a relation between the sums $\sum \sigma_k(n)n^{-k/2}f(n)$ and $\sum \sigma_k(n)n^{-k/2}g(n)$, where $f(x)$ and $g(x)$ are Hankel transforms and $k > 1$. It is unnecessary to consider negative k separately since

$$\sigma_{-k}(n) = \sum_{lm=n} m^{-k} = \sum_{lm=n} l^k(lm)^{-k} = n^{-k}\sum_{lm=n} l^k,$$

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and hence $\sigma_k(n)n^{-k/2}$ is unchanged when k is replaced by $-k$. The case $k = 1$ shall be given elsewhere, since it presents special difficulties at the origin.

2. The kernel. Let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, $R(s) > 1$, be the Riemann zeta-function, and let

$$K(s) = \frac{\psi(1-s)}{\psi(s)}, \quad \text{where } \psi(s) = \zeta(s - k/2)\zeta(s + k/2), \quad k \geqq 0.$$

Then define

$$A_{k+1}(x) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{1/2-iT}^{1/2+iT} \frac{K(s)}{k/2+1-s} x^{1-s}\, ds. \tag{2.1}$$

Now $K(s)K(1-s) = 1$ and $|K(1/2+it)| = 1$, hence $K(s)/(k/2+1-s)$, the Mellin transform of $A_{k+1}(x)/x$, belongs to $L^2(1/2 - i\infty, 1/2 + i\infty)$. Further, the integral in (2.1) exists in the mean square and $A_{k+1}(x)/x \in L^2(0, \infty)$; see (**9**, § 8.5). We shall call $A_{k+1}(x)$ the truncated Hankel kernel of order $k + 1$, since it is expressed in terms of truncated Bessel functions of order $k + 1$. By using the functional equation of $\zeta(s)$, we can write

$$A_{k+1}(x) = \frac{-1}{\pi i} \int_{1/2-i\infty}^{1/2+i\infty} (2\pi)^{-2s}\Gamma(s + k/2)\Gamma(s - k/2 - 1) \times \{\cos \pi s + \cos \pi k/2\} x^{1-s}\, ds. \tag{2.2}$$

Considering Mellin's inversion formulae of the Bessel functions $J_{k+1}(x)$, $Y_{k+1}(x)$, and $K_{k+1}(x)$ and shifting their lines of integration to $R(s) = 1/2$, the integral in (2.2) can be evaluated to yield

$$\begin{aligned} x^{-1/2}A_{k+1}(x) = {} & -\sin \tfrac{1}{2}\pi k J_{k+1}(4\pi x^{1/2}) \\ & - \cos \tfrac{1}{2}\pi k \left\{ Y_{k+1}(4\pi x^{1/2}) + \frac{2}{\pi} K_{k+1}(4\pi x^{1/2}) \right\} \\ & - \frac{2}{\pi} \cos \tfrac{1}{2}\pi k \sum_{n=0}^{[k/4+1/4]} \frac{\Gamma(k+1-2n)}{n!} (2\pi x^{1/2})^{n-k-1}. \end{aligned}$$

Now put

$$F_{k+3}(x) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{1/2-iT}^{1/2+iT} \frac{K(s)(k/2+s)}{(k/2+1-s)(k/2+2-s)} x^{1-s}\, ds.$$

Then

$$F_{k+3}(x) = (k+2)x^{-(k/2+1)} \int_0^x A_{k+1}(t)t^{k/2}\, dt - A_{k+1}(x).$$

Repeating this process λ times, we obtain:

$$F_{k+2\lambda+3}(x) = (k + 2\lambda + 2)x^{-(k+2\lambda+2)/2} \int_0^x F_{k+2\lambda+1}(t)t^{(k+2\lambda)/2}\, dt - F_{k+2\lambda+1}(x),$$

where

$$F_{k+2\lambda+3}(x) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{1/2-iT}^{1/2+iT} \frac{\Gamma(k/2+s+\lambda+1)\Gamma(k/2-s+1)}{\Gamma(k/2+s)\Gamma(k/2-s+\lambda+3)} \times K(s)x^{1-s}\, ds. \tag{2.3}$$

On the line $s = 1/2 + it$, by Sterling's approximation of $\Gamma(s)$,

$$\left|\frac{\Gamma(k/2 + s + \lambda + 1)\Gamma(k/2 - s + 1)}{\Gamma(k/2 + s)\Gamma(k/2 - s + \lambda + 3)} K(s) = O(t^{-1}),\right.$$

and therefore belongs to $L^2(1/2 - i\infty, 1/2 + i\infty)$, when integrated with respect to t. Hence, the integral in (2.3) converges in mean square and

$$\frac{F_{k+2\lambda+3}(x)}{x} \in L^2(0, \infty).$$

Furthermore, $F_{k+2\lambda+3}(x)$ is a Hankel kernel of order $k + 2\lambda + 3$. By considering Mellin's inversion formulae of the Bessel functions $J_{k+2\lambda+3}(x)$, $Y_{k+2\lambda+3}(x)$, and $K_{k+2\lambda+3}(x)$, and shifting their line of integration to $R(s) = 1/2$, the integral in (2.3) can be evaluated. This yields an expression for $F_{k+2\lambda+3}(x)$ in terms of truncated Bessel functions of order $k + 2\lambda + 3$.

