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METRIC SPACES WITHOUT LARGE CLOSED DISCRETE SETS

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We investigate the structure of those non-separable metric spaces X, and their Stone-Čech compactifications, for which X has no closed discrete subspace of power equal to the weight of X. (Throughout this paper we denote the weight of X—the smallest power of a base for the topology of X—by the symbol $\mathbf{w}X$.)

For a metric space X as above the cardinal number $\mathbf{w}X$ is a sequential cardinal, i.e., $\mathbf{w}X = \sum_{n<\omega}\mathfrak{m}_n$ with $\mathfrak{m}_n < \mathfrak{m}_{n+1}$ for all $n < \omega$. Further, for every sequential cardinal $\mathfrak{m}$ the hedgehog $H(\mathfrak{m})$, consisting of a center p and, for $n < \omega$, $\mathfrak{m}_n$ isolated points at distance $1/n$ from p, is such a space. A more complex example, $H(\mathfrak{m}, K)$, is obtained by replacing the center p by an arbitrary compact metric space K. These spaces almost exhaust the examples: for any such X with weight $\mathfrak{m}$, the subset $K(X)$ of points each of whose neighborhoods has weight $\mathfrak{m}$ is compact, and X contains a dense subset which is a one-to-one uniformly continuous image of $H(\mathfrak{m}, K(X))$ under a map which takes $K(X)$ onto $K(X)$ and which takes isolated points to points of X with neighborhoods of small weight.

As to Stone-Čech compactifications: the space $\beta H(\mathfrak{m})$ is the one-point compactification of the "disjoint union" $\sum_{n<\omega}\beta\mathfrak{m}_n$, the "point at infinity" being the point p; here $\beta\mathfrak{m}_n$ denotes the Stone-Čech compactification of the discrete space of power $\mathfrak{m}_n$. Similarly, $\beta H(\mathfrak{m}, K)$ is obtained by compactifying $\sum_{n<\omega}\beta\mathfrak{m}_n$ with K; and for a general space X as above such that $\mathbf{w}X = \mathfrak{m}$, the space βX is a continuous image of $\beta H(\mathfrak{m}, K(X))$. It follows that every point of $\beta X\backslash X$ is in the closure in βX of a subset of X of power $< \mathfrak{m}$. Thus $|\beta X| = \sum_{n<\omega} \exp \exp \mathfrak{m}_n$, and $\mathbf{w}\beta X = \sum_{n<\omega} \exp \mathfrak{m}_n$. Under the generalized continuum hypothesis, then, $|\beta X| = \mathbf{w}\beta X = \mathbf{w}X$. This is in marked contrast to the situation for a metric space Y which does contain a closed discrete set of power $\mathbf{w}Y$; for such Y, one has

$$|\beta Y| = \exp \exp \mathbf{w}Y \geqq \exp |Y| = \mathbf{w}\beta Y > \mathbf{w}Y.$$

1. Preliminary remarks. The subset S of the topological space X is said to be *discrete* if S is discrete in the relative topology. Clearly, S is closed and discrete if and only if each point of X has a neighborhood meeting S in at

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most one point. A subset S of the metric space X is *ϵ-discrete* (for $\epsilon > 0$) if any two different points of S have distance at least ϵ. A subset S is *metrically discrete* if it is ϵ-discrete for some $\epsilon > 0$. Such a subset is closed and discrete. On the other hand, if S is a closed, discrete subset of the metrizable space X, then according to a well-known theorem of Hausdorff (see, for example, [**14**, Exercise 22.E] for a proof) the "discrete metric" on S (which assigns distance 1 to every two distinct points of S) may be extended to a compatible metric d on all of X.

The *density* of the topological space X, denoted δX, and sometimes called the *density character* of X, is the minimum cardinal of a dense subset of X.

1.1. LEMMA. *For metrizable X, the following cardinals coincide.*

(a) $\mathbf{w}X$;

(b) δX;

(c) *the supremum of the powers of closed discrete sets;*

(d) *the supremum of the powers of metrically discrete sets (with respect to any one fixed compatible metric).*

Proof. This is well-known. For $\mathbf{w}X = \delta X$, see for example [**6**, § 4.1, Theorem (6)]. We sketch the rest of the proof, assuming throughout that $|X| \geqq \omega$ since otherwise the cardinals in question are (obviously) all equal to $|X|$. Let α and β be the numbers in (c) and (d), respectively. Then $\beta \leqq \alpha$ because any metrically discrete set is closed. If S is closed and discrete, and if $\mathscr{B}$ is a base for X such that $|\mathscr{B}| = \mathbf{w}X$, then for $x \in S$ there is $B_x \in \mathscr{B}$ such that $B_x \cap S = \{x\}$ and it follows that

$$|S| = |\{B_x : x \in S\}| \leqq \mathbf{w}X;$$

hence $\alpha \leqq \mathbf{w}X = \delta X$. To prove, finally, the inequality $\delta X \leqq \beta$, we record the following lemma.

1.2. LEMMA. *Let (X, d) be an infinite metric space and let β be as defined above. For $n < \omega$, let D_n be a maximal $1/n$-discrete set, and set $D = \bigcup_{n<\omega} D_n$. Then*

(a) D is dense and $|D| = \delta X$; and

(b) $\delta X = \sum_{n<\omega}|D_n| = \beta$.

Proof. The sets D_n are chosen by Zorn's lemma; their maximality gives (a). Since $|D_n| \leqq \beta$ for all $n < \omega$ we have

$$\delta X \leqq |D| \leqq \textstyle\sum_{n<\omega} |D_n| \leqq \omega \cdot \beta = \beta \leqq \delta X,$$

and hence (b). The proofs of 1.1 and 1.2 are complete.

1.3. PROPOSITION. *Let X be a metrizable space with no closed, discrete subset of power $\mathbf{w}X$. Then $\mathbf{w}X$ is a sequential cardinal.*

Proof. Clearly $|X| \geqq \omega$. With D_n as in Lemma 1.2 and $\mathfrak{m}_n = |D_n|$ we have $\mathbf{w}X = \sum_{n<\omega}\mathfrak{m}_n$ and

$$\mathfrak{m}_n < \mathbf{w}X.$$

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Evidently there is a sequence $\{n_k : k < \omega\}$ such that $\mathfrak{m}_{n_k} < \mathfrak{m}_{n_{k+1}}$ (and $\mathbf{w}X = \sum_{k<\omega}\mathfrak{m}_{n_k}$).

2. Some examples. We construct some simple spaces whose weight is a given sequential cardinal $\mathfrak{m}$, and which have no closed discrete subsets of power $\mathfrak{m}$.

Throughout this section, $\mathfrak{m}$ denotes a sequential cardinal, and we fix $\{\mathfrak{m}_n : n < \omega\}$ such that $\mathfrak{m} = \sum_{n<\omega}\mathfrak{m}_n$ and $\mathfrak{m}_n < \mathfrak{m}_{n+1}$ for all $n < \omega$.

2.1. *The hedgehog* $H(\mathfrak{m})$. Let $\{D_n : n < \omega\}$ be a sequence of disjoint sets such that $|D_n| = \mathfrak{m}_n$ for $n < \omega$, let $p \notin \bigcup_{n<\omega}D_n$, and let X be the set $X = \bigcup_{n<\omega}D_n \cup \{p\}$. Define d on $X \times X$ as follows:

$$d(p, x) = d(x, p) = 1/n \quad \text{for } x \in D_n,\ n < \omega;$$
$$d(x, y) = d(y, x) = 1/n + 1/m \quad \text{for } x \in D_n, y \in D_m,\ n \leqq m < \omega, x \neq y;$$
$$d(x, x) = d(p, p) = 0 \quad \text{for } x \in \bigcup_{n<\omega} D_n.$$

It is readily verified that d is a metric. The resulting metric space, $H(\mathfrak{m})$, has weight and power $\mathfrak{m}$—indeed, the weight of any neighborhood of p is $\mathfrak{m}$—and no closed discrete set of power $\mathfrak{m}$. The proofs are deferred to 2.3 (and are very easy).

The space $H(\mathfrak{m})$ is a closed subset of the "hedgehog with $\mathfrak{m}$ spines" introduced (in different terminology) by Urysohn [**13**, pp. 50-51] and Schmidt [**11**, footnote 4]. See [**6**] or [**14**] for discussions of this space, which consists of $\mathfrak{m}$ unit intervals, the spines, attached together at 0, with distance $d(x, y) = |x - y|$ if x and y lie on the same spine, $d(x, y) = |x| + |y|$ if not.

The space $H(\omega)$ is easily seen to be a convergent sequence. Next, we fatten the hedgehog.

2.2 *The hedgehog* $H(\mathfrak{m}, K, A)$. Let K be a compact metric space with metric ρ, and let A be a dense subspace of K. We erect pairwise disjoint hedgehogs, each isometric to $H(\mathfrak{m})$, over the points of A as follows.

Let $\{H_a : a \in A\}$ be a family of $|A|$ disjoint copies of $H(\mathfrak{m})$, let d_a be the metric in H_a, identify the center of H_a with $a \in A \subset K$, and on this "quotient" of $K \cup \bigcup \{H_a : a \in A\}$ define d as follows:

$$d(x, y) = d_a(x, y) \quad \text{for } x, y \in H_a, a \in A;$$
$$d(x, y) = d_a(x, a) + \rho(a, b) + d_b(b, y) \quad \text{for } x \in H_a, y \in H_b, a, b \in A \text{ and } a \neq b;$$
$$d(p, q) = \rho(p, q) \quad \text{for } p, q \in K.$$

It is readily verified that d is a metric. The resulting metric space, $H(\mathfrak{m}, K, A)$, has the following properties.

(a) $\mathbf{w}H(\mathfrak{m}, K, A) = \mathfrak{m} \cdot |A|$, and $|H(\mathfrak{m}, K, A)| = |K| + \mathfrak{m}$.