We now define $\chi_k(x)$ by

$$x^{k/2}A_{k+1}(x) = \int_0^x t^{k/2}\chi_k(t)\,dt, \qquad k > 0,$$

whence

$$\begin{aligned}(2.4)\quad \chi_k(x) = {}& -2\pi \sin \tfrac{1}{2}\pi k J_k(4\pi x^{1/2}) \\ & - 2\pi \cos \tfrac{1}{2}\pi k\left\{ Y_k(4\pi x^{1/2}) - \frac{2}{\pi} K_k(4\pi x^{1/2})\right\} \\ & - \frac{2}{\pi} \cos \tfrac{1}{2}\pi k \sum_{n=0}^{[k/4+1/4]} \frac{\Gamma(k+1-2n)}{(2n-1)} (2\pi)^{4n-k-1} x^{2n-k/2-1}.\end{aligned}$$

We find that $\chi_k(x)$ belongs to a class of kernels, D_k^2, defined by Miller (**7**), since the following conditions are satisfied:

(1) There is defined $K(s) = \psi(1 - s)/\psi(s)$, where

$$\psi(s) = \zeta(s - k/2)\zeta(s + k/2),$$

with the properties $K(s)K(1 - s) = 1$ and $|K(1/2 + it)| = 1$;

(2) Let $W_{k+2\lambda+2}(x)$ be defined by

$$F_{k+2\lambda+3}(x) = x^{-(k/2+2\lambda+2)} \int_0^x t^{(k+2\lambda+2)/2} W_{k+2\lambda+2}(t)\,dt.$$

Then $W_{k+2\lambda+2}(x)$ is bounded and continuous in $(0, \infty)$ and

$$\int_x^\infty \frac{W_{k+2\lambda+2}(x)}{t}\,dt = O(x^{-3/4}) \quad \text{as } x \to \infty;$$

(3) The function $A_{k+1}(x) = O(x^{-1/4})$ as $x \to \infty$.

Next we define a class of functions $G_\lambda^2(0, \infty)$, which was first defined by Guinand (**6**) and Miller (**7**).

Definition. A function $f(x)$ belongs to $G_\lambda^2(0, \infty)$ if

(i) there exists almost everywhere in $(0, \infty)$ a function $f^{(\lambda)}(x)$, where $f^{(\lambda)}(x)$

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denotes the λth derivative of $f(x)$ (a part from a factor $(-1)^\lambda$) such that

$$f(x) = \frac{1}{\Gamma(\lambda)} \int_x^\infty (t-x)^{\lambda-1} f^{(\lambda)}(t)\, dt$$

holds everywhere in $x > 0$,

(ii) $x^\lambda f^{(\lambda)}(x) \in L^2(0, \infty)$.

It can be shown that if $f(x) \in G_\lambda^2(0, \infty)$, then:

(i) $f(x)$ is continuous and approaches zero as $x \to \infty$,

(ii) $f(x) \in L^2(0, \infty)$, and

(iii) $x^{\xi+1/2} f^{(\xi)}(x) \to 0$ as $x \to 0$ or ∞, $0 \leqq \xi < \lambda$.

3. Preliminary lemmas. The following result is due to Miller (**7**).

LEMMA 3.1. *Let $\chi_k(x) \in D_k^2$. Then for a function $f(x) \in G_\lambda^2$, $\lambda > 1/2$, there exists $g(x)$, defined by*

$$g(x) = \int_{\to 0}^{\to\infty} f(t)\chi_k(xt)\, dt, \qquad x > 0,$$

also belonging to $G_\lambda^2(0, \infty)$. Furthermore,

$$f(x) = \int_{\to 0}^{\to\infty} g(t)\chi_k(xt)\, dt, \qquad x > 0.$$

LEMMA 3.2. *If the functions $f(x)$, $g(x)$, and $\chi_k(x)$ satisfy the conditions of Lemma 3.1, then*

$$F(x) = x^{1+k/2+\lambda}\left(\frac{d}{dx}\right)^{\lambda+1} \{x^{-k/2} f(x)\} \quad \text{and}$$

$$G(x) = x^{1+k/2+\lambda}\left(\frac{d}{dx}\right)^{\lambda+1} \{x^{-k/2} g(x)\} \tag{3.1}$$

are transforms of $L^2(0, \infty)$, with respect to the Hankel kernel $F_{k+2\lambda+3}(x)$.

Proof. Let $\mathfrak{F}(s)$ and $\mathfrak{G}(s)$ denote the Mellin transforms of $f(x)$ and $g(x)$, respectively. Now $F(x) \in L^2(0, \infty)$, and therefore has a Mellin transform given by

$$F^*(s) = (-1)^\lambda \frac{\Gamma(s + k/2 + \lambda + 1)}{\Gamma(s + k/2)} \mathfrak{F}(s),$$

belonging to $L^2(1/2 - i\infty, 1/2 + i\infty)$. By the Parseval Theorem for Mellin transforms,

$$\begin{aligned} x^{1+k/2+\lambda} &\int_0^\infty t^{-1} F(t) F_{k+2\lambda+3}(xt)\, dt \\ &= x^{1+k/2+\lambda} \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F^*(1-s) \frac{\Gamma(k/2+s+\lambda+1)\Gamma(k/2-s+1)}{\Gamma(k/2+s)\Gamma(k/2-s+\lambda+3)} \\ &\qquad\qquad \times K(s) x^{1-s}\, ds \\ &= \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\mathfrak{F}(1-s)K(s)\Gamma(k/2+s+\lambda+1)}{(k/2-s+\lambda+2)\Gamma(k/2+s)} x^{k/2-s+\lambda+2}\, ds \\ &= \int_0^x t^{k/2+\lambda+1} G(t)\, dt, \end{aligned}$$

since $F(1-s)K(s) = G(s)$, and the Mellin transform of $G(x)$ is

$$\frac{(-1)^\lambda \Gamma(s+k/2+\lambda+1)}{\Gamma(s+k/2)} G(s).$$

A similar relation with $F(x)$ and $G(x)$ interchanged can also be established likewise. Hence, the theorem follows.