(b) Each closed discrete set is metrically discrete, and has power $\leqq \mathfrak{m}_n \cdot |A|$ for some $n < \omega$.

(c) $K = \{x \in H(\mathfrak{m}, K, A)$: the weight of each neighborhood of x is $\geqq \mathfrak{m}\}$, and each point of $H(\mathfrak{m}, K, A)\backslash K$ is isolated.

(d) If $|A| < \mathfrak{m}$, then $\mathbf{w}H(\mathfrak{m}, K, A) = \mathfrak{m}$ and $H(\mathfrak{m}, K, A)$ has no closed discrete subset of power $\mathfrak{m}$.

The space $H(\mathfrak{m}, K, A)$ is easy to visualize, and since the proofs of (a), (b), (c) and (d) are easy we omit them. Apart from a hedge concerning the continuum hypothesis (see the next paragraph), the space $H(\mathfrak{m}, K, A)$ is homeomorphic to the space $H(\mathfrak{m}, K)$ defined in 2.3 (as Theorem 3.6 will show).

We note that $(2^\omega)^\omega = 2^\omega$, while $\mathfrak{m}^\omega > \mathfrak{m}$. (For a proof of this familiar consequence of König's theorem, see for example [**4**, Corollary 1.20].) It follows that if the continuum hypothesis is assumed (and $\mathfrak{m} > \omega$) then $\mathfrak{m} > 2^\omega$, so that (since $|K| \leqq 2^\omega$) there is no dense $A \subset K$ such that $|A| \geqq \mathfrak{m}$; see (a) and (d) above. On the other hand it follows from the theorem of Easton [**5**] that it is consistent with the usual axioms of Zermelo-Fraenkel set theory that there are (sequential) $\mathfrak{m}$ such that $\omega < \mathfrak{m} < 2^\omega$. In this case for any compact K such that $|K| = 2^\omega$ there is dense $A \subset K$ such that $|A| = \mathfrak{m}$, and although $\mathbf{w}H(\mathfrak{m}, K, A) = \mathfrak{m}$ (from (a) above) in this case there is a closed discrete subset S such that $|S| = \mathfrak{m}$; indeed if

$$D_a = \{x \in H_a : d_a(x, a) = 1\}$$

for $a \in A$ then $\bigcup_{a \in A} D_a$ is such a set.

2.3 *The hedgehog* $H(\mathfrak{m}, K)$. Let K be a compact metric space with metric ρ. Let $\{\epsilon_n\}$ be any sequence of positive reals with $\epsilon_n \to 0$, and for each n select a finite ϵ_n-net N_n in K, in such a way that $N_n \subset N_{n+1}$ for all $n < \omega$.

(The statement that N is an ϵ-net in K means that for each $q \in K$ there is $p \in N$ such that $\rho(p, q) < \epsilon$. The family $\{N_n : n < \omega\}$ is defined recursively: N_0 is any maximal ϵ_0-discrete subset of K, and then, N_n having been defined, N_{n+1} is any maximal ϵ_{n+1}-discrete subset of K such that $N_n \subset N_{n+1}$. Maximality makes N_n an ϵ_n-net, and compactness of K makes N_n finite.)

Let $|N_n| = k_n$, write $N_n = \{p_i^n : 1 \leqq i \leqq k_n\}$, and let $\{D_i^n | 1 \leqq i \leqq k_n\}$ be k_n disjoint sets each of power $\mathfrak{m}_n$, with $D_i^m \cap D_j^n = \emptyset$ for $m \neq n$, as well. On the set $K \cup \bigcup_{n,i} D_i^n$ define d as follows:

$$d(x, p) = 1/n + \rho(p, p_i^n) \quad \text{for } x \in D_i^n, p \in K;$$

$$d(p, q) = \rho(p, q) \quad \text{for } p, q \in K;$$

$$d(x, y) = 1/n + \rho(p_i^n, p_j^m) + 1/m \quad \text{for } x \in D_i^n, y \in D_j^m$$

(*i.e.*, we measure down, over, and up).

It is readily verified that d is a metric. The resulting metric space, $H(\mathfrak{m}, K)$, has the following properties.

(a) $\mathbf{w}H(\mathfrak{m}, K) = \mathfrak{m}$, and $|H(\mathfrak{m}, K)| = |K| + \mathfrak{m}$.

(b) Each closed discrete set is metrically discrete, and has power $< \mathfrak{m}$.

(c) $K = \{x \in H(\mathfrak{m}, K)$: the weight of each neighborhood of x is $\mathfrak{m}\}$, and each point of $H(\mathfrak{m}, K)\backslash K$ is isolated.

Proof. Clearly $H(\mathfrak{m}, K)\backslash K$ has power $\mathfrak{m}$, and hence $|H(\mathfrak{m}, K)| = |K| + \mathfrak{m}$. Since $H(\mathfrak{m}, K)\backslash K$ is discrete, it has weight $\mathfrak{m}$, so that

$$\mathbf{w}H(\mathfrak{m}, K) \geqq \mathbf{w}(H(\mathfrak{m}, K)\backslash K) = \mathfrak{m}.$$

For the reverse inequality, we note that $H(\mathfrak{m}, K)\backslash K$, together with a countable dense subset of K, is dense in $H(\mathfrak{m}, K)$; hence

$$\mathbf{w}H(\mathfrak{m}, K) = \delta H(\mathfrak{m}, K) \leqq \mathfrak{m}.$$

If $p \in K$, then any neighborhood of p contains $\mathfrak{m}$ isolated points, and thus has weight $\mathfrak{m}$.

Now let F be a closed, discrete subset of $H(\mathfrak{m}, K)$. Since $|F \cap K| < \omega$, it is enough to show that F is metrically discrete under the additional assumption that $F \cap K = \emptyset$. The function $p \to d(p, F)$ is continuous and never zero on K. Thus there is $n < \omega$ such that $d(p, F) \geqq 1/n$ for all $p \in K$. Now

$$F \subset \bigcup \{D_i^{\,l} : 1 \leqq l \leqq n, 1 \leqq i \leqq k_l\},$$

and since this latter set is $2/n$-discrete and of power

$$\sum\{\mathfrak{m}_l : 1 \leqq l \leqq n\} = \mathfrak{m}_n < \mathfrak{m},$$

the same is true of F.

The proof is complete.

2.4 *Remark.* The space $H(\mathfrak{m}, K)$ apparently depends on the representation of $\mathfrak{m}$ as $\sum_{n<\omega}\mathfrak{m}_n$ (as does $H(\mathfrak{m})$), and the sequence $\{N_n\}$ (and the sequence $\{\epsilon_n\}$). Theorem 3.6 below shows that this dependence is illusory.

3. Structure of general spaces. We show now how an arbitrary metrizable space X with no closed discrete subset of power $\mathbf{w}X$ looks rather like one of the spaces $H(\mathfrak{m}, K)$.

For an infinite cardinal $\mathfrak{n}$, a space X is $\mathfrak{n}$-*compact* if every open cover of X has a subcover of power $< \mathfrak{n}$. It is well-known (see for example [**6**]) that (metrizable) X has $\mathbf{w}X \leqq \mathfrak{n}$ if and only if X is $\mathfrak{n}^+$-compact (where $\mathfrak{n}^+$ denotes the cardinal successor of $\mathfrak{n}$).

3.1. Lemma. *Let X be metrizable. Then X has no closed discrete set of power* $\mathbf{w}X$ *if and only if X is* $\mathbf{w}X$-*compact.*

Proof. If S is a closed, discrete subset of X such that $|S| = \mathbf{w}X$, and if $\{U_x : x \in S\}$ is a family of open subsets of X such that $U_x \cap S = \{x\}$ for $x \in S$, then $\{X\backslash S\} \cup \{U_x : x \in S\}$ is an open cover of X with no subcover of power $< \mathbf{w}X$.

Let $\mathscr{U}$ be an open cover of X with no subcover of power $< \mathbf{w}X$. Since X is paracompact, we may assume without loss of generality that $\mathscr{U}$ is locally finite. We choose $x_U \in U$ for $U \in \mathscr{U}$, and we note that $\{x_U : U \in \mathscr{U}\}$ is closed and discrete in X, and of power $\mathbf{w}X$.

3.2. THEOREM. *Let X be a metrizable space with no closed discrete subset of power $\mathbf{w}X$, and define*

$$K(X) = \{x \in X\colon \textit{the weight of each neighborhood of } x \textit{ is } \mathbf{w}X\}.$$

Then

(a) *$K(X)$ is non-empty and compact; and*

(b) *if $\mathbf{w}X > \omega$, then $K(X)$ is nowhere dense.*

Proof. We write $\mathbf{w}X = \mathfrak{m} = \sum_{n<\omega}\mathfrak{m}_n$ with $\mathfrak{m}_n < \mathfrak{m}_{n+1}$ for $n < \omega$ (using Proposition 1.3).

(a) Suppose first that there is $\mathfrak{k} < \mathfrak{m}$ such that each point has a neighborhood of weight $\leqq \mathfrak{k}$. Then by 3.1, X has a cover $\mathscr{U}$ by open sets of weight $\leqq \mathfrak{k}$, with $|\mathscr{U}| < \mathfrak{m}$. For $U \in \mathscr{U}$, let $\mathscr{B}_U$ be a base for U with $|\mathscr{B}_U| \leqq \mathfrak{k}$; then $\bigcup\{\mathscr{B}_U | \ U \in \mathscr{U}\}$ is a base for X of power $\leqq \mathfrak{k}\cdot |\mathscr{U}| < \mathfrak{m}$, a contradiction. It follows that for each $n < \omega$ there is $x_n \in X$ such that each neighborhood of x_n has weight $\geqq \mathfrak{m}_n$. If the set $\{x_n : n < \omega\}$ is not closed and discrete, then there is $x \in X$ such that every neighborhood of x contains infinitely many of the points x_n; clearly such a point x is an element of $K(X)$. We assume, then, that $\{x_n : n < \omega\}$ is closed and discrete, and we construct by recursion a sequence $\{U_n : n < \omega\}$ of disjoint open sets such that

U_n is a neighborhood of x_n, and the diameter of U_n is $\leqq 1/n$.