Let

$$(3.2) \quad \phi(x) = \left\{\frac{1}{\Gamma(\lambda+1)} \sum_{n \leqq x} \sigma_k(n)(x-n)^\lambda - \frac{\zeta(1+k)\Gamma(1+k)}{\Gamma(2+k+\lambda)} x^{1+k+\lambda}\right\} \times x^{-(1+k/2+\lambda)}$$
$$= D_\lambda(x) x^{-(1+k/2+\lambda)}.$$

It is known that (**11**)

$$D_\lambda(x) = O(x^{1+\lambda}) \quad \text{as } x \to \infty \text{ when } \lambda > k + 1/2, \qquad k > 0.$$

Therefore,

$$\phi(x) = O(x^{-k/2}) \quad \text{as } x \to \infty,$$
$$= O(x^{k/2}) \quad \text{as } x \to 0.$$

Hence, $\phi(x) \in L^2(0, \infty)$ when $\lambda > k + 1/2$, $k > 1$, and therefore has a Mellin transform $\Phi(s) \in L^2(1/2 - i\infty, 1/2 + i\infty)$, and

$$\Phi(s) = \int_0^\infty \phi(x) x^{s-1}\, dx;$$

the integral converges for $-k/2 < R(s) < k/2$, $k > 1$, hence includes the line $R(s) = 1/2$.

$$\Phi(s) = \int_0^\infty D_\lambda(x) x^{s-k/2-\lambda-2}\, dx - \frac{\zeta(1+k)\Gamma(1+k)}{\Gamma(2+k+\lambda)} \int_0^1 x^{s+k/2-1}\, dx$$
$$= \int_0^\infty D_\lambda(x) x^{s-k/2-\lambda-2}\, dx - \frac{\zeta(1+k)\Gamma(1+k)}{\Gamma(2+k+\lambda)} \frac{1}{s+k/2}.$$

The integral converges for $R(s) < k/2$, and this yields an analytic continuation of $\Phi(s)$ into $R(s) < -k/2$ and in this region

$$\Phi(s) = \frac{1}{\Gamma(\lambda+1)} \int_1^\infty \sum_{n \leqq x} \sigma_k(n)(x-n)^\lambda x^{s-k/2-\lambda-2}\, dx$$
$$- \frac{\zeta(1+k)\Gamma(1+k)}{\Gamma(2+k+\lambda)} \int_1^\infty x^{s+k/2-1}\, dx - \frac{\zeta(1+k)\Gamma(1+k)}{\Gamma(2+k+\lambda)} \frac{1}{s+k/2}.$$

The last two terms cancel, and we obtain:

$$(3.3)\qquad \Phi(s) = \frac{1}{\Gamma(\lambda+1)} \int_1^\infty \sum_{n\leq x} \sigma_k(n)(x-n)^\lambda x^{s-k/2-\lambda-2}\,dx$$
$$= \frac{1}{\Gamma(\lambda)} \int_1^\infty x^{s-k/2-\lambda-2}\,dx \int_1^x \sum_{n\leq t} \sigma_k(n)(x-t)^{\lambda-1}\,dt$$
$$= \frac{1}{\Gamma(\lambda)} \int_1^\infty \sum_{n\leq t} \sigma_k(n)\,dt \int_t^\infty (x-t)^{\lambda-1} x^{s-k/2-\lambda-2}\,dx$$
$$= \frac{\Gamma(2+k/2-s)}{\Gamma(2+k/2+\lambda-s)} \int_1^\infty \sum_{n\leq t} \sigma_k(n) t^{s-k/2-2}\,dt$$
$$= \frac{\Gamma(1+k/2-s)}{\Gamma(2+k/2+\lambda-s)} \sum_{n=1}^\infty \sigma_k(n) n^{s-k/2-1}$$
$$= \frac{\Gamma(1+k/2-s)}{\Gamma(2+k/2+\lambda-s)} \zeta(1+k/2-s)\zeta(1-k/2-s).$$

Consider the integral

$$x^{1+k/2+\lambda} \int_0^\infty t^{-1}\phi(t) F_{k+2\lambda+3}(xt)\,dt.$$

By Parseval's theorem for Mellin transforms of L^2-functions, the above integral is equal to

$$x^{1+k/2+\lambda} \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \Phi(1-s) \frac{\Gamma(k/2+s+\lambda+1)\Gamma(k/2-s+1)}{\Gamma(k/2+s)\Gamma(k/2-s+\lambda+3)} K(s) x^{1-s}\,ds$$
$$= \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\zeta(1-s-k/2)\zeta(1-s+k/2)\Gamma(1+k/2-s)}{\Gamma(2+k/2+\lambda-s)(2+k/2+\lambda-s)} x^{2+k/2+\lambda-s}\,ds.$$

Further,

$$K(s) = \frac{\zeta(1-s-k/2)\zeta(1-s+k/2)}{\zeta(s-k/2)\zeta(s+k/2)}$$
$$= \int_0^x \phi(t) t^{1+k/2+\lambda}\,dt.$$

Thus, we have the following result.