Then $\mathbf{w}U_n \geqq \mathfrak{m}_n$, and by Lemma 1.2 (applied to the space U_n) for $n > 0$ there is a closed discrete subset D_n of U_n such that $|D_n| \geqq \mathfrak{m}_{n-1}$. Since $\bigcup_{1\leqq n<\omega}D_n$ is discrete and of power $\mathfrak{m}$, this set has a cluster point x. It is clear that $x \in K(X)$ (and in fact that x is a cluster point of $\{x_n : n < \omega\}$).

If $K(X)$ were not compact, there would be an infinite closed discrete subset $\{x_n : n < \omega\}$ of $K(X)$, and as above there would be a sequence $\{U_n : n < \omega\}$ of disjoint closed sets such that

U_n is a neighborhood of x_n, and the diameter of U_n is $\leqq 1/n$.

Then with D_n as above the set $\bigcup_{n<\omega}D_n$ is closed and discrete of power $\mathfrak{m}$, a contradiction.

(b) If $\operatorname{int}K(X) \neq \emptyset$ then $\mathbf{w}K(X) = \mathfrak{m}$ and we have the contradiction $\omega = \mathbf{w}K(X) = \mathfrak{m} > \omega$.

The proof is complete.

Recall that a space X is *weight-homogeneous* if $\mathbf{w}U = \mathbf{w}X$ for every non-empty open subset U of X.

3.3 COROLLARY. *No nonseparable metrizable space X without a closed discrete set of power $\mathbf{w}X$ is weight-homogeneous.*

Corollary 3.3 also can be derived, somewhat elaborately, as follows:

(a) If $\mathbf{w}X = \mathfrak{m}$, and X has no closed discrete set of power $\mathfrak{m}$, then the completion of X has these properties also.

(b) A complete metric space which is weight-homogeneous and of sequential weight $\sum_{n<\omega}\mathfrak{m}_n$ contains a closed copy of $\prod_{n<\omega}D(\mathfrak{m}_n)$, where $D(\mathfrak{m}_n)$ denotes the discrete space of power $\mathfrak{m}_n$ (see [**11**] or [**12**] for a proof).

(c) $\prod_{n<\omega}D(\mathfrak{m}_n)$ contains a closed discrete set of power $\mathfrak{m}$. (Such a set may be constructed as follows. Assume without loss of generality that $\mathfrak{m}_0 = \omega$ and for $n < \omega$ and $\xi < \mathfrak{m}_n$ define

$$U(n, \xi) = \{p \in \textstyle\prod_{n<\omega}D(\mathfrak{m}_n): p_0 = n, p_n = \xi).$$

Then $\{U(n, \xi) : n < \omega, \xi < \mathfrak{m}_n\}$ is a cover of $\prod_{n<\omega}D(\mathfrak{m}_n)$ by non-empty, pairwise disjoint, open (and closed) sets. The required discrete set is obtained by choosing a point from each of the sets $U(n, \xi)$.)

We note the following consequence of (b) and (c): If $\mathfrak{m}$ is an uncountable cardinal and $\{X_n : n < \omega\}$ is a sequence of metric spaces each of weight $\mathfrak{m}$, then $\prod_{n<\omega}X_n$ contains a closed discrete subset of power $\mathfrak{m} = \mathbf{w}(\prod_{n<\omega}X_n)$. For either there is $n < \omega$ such that X_n contains such a set, or the product of the completions of the spaces X_n is a weight-homogeneous space of sequential weight $\mathfrak{m}$ (so that (b) and hence (c) apply).

3.4. Proposition. *Let X be a metric space with no closed discrete set of power $\mathbf{w}X$. There are a metric space Y and a function $h : Y \to X$ such that*

(a) $\mathbf{w}Y = \mathbf{w}X$;

(b) *each closed discrete subset of Y is metrically discrete, and has power $< \mathbf{w}Y$;*

(c) $K(Y) = K(X)$, *and each element of $Y\backslash K(Y)$ is isolated; and*

(d) *h is a one-to-one uniformly continuous function onto a dense subspace of X.*

Proof. If $\mathbf{w}X = \omega$ we set $Y = X$ and we take for h the identity function on X. We assume in what follows that $\mathbf{w}X = \mathfrak{m} > \omega$.

Let $\mathfrak{m} = \sum_{n<\omega}\mathfrak{m}_n$ with $\mathfrak{m}_n < \mathfrak{m}_{n+1} < \mathfrak{m}$ for $n < \omega$, set $K(X) = K$ and fix a metric d for X.