LEMMA 3.3. *Let $\phi(x)$ be defined as in* (3.2). *Then $\phi(x)$ is self-reciprocal with respect to the kernel $F_{k+2\lambda+3}(x)$, given by* (2.5).

This lemma is similar to a lemma of Busbridge (**2**). Next consider the function

$$(3.4)\qquad L(x) = \left(\cos 2\pi x - \sum_{n=0}^{l} \frac{(-1)^n(2\pi x)^{2n}}{(2n)!}\right) x^{-(1+k/2+\lambda)}$$
$$= q(x) x^{-(1+k/2+\lambda)}, \quad \text{say},$$
$$= O(x^{2l-k/2+\lambda+1}) \quad \text{as } x \to 0,$$
$$= O(x^{2l-k/2-\lambda-1}) \quad \text{as } x \to \infty.$$

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Therefore, $L(x) \in L^2(0, \infty)$, whenever

(i) $(k + 2\lambda - 3)/4 < l < (k + 2\lambda + 1)/4$,

(ii) λ is a positive odd integer, and

(iii) $k \neq 4n + 1, n = 0, 1, \ldots$.

Let $H(s)$ denote the Mellin transform of $L(x)$. Then

$$H(s) = \int_0^\infty L(x)x^{s-1}\,dx.$$

The integral converges for $k/2 - 2l + \lambda - 1 < R(s) < k/2 - 2l + \lambda + 1$, this range includes the line $R(s) = 1/2$. Thus,

$$H(s) = \int_0^\infty q(x)x^{s-k/2-\lambda-2}\,dx.$$

Integrating by parts $2l + 1$ times, we obtain:

$$H(s) = \left[\sum_{r=0}^{2l} \left(\frac{d}{dx}\right)^r q(x) \right.$$
$$\left. \times \frac{x^{s-k/2-\lambda+r-1}}{(s - k/2 - \lambda - 1)(s - k/2 - \lambda)\ldots(s - k/2 - \lambda + r - 1)}\right]_0^\infty$$
$$+ \frac{(-1)^l(2\pi)^{2l+1}}{(s - k/2 - \lambda - 1)\ldots(s - k/2 - \lambda + 2l - 1)} \int_0^\infty \sin 2\pi x\, x^{s-k/2-\lambda+2l-1}\,dx.$$

The integrated terms at the lower limit are

$$O(x^{2l-k/2-\lambda+s+1}) = 0 \quad \text{as } x \to 0, \text{ for } R(s) > k/2 + \lambda - 2l - 1,$$

and at the upper limit the integrated terms are

$$O(x^{2l-k/2-\lambda+s-1}) \to 0 \quad \text{as } x \to \infty, \text{ for } R(s) < k/2 + \lambda - 2l + 1.$$

Thus, the integrated terms vanish at both the limits. Now the integral can be evaluated to yield:

$$H(s) = \frac{(2\pi)^{k/2+\lambda-s+1} \sin \frac{1}{2}\pi(s - k/2 - \lambda)\Gamma(s - k/2 - \lambda + 2l)}{(s - k/2 - \lambda - 1)(s - k/2 - \lambda)\ldots(s - k/2 - \lambda + 2l - 1)}$$
$$= (2\pi)^{k/2+\lambda-s+1} \sin \tfrac{1}{2}\pi(s - k/2 - \lambda)\Gamma(s - k/2 - \lambda - 1).$$

Now apply Parseval's theorem for Mellin transforms of L^2-functions to $L(x)$ and $x^{-1}F_{k+2\lambda+3}(x)$; we then obtain:

$$x^{1+k/2+\lambda} \int_0^\infty t^{-1}h(t)F_{k+2\lambda+3}(xt)\,dt$$
$$= x^{1+k/2+\lambda} \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} H(1 - s) \frac{\Gamma(k/2 + \lambda + s + 1)\Gamma(k/2 - s + 1)}{\Gamma(s + k/2)\Gamma(k/2 + \lambda - s + 3)}$$
$$\times K(s)x^{1-s}\,ds.$$

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Now using the definition of $K(s)$ and the functional equation

$$\Gamma(z)\Gamma(1-z) = \pi \csc \pi z$$

and the fact that λ is odd, the above becomes

$$\frac{-1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} (2\pi)^{1+k/2+\lambda-s}\Gamma(s-k/2-\lambda-1)\sin \tfrac{1}{2}\pi(k/2+\lambda-s) \times \frac{x^{k/2+\lambda-s+2}}{k/2+\lambda-s+2}\,ds = \int_0^x L(t)t^{k/2+\lambda+1}\,dt,$$

by Parseval's theorem, since the Mellin transform of the function $t^{k/2+\lambda+1}$, $t < x$, is given by $x^{s+k/2+\lambda+1}/(s+k/2+\lambda+1)$. Thus, we have the following result.

LEMMA 3.4. *Let $L(x)$ be defined as in* (3.4). *Then $L(x)$ is self-reciprocal with respect to the kernel $F_{k+2\lambda+3}(x)$, given by* (2.5).

4. The main theorem (Theorem 4.1). Let

$$\psi(x) = \phi(x) - (2\pi)^{-(\lambda+1)}(-1)^{(\lambda+1)/2}\zeta(1-k)L(x),$$

where $\phi(x)$ and $L(x)$ are defined by (3.2) and (3.4), respectively. By Lemmas 3.3 and 3.4, $\psi(x) \in L^2(0,\infty)$ and is self-reciprocal with respect to $F_{k+2\lambda+3}(x)$, whenever

(i) $k \neq 4n+1$, $n = 0, 1, 2, \ldots$,

(ii) $1 < k < \lambda - 1/2$,

(iii) λ is an odd integer, and

(iv) $l = [(2\lambda+k+1)/4]$. Here the notation $[p]$ stands for the greatest integer less than p. Now let

$$\Delta_0(x) = \sum_{n\leqq x}\sigma_k(n) - \frac{\zeta(1+k)}{1+k}x^{1+k} - \frac{1}{2\pi}\zeta(1-k) \times \left\{\sin 2\pi x - \sum_{n=0}^{[(k-1)/4]}\frac{(-1)^n(2\pi x)^{2n+1}}{(2n+1)!}\right\}.$$

If $\Delta_\lambda(x)$ denotes the λth integral of $\Delta_0(x)$, then

$$\psi(x) = \Delta_\lambda(x)x^{-(1+k/2+\lambda)}.$$

Let $f(x)$ and $g(x) \in G_{\lambda+\tau}^2$; then these certainly belong to G_λ^2, for $\tau > 0$; hence, Lemma 3.1 could be applied.

Applying Parseval's theorem for the pairs of $F_{k+2\lambda+3}$-transforms, $\psi(x)$, $\psi(x)$ and $F(x)$, $G(x)$, the latter defined in (3.1), we have:

$$\int_0^\infty \psi(x)F(x)\,dx = \int_0^\infty \psi(x)G(x)\,dx.$$

Or,

$$\text{(4.1)}\quad \int_0^\infty \Delta_\lambda(x)\left(\frac{d}{dx}\right)^{\lambda+1}\{x^{-k/2}f(x)\}\,dx = \int_0^\infty \Delta_\lambda(x)\left(\frac{d}{dx}\right)^{\lambda+1}\{x^{-k/2}g(x)\}\,dx.$$

The left-hand side can be written as

$$\lim_{N\to\infty}\int_0^N \Delta_\lambda(x)D^{(\lambda+1)}\{x^{-k/2}f(x)\}\,dx,$$

where $D = d/dx$. Integrating by parts $\lambda + 1$ times, the above integral yields:

$$(4.2)\quad \lim_{N\to\infty}\left\{\left[\sum_{r=0}^{\lambda}(-1)^r\Delta_{\lambda-r}(x)D^{(\lambda-r)}\{x^{-k/2}f(x)\}\right]_0^N + (-1)^{\lambda+1}\int_0^N x^{-k/2}f(x)d(\Delta_0(x))\right\}.$$

Since $f(x) \in G_{\lambda+\tau}{}^2$, we have:

$$f^{(n)}(x) = O(x^{-n-1/2}) \quad \text{as } x \to 0 \text{ or } \infty, \text{ if } 0 \leqq n < \lambda + \tau,$$

and

$$D^{(\lambda-r)}\{x^{-k/2}f(x)\} = O(x^{-k/2-\lambda+r-1/2}) \quad \text{as } x \to 0 \text{ or } \infty.$$

Furthermore,

$$\Delta_{\lambda-r}(x) = O(x^{1+k+\lambda-r}) + O(x^{2l-r+2}) \quad \text{as } x \to 0.$$

Hence, at the lower limit, the integrated terms in (4.2) are

$O(x^{k/2+1/2}) + O(x^{2l-k/2-\lambda+3/2}) = 0$ as $x \to 0$ for $k > 0$ and $l > (k + 2\lambda - 3)/4$.

We shall now show that the integrated terms in (4.2) at the upper limit, $x = N$, are limitable by Riesz means (R, N, τ) to zero as $N \to \infty$ for a large τ. Let

$$S(N) = \sum_{r=0}^{\lambda}(-1)^r\Delta_{\lambda-r}(N)D^{(\lambda-r)}\{N^{-k/2}f(N)\}.$$

It is required to show that

$$\lim_{N\to\infty}\tau N^{-\tau}\int_0^N S(t)(N-t)^{\tau-1}\,dt = 0.$$

The left-hand side is

$$\sum_{r=0}^{\lambda}(-1)^r\tau\lim_{N\to\infty}N^{-\tau}\int_0^N \Delta_{\lambda-r}(t)D^{(\lambda-r)}\{t^{-k/2}f(t)\}(N-t)^{\tau-1}\,dt.$$