The set $X_1 = \{x \in X: d(x, K) \geqq 1\}$ is closed in X and hence has no closed discrete set of power $\mathfrak{m}$. Since $X_1 \cap K = \emptyset$, X_1 has no points each of whose neighborhoods has weight $\mathfrak{m}$. From 3.2 (or 3.1) we have $\mathbf{w}X_1 < \mathfrak{m}$, so there is $n_1 < \omega$ such that $\mathbf{w}X_1 \leqq \mathfrak{m}_{n_1}$. Let Y_1 be a dense subset of X_1 of power $\leqq \mathfrak{m}_{n_1}$. (The possibility of choosing $Y_1 = X_1$—and more generally of choosing $Y = X$ and h a function onto X—is discussed in Corollary 3.5 and the remark preceding it.) By the same argument, X_2 has a dense subset Y_2 of power $\leqq \mathfrak{m}_{n_2}$.

By induction, we construct for each k a dense subset Y_k of $\{x \in X: 1/(k-1) > d(x, K) \geqq 1/k\}$ such that $|Y_k| \leqq \mathfrak{m}_{n_k}$.

Evidently the sets Y_k are pairwise disjoint, $\bigcup_{k<\omega}Y_k$ is dense in $X\backslash K$, and $(\bigcup_{k<\omega}Y_k) \cup K$ is dense in X.

It follows that $\sup\{|Y_k|: k < \omega\} = \mathfrak{m}$.

Let Y be the space $(\bigcup_{k<\omega}Y_k) \cup K$ with all points of $\bigcup_{k<\omega}Y_k$ made isolated.

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The following function e is easily seen to be a metric generating this topology.

$$e(y, p) = e(p, y) = d(y, p) + 1/k \quad \text{for } p \in K, y \in Y_k;$$
$$e(y, z) = d(y, z) + 1/k + 1/l \quad \text{for } y \in Y_k, z \in Y_l.$$

Since the neighborhoods in Y of points $p \in K$ are exactly the sets $U \cap Y$ with U a neighborhood of p in X, and since $\sup\{|Y_k|: k < \omega\} = \mathfrak{m}$, every Y-neighborhood of every point of K has weight $\mathfrak{m}$; in particular we have $\mathbf{w}Y = \mathbf{w}X$, which is (a).

With respect to e, the set Y_k is $2/k$-discrete; it follows that the larger set $\bigcup_{l \leqq k} Y_l$ is also $2/k$-discrete.

Now let F be a closed discrete subset of Y. Since $|F \cap K| < \omega$, in order to prove (b) it is enough to show that F is metrically discrete (and that $|F| < \mathfrak{m}$) under the additional assumption that $F \cap K = \emptyset$. We claim that there is $n < \omega$ such that $F \subset \bigcup_{l \leqq n} Y_l$. If the claim fails there are a sequence $\{n_k : k < \omega\}$ with $n_k < n_{k+1}$ for $k < \omega$ and $y_{n_k} \in Y_{n_k}$ such that $y_{n_k} \in F$. For $k < \omega$ there is $p_{n_k} \in K$ such that

$$d(y_{n_k}, p_{n_k}) < 1/(n_k - 1),$$

and from

$$e(y_{n_k}, p_{n_k}) = d(y_{n_k}, p_{n_k}) + 1/n_k$$

it follows that $e(y_{n_k}, p_{n_k}) \to 0$. Since $\{p_{n_k} : k < \omega\} \subset K$ and K is compact, this sequence has a subsequence converging to some point p of K; the corresponding subsequence of $\{y_{n_k} : k < \omega\}$ then also converges to p and we have $p \in F \cap K$, a contradiction. Thus it follows that there is $n < \omega$ such that $F \subset \bigcup_{l \leqq n} Y_l$, so that F is $2/n$-discrete and $|F| \leqq \mathfrak{m}_n < \mathfrak{m}$.

The proof of (b) is complete, and (c) is obvious.

For h in (d) we take the identity function on Y. Then h is continuous because the topology of Y is finer than the topology inherited from X. That h is in fact uniformly continuous follows from the fact that h decreases distances.

Remark. According to Atsuji [**1**] and Rainwater [**10**], the property (in (b), and of the spaces in § 2) that each closed discrete set be metrically discrete is characteristic of those metric spaces for which the metric uniformity is the finest uniformity compatible with the topology.

Remark. If in Proposition 3.4 the sequentially accessible cardinal $\mathfrak{m} = \mathbf{w}X$ has the property that $2^{\mathfrak{k}} < \mathfrak{m}$ whenever $\mathfrak{k} < \mathfrak{m}$ (i.e., if $\mathfrak{m}$ is a strong limit cardinal) then the function h may be taken onto X (by defining $Y_n = X_n$ for all $n < \omega$). But if $\mathfrak{m}$ is a sequentially accessible cardinal, not a strong limit cardinal, then it is easy to construct a metric space X as in 3.4 with $\mathbf{w}X = \mathfrak{m}$ and with $|X_k| \geqq \mathfrak{m}$ for some $k < \omega$; since Y_k is a metrically discrete subset of Y, the choice $Y_k = X_k$ (and *a fortiori* the definition $Y = X$) is then incompatible with condition (b) of 3.4. Since $h[Y]$ is dense in X (and since $|Z| \leqq \exp \exp \delta Z$ for every regular Hausdorff space Z) we have $|X| = |Y| = \mathfrak{m}$ in case $\mathfrak{m}$ is a strong limit cardinal. We state this formally.