Integrating by parts $\tau - 1$ times, we obtain:

$$(4.4)\quad \sum_{r=0}^{\lambda}(-1)^r\tau\lim_{N\to\infty}N^{-\tau}\left\{\left[\sum_{m=1}^{\tau-1}(-1)^{m+1}\Delta_{\lambda-r+m}(t)D^{(m-1)} \times [(N-t)^{\tau-1}D^{(\lambda-r)}\{t^{-k/2}f(t)\}]\right]_0^N + (-1)^{\tau+1}\int_0^N \Delta_{\lambda-r+\tau-1}(t)D^{(\tau-1)}[(N-t)^{\tau-1}D^{(\lambda-r)}\{t^{-k/2}f(t)\}]\,dt\right\}.$$

Now $D^{(m-1)}[(N-t)^{\tau-1}D^{(\lambda-r)}\{t^{-k/2}f(t)\}]$ can be expressed in terms of finite series of the form

$$(N-t)^{\tau-1-n}D^{(\lambda-r+m-1-n)}\{t^{-k/2}f(t)\},$$

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where $0 \leqq n \leqq m-1, 0 \leqq r \leqq \lambda$, and $1 \leqq m \leqq \tau - 1$, which vanishes at $t = N$. Furthermore, this contributes a term of the order

$$N^{\tau-n-1}O(t^{n-k/2-\lambda+r-m+1/2}) \quad \text{as } t \to 0,$$

from (4.3).

From the definition of $\Delta_\lambda(t)$, we have:

$$\Delta_{\lambda-r+m}(t) = O(t^{k+\lambda-r+m+1}) + O(t^{2l-r+m+2}) \quad \text{as } t \to 0.$$

Therefore, the integrated terms in (4.4), at the lower limit, are

$$\lim_{N\to\infty} \tau N^{-(n+1)}\{O(t^{k/2+n+3/2}) + O(t^{2l-k/2-\lambda+n+5/2})\} = 0,$$

as $t \to 0$, where $0 \leqq n \leqq m-1$, $k > 0$, and $l > (k + 2\lambda - 5)/4$. Now the expression (4.4) reduces to

$$\sum_{r=0}^{\lambda} (-1)^{r+\tau+1}\tau \lim_{N\to\infty} N^{-\tau}\int_0^N \Delta_{\lambda-r+\tau-1}(t)D^{(\tau-1)}[(N-t)^{\tau-1}D^{(\lambda-r)}\{t^{-k/2}f(t)\}]\,dt.$$

Splitting the range of integration into $(0, \delta)$, (δ, N), $\delta > 0$, and writing $D^{(\tau-1)}[(N-t)^{\tau-1}D^{(\lambda-r)}\{t^{-k/2}f(t)\}]$ as a sum of finite series of the form $(N-t)^nD^{(\lambda-r+n)}t^{-k/2}f(t)$, where $0 \leqq n \leqq \tau - 1$, the above integral can be written as

$$\text{(4.5)} \qquad \lim_{N\to\infty} N^{-\tau}\left(\int_0^\Delta + \int_0^N\right)\Delta_{\lambda-r+\tau-1}(t)(N-t)^nD^{(\lambda-r+n)}\{t^{-k/2}f(t)\}\,dt,$$

where $0 \leqq r \leqq \lambda, 0 \leqq n \leqq \tau - 1$.

Now from (4.3),

$$\text{(4.6)} \qquad D^{(\lambda-r+n)}\{t^{-k/2}f(t)\} = O(t^{-k/2-\lambda+r-n-1/2}) \quad \text{as } t \to 0 \text{ or } \infty$$

and

$$\Delta_{\lambda-r+\tau-1}(t) = O(t^{k+\lambda-r+\tau}) + O(t^{2l-r+\tau+1}) \quad \text{as } t \to 0.$$

Hence, the integral with the range $(0, \delta)$ contributes terms of the type

$$\lim_{N\to\infty} N^{n-\tau}\{O(t^{k/2-n+\tau+1/2}) + O(t^{2l-k/2-\lambda+\tau-n+3/2})\} = 0$$

as $t \to 0$, since $0 \leqq n \leqq \tau - 1$, $k > 0$, and $l > (k + 2\lambda - 2\tau + 2n - 3)/4$. The last condition holds since $l > (k + 2\lambda - 3)/4$.

We know that (**11**)

$$\frac{1}{\Gamma(\lambda+1)}\sum_{n\leqq x}\sigma_k(n)(x-n)^\lambda - \frac{\zeta(1+k)\Gamma(1+k)}{\Gamma(2+k+\lambda)}x^{1+k+\lambda} = O(x^{1+\lambda})$$

as $x \to \infty$ and $0 < k < \lambda - 1/2$. Therefore,

$$\Delta_{\lambda-r+\tau-1}(t) = O(t^{\lambda-r+\tau}) + O(t^{2l-r+\tau-1})$$

as $t \to \infty$ and $0 < k < \lambda - r + \tau - 3/2$, $0 \leqq r \leqq \lambda$. To ensure this when

$r = \lambda$, let $\tau > k + 3/2$ and $k > 1$. Then the integral with the range (δ, N) yields:

$$\lim_{N\to\infty} N^{-\tau} \int_\delta^N \{O(t^{\lambda-r+\tau}) + O(t^{2l-r+\tau-1})\}(N-t)^n D^{(\lambda-r+n)}\{t^{-k/2}f(t)\}\, dt.$$