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3.5. COROLLARY. *Let X be a metrizable space with no closed discrete set of power* $\mathbf{w}X$, *and suppose that the (sequentially accessible) cardinal* $\mathbf{w}X$ *is an uncountable strong limit cardinal. Then* $|X| = \mathbf{w}X$.

In view of remark (a) following Corollary 3.3, it follows from 3.5 that if X is as in 3.5 (and with $\mathbf{w}X$ an uncountable strong limit cardinal), then every completion of X (with respect to a compatible metric) has power $\mathfrak{m}$. This is, perhaps, surprising, since one might expect completions of power $\mathfrak{m}^\omega$; that $\mathfrak{m}^\omega > \mathfrak{m}$ for sequentially accessible $\mathbf{m}$ has been remarked above.

The observation of the preceding paragraph is closely related to Theorem 2.3 of [**7**].

We note that under the generalized continuum hypothesis every sequentially accessible cardinal—indeed, every limit cardinal—is a strong limit cardinal.

We show now that the space Y of Proposition 3.4 is one of the spaces $H(\mathfrak{m}, K)$.

3.6. THEOREM. *Let Y be a metrizable space with no closed discrete subset of power* $\mathbf{w}Y = \mathfrak{m} > \omega$ *and suppose that every point not in* $K(Y)$ *(as defined in Theorem* 3.2) *is isolated. Then Y is homeomorphic to* $H(\mathfrak{m}, K(Y))$.

Proof. The set $K(Y)$ is precisely the set of nonisolated points of Y, and is compact by Theorem 3.2(a). According to Atsuji [**1**] and Rainwater [**10**], a metrizable space in which the set of nonisolated points is compact has a compatible metric with respect to which any closed discrete set is metrically discrete. We equip Y with such a metric, d. We set $K = K(Y)$.

By Proposition 1.3 there is a sequence $\{\mathfrak{m}_n : n < \omega\}$ of cardinals such that

$$\mathfrak{m}_n < \mathfrak{m}_{n+1} < \mathfrak{m} \quad \text{for } n < \omega, \quad \text{and}$$
$$\mathfrak{m} = \textstyle\sum_{n<\omega}\mathfrak{m}_n.$$

As in 2.3 we select and fix a sequence $\{\epsilon_n : n < \omega\}$ of positive real numbers and for $n < \omega$, a finite ϵ_n-net N_n in K such that

$$\epsilon_n \to 0, \quad \text{and } N_n \subset N_{n+1} \quad \text{for } n < \omega;$$

as before we write $N_n = \{p_i^n : 1 \leqq i \leqq k_n\}$.

We are going to homeomorphically coordinatize $Y \backslash K$ by the sets D_i^n used in constructing $H(\mathfrak{m}, K)$. In this recursive process we appeal repeatedly to the following lemma.

3.7. LEMMA. *Let (Z, d) be a metric space with no closed discrete set of power* $\mathbf{w}Z > \omega$, *let* $K = K(Z)$, *let* $\{p_i : i \leqq k\}$ *be a finite subset of Z, and for* $b > 0$ *define*

$$Z(b) = \{x \in Z : d(x, K) \leqq b\}.$$

If $\delta > 0$ *and* $\mathfrak{n} < \mathfrak{m}$ *then there is* $b > 0$ *such that* $b \leqq \delta$ *and the cover* $\{E_i : i \leqq k\}$ *of* $Z(b)$ *defined by*

$$E_i = \{x \in Z(b) : i = \min\{j : d(x, p_j) = \min_l d(x, p_l)\}\}$$

decomposes $Z(b)$ *into sets* E_i *such that* $|E_i| \geqq \mathfrak{n}$ *for all* $i \leqq k$.

Proof. (Note that for $x \in Z(b)$ and $i \leqq k$, $x \in E_i$ if and only if i is the first index such that $d(x, p_i) \leqq d(x, p_j)$ for all $j \leqq k$.)

For $c > 0$ define

$$E_i^0(c) = \{x \in Z(c) : d(x, p_i) < d(x, p_j) \quad \text{for } j \leqq k, j \neq i\}.$$

If $c' < c$ we have $E_i^0(c') \supset E_i^0(c)$ for all $i \leqq k$, so it is enough to show that for $i \leqq k$ there is $c_i > 0$ such that $|E_i^0(c_i)| \geqq \mathfrak{n}$; for then the number $b = \min\{c_i : i \leqq k\}$ will be as required.