From (4.6), the above is

$$\lim_{N\to\infty} N^{-\tau} \int_\delta^N \{O(t^{t-k/2-n-1/2}) + O(t^{2l-k/2-\lambda+\tau-n-3/2})\}(N-t)^n\, dt$$

$$= \lim_{N\to\infty} N^{(1-k)/2} O\left\{\int_{\delta/N}^1 x^{\tau-k/2-n-1/2}(1-x)^n\, dx\right\}$$

$$+ \lim_{N\to\infty} N^{2l-k/2-\lambda-1/2} O\left\{\int_{\delta/N}^1 x^{2l-k/2-\lambda+\tau-n-3/2}(1-x)^n\, dx\right\}, \quad 0 \leqq n \leqq \tau - 1.$$

The first integral is:

$$\lim_{N\to\infty} O(N^{1/2-k/2}) \quad \text{if } \tau - k/2 - n + 1/2 > 0,$$

$$\lim_{N\to\infty} O(N^{n-\tau}) \quad \text{if } \tau - k/2 - n + 1/2 < 0,$$

and

$$\lim_{N\to\infty} O(N^{n-\tau}\log N) \quad \text{if } \tau - k/2 - n + 1/2 = 0,$$

all of which vanish for $0 \leqq n \leqq \tau - 1$ and $k > 1$.

The second integral contributes a term which is

$$\lim_{N\to\infty} O(N^{2l-k/2-\lambda-1/2}) \quad \text{if } 2l - k/2 - \lambda + \tau - n - 1/2 > 0,$$

$$\lim_{N\to\infty} O(N^{n-\tau}) \quad \text{if } 2l - k/2 - \lambda + \tau - n - 1/2 < 0,$$

or

$$\lim_{N\to\infty} O(N^{n-\tau}\log N) \quad \text{if } 2l - k/2 - \lambda + \tau - n - 1/2 = 0,$$

which consequently vanishes for $0 \leqq n \leqq \tau - 1$ and $l < (k + 2\lambda + 1)/4$. Hence, the integral with the range (δ, N) in (4.5) vanishes, and thus (4.5) vanishes as well. Ultimately, we have shown that

$$\lim_{N\to\infty} tN^{-\tau} \int_0^N S(t)(N-t)^{\tau-1}\, dt = 0,$$

as required, whenever $\tau > k + 3/2$.

Now the integral in (4.2) yields:

$$\lim_{N\to\infty} tN^{-\tau} \int_0^N (N-t)^{\tau-1}\, dt \int_0^t x^{-k/2} f(x)\, d(\Delta_0(x)).$$

By substituting the value of $\Delta_0(x)$ in it, the above can be written as

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$$\lim_{N\to\infty} tN^{-\tau}\Big[\int_0^N (N-t)^{\tau-1}\,dt \int_0^t x^{-k/2}f(x)\,d\Big(\sum_{n\leqq x}\sigma_k(n)\Big)$$
$$-\zeta(1+k)\int_0^N (N-t)^{\tau-1}\,dt\int_0^t x^{k/2}f(x)\,dx$$
$$-\zeta(1-k)\int_0^N (N-t)^{\tau-1}\,dt\int_0^t x^{-k/2}f(x)$$
$$\times\Big\{\cos 2\pi x-\sum_{n=0}^{[(k-1)/4]}\frac{(-1)^n(2\pi x)^{2n}}{(2n)!}\Big\}\,dx\Big]$$
$$=\lim_{N\to\infty} tN^{-\tau}\Big[\int_0^N x^{-k/2}f(x)\,d\Big(\sum_{n\leqq x}\sigma_k(n)\Big)\int_x^N (N-t)^{\tau-1}\,dt$$
$$-\zeta(1+k)\int_0^N x^{k/2}f(x)\,dx\int_x^N (N-t)^{\tau-1}\,dt-\zeta(1-k)\int_0^N x^{-k/2}f(x)$$
$$\times\Big\{\cos 2\pi x-\sum_{n=0}^{[(k-1)/4]}\frac{(-1)^n(2\pi x)^{2n}}{(2n)!}\Big\}\,dx\int_x^N (N-t)^{\tau-1}\,dt\Big].$$

The inversion is justified because of absolute convergence. Evaluating the integrals with range (x, N) and using the fact that $\sum_{n\leqq x}\sigma_k(n)$ is a step function, and non-decreasing, the above can be written (by Stieltjes integral) as:

$$\lim_{N\to\infty}\Big[\sum_{n=1}^{N}\sigma_k(n)n^{-k/2}f(n)\Big(1-\frac{n}{N}\Big)^\tau-\zeta(1+k)\int_0^N x^{k/2}f(x)\Big(1-\frac{x}{N}\Big)^\tau dx$$
$$-\zeta(1-k)\int_0^N x^{-k/2}f(x)\Big\{\cos 2\pi x-\sum_{n=0}^{[(k-1)/4]}\frac{(-1)^n(2\pi x)^{2n}}{(2n)!}\Big\}\Big(1-\frac{x}{N}\Big)^\tau dx\Big].$$

Treating the right-hand side of (4.1) in the same manner, we obtain a similar expression involving $g(x)$. Thus, we have the following result.

THEOREM 4.1. *Let* $\chi_k(x)$ *be as defined by* (2.4). *If* $f(x)\in G_{\lambda+\tau^2}$, *then there exists* $g(x)\in G_{\lambda+\tau^2}$, *such that*

$$g(x)=\int_{\to 0}^{\to\infty} f(t)\,\chi_k(xt)\,dt,\qquad x>0,$$

and

$$f(x)=\int_{\to 0}^{\to\infty} g(t)\,\chi_k(xt)\,dt,\qquad x>0.$$

Further, if (i) $1<k<\lambda-1/2$, (ii) $k\neq 4n+1$, $n=0,1,\ldots$, (iii) λ *is a positive odd integer, and* (iv) $\tau>k+3/2$, *then*

$$\lim_{N\to\infty}\Big[\sum_{n=1}^{N}\sigma_k(n)n^{-k/2}f(n)\Big(1-\frac{n}{N}\Big)^\tau-\zeta(1+k)\int_0^N x^{k/2}f(x)\Big(1-\frac{x}{N}\Big)^\tau dx$$
$$-\zeta(1-k)\int_0^N x^{-k/2}f(x)\Big\{\cos 2\pi x-\sum_{n=0}^{[(k-1)/4]}\frac{(-1)^n(2\pi x)^{2n}}{(2n)!}\Big\}\Big(1-\frac{x}{N}\Big)^\tau dx\Big]$$
$$=\lim_{N\to\infty}\Big[\sum_{n=1}^{N}\sigma_k(n)n^{-k/2}g(n)\Big(1-\frac{n}{N}\Big)^\tau-\zeta(1+k)\int_0^N x^{k/2}g(x)\Big(1-\frac{x}{N}\Big)^\tau dx$$
$$-\zeta(1-k)\int_0^N x^{-k/2}g(x)\Big\{\cos 2\pi x-\sum_{n=0}^{[(k-1)/4]}\frac{(-1)^n(2\pi x)^{2n}}{(2n)!}\Big\}\Big(1-\frac{x}{N}\Big)^\tau dx\Big].$$

For the case that $k = 4n + 1$, $n = 0, 1, 2, \ldots$, note that $\psi(x) = \phi(x)$. Now using the same technique which has been used to prove the main theorem, for the two pairs of $F_{k+2\lambda+3}$-transforms, $\phi(x)$, $\phi(x)$ and $F(x)$, $G(x)$, we obtain the following result.

THEOREM 4.2. *Let $\chi_k(x)$, $f(x)$, and $g(x)$ be as defined in Theorem* 4.1. *Then if*

(i) $k = 4n + 1$, $n = 1, 2, \ldots$,

(ii) $1 < k < \lambda - 1/2$, *and*

(iii) $\tau > k + 3/2$,

then

$$\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\sigma_k(n)n^{-k/2}f(n)\left(1-\frac{n}{N}\right)^{\tau} - \zeta(1+k)\int_0^N x^{k/2}f(x)\left(1-\frac{x}{N}\right)^{\tau}dx\right\}$$

$$= \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\sigma_k(n)n^{-k/2}g(n)\left(1-\frac{n}{N}\right)^{\tau} - \zeta(1+k)\int_0^N x^{k/2}g(x)\left(1-\frac{x}{N}\right)^{\tau}dx\right\}.$$

5. An example. Let

$$f(x) = x^{k/2}e^{-xy},\ y > 0, \quad \text{and} \quad g(x) = \int_0^{\infty} f(t)\chi_k(xt)\,dt.$$

The kernel $\chi_k(x)$ can be evaluated explicitly from (4.3) when k is an odd integer and is given by

$$(2\pi)(-1)^{(k+1)/2}J_k(4\pi x^{1/2});$$

then

$$g(x) = 2\pi(-1)^{(k+1)/2}\int_0^{\infty} t^{k/2}e^{-ty}J_k(4\pi\sqrt{(xt)})\,dt$$
$$= (2\pi)^{k+1}(-1)^{(k+1)/2}y^{-(1+k)}x^{k/2}e^{-4\pi^2x/y}.$$

The functions $f(x)$ and $g(x)$ satisfy all the conditions of the summation formula which becomes:

$$\sum_{n=1}^{\infty}\sigma_k(n)e^{-ny} - (-1)^{(k+1)/2}(2\pi)^{k+1}y^{-(1+k)}\sum_{n=1}^{\infty}\sigma_k(n)e^{-4\pi^2n/y}$$
$$= \zeta(1+k)\int_0^{\infty}x^ke^{-xy}\,dx + \zeta(1+k)(-1)^{(k+1)/2}(2\pi)^{k+1}y^{-(1+k)}\int_0^{\infty}e^{-4\pi^2x/y}x^k\,dx,$$

whenever k is a positive odd integer. Evaluating the integrals in the above equation, the summation formula can be expressed as

$$\sum_{n=1}^{\infty}\sigma_k(n)e^{-ny} - (-1)^{(k+1)/2}(2\pi)^{k+1}y^{-(1+k)}\sum_{n=1}^{\infty}\sigma_k(n)e^{-4\pi^2n/y}$$
$$= \frac{(-1)^{k+1}}{2(k+1)}B_{k+1} + \frac{(-1)^{(k-1)/2}}{2(k+1)}(2\pi)^{k+1}y^{-(1+k)}B_{k+1},$$

where $B_1, B_2, \ldots$, are Bernoulli's numbers, such that

$$\frac{z}{e^z - 1} = 1 - z/2 + B_1 \frac{z^2}{2!} - B_2 \frac{z^4}{4!} + \ldots .$$

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University of Calgary,
Calgary, Alberta