We fix $i \leqq k$, and we define $a = \min\{d(p_i, p_j) : j \neq i\}$. Then $a > 0$, and

$$S(p_i, a/2) \cap Z(c) \subset E_i^0(c) \quad \text{for all } c > 0.$$

Thus, if $|S(p_i, a/2) \cap Z(c)| < \mathfrak{n}$ for all $c > 0$, then choosing a sequence $\{c_n : n < \omega\}$ such that $c_n \to 0$, we have

$$S(p_i, a/2)\backslash K = \cup_{n<\omega}(S(p_i, a/2) \cap Z(c_n)),$$

and thus $|S(p_i, a/2)\backslash K| \leqq \omega \cdot \mathfrak{n} < \mathfrak{m}$. Since $\mathbf{w}K = \omega$, it then follows that $\mathbf{w}(S(p_i), a/2) < \mathfrak{m}$, contradicting that $p_i \in K$.

The proof of Lemma 3.7 is complete, and we return to Theorem 3.6.

We choose a sequence $\{\delta_n : n < \omega\}$ such that $0 < \delta_n$ for all $n < \omega$ and $\delta_n \to 0$.

We apply Lemma 3.7 with $Z = Z_1 = Y$, with $\{p_i : i \leqq k\} = \{p_i^1 : i \leqq k_1\}$, with $\delta = \delta_1$ and $\mathfrak{n} = \mathfrak{m}_1$. There is $b_1 > 0$ such that $b_1 \leqq \delta_1$ and the disjoint sets $\{E_i^1 : i \leqq k_1\}$, with union

$$Z(b_1) = \{x \in Y : d(x, K) \geqq b_1\},$$

are of power $\geqq \mathfrak{m}_1$.

We note that $Z(b_1)$ is discrete and closed in Y, so there is $r_1 < \omega$ such that $|Z(b_1)| \leqq \mathfrak{m}_{r_1}$.

Now recursively for $1 < n < \omega$ we set

$$Z_n = Y\backslash\cup_{k<n}Z(b_k) = \{x \in Y : d(x, K) < b_{n-1}\}$$

and, noting that $K(Z_n) = K$, we apply Lemma 3.7 with $\{p_i : i \leqq k\} = \{p_i^n : i \leqq k_n\}$, with $\delta = \delta_n$ and $\mathfrak{n} = \mathfrak{m}_n$. There is $b_n > 0$ such that $b_n \leqq \delta_n$ and the disjoint sets $\{E_i^n : i \leqq k_n\}$, with union

$$Z(b_n) = \{x \in Y : b_{n-1} > d(x, K) \geqq b_n\},$$

are of power $\geqq \mathfrak{m}_n$. As above, there is $r_n < \omega$ such that $|Z(b_n)| \leqq \mathfrak{m}_{r_n}$.

We note next that if $x \in E_i^n$ then $d(x, p_i^n) \leqq \delta_{n-1} + \epsilon_n$. Indeed, there is $p \in K$ such that $d(x, p) = d(x, K)$, and since N_n is an ϵ_n-net in K there is $j \leqq k_n$ such that $d(p, p_j^n) < \epsilon_n$; then from the definition of E_i^n we have

$$d(x, p_i^n) \leqq d(x, p_j^n) \leqq d(x, p) + d(p, p_j^n) < b_{n-1} + \epsilon_n \leqq \delta_{n-1} + \epsilon_n.$$

We now exhibit the homeomorphism of Y onto $H(\mathfrak{m}, K)$. The proof is

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particularly simple if it should happen that $|E_i^n| = \mathfrak{m}_n$ for all $n < \omega$, $i \leqq k_n$. (This is indeed the case for the spaces $Y = H(\mathfrak{m}, K, A)$ of 2.2, if $|A| < \mathfrak{m}$.) We first sketch the proof under this assumption.

Given n, and i with $1 \leqq i \leqq k_n$, we identify the set E_i^n in Y with the set D_i^n in $H(\mathfrak{m}, K)$, and we identify the sets K in each space. Thus as sets we have the equality $Y = H(\mathfrak{m}, K)$. We must show that the metrics are topologically equivalent.

Let h denote the metric of $H(\mathfrak{m}, K)$, defined as in 2.3; ρ is the metric of K, $h(p, q) = \rho(p, q)$ for all $p, q \in K$, and the metric d of Y is compatible with ρ on K.

To establish the required homeomorphism it is clearly sufficient to show that if $x_l \notin K$ and $p \in K$, then $h(x_l, p) \rightarrow 0$ if and only if $d(x_l, p) \rightarrow 0$.

For $l < \omega$ there are $n(l) < \omega$ and $i(l) \leqq k_{n(l)}$ such that $x_l \in D_{i(l)}{}^{n(l)}$.

Suppose first that $h(x_l, p) \rightarrow 0$. Since

$$h(x_l, p) = 1/n(l) + \rho(p, p_{i(l)}{}^{n(l)}),$$

we have $n(l) \rightarrow \infty$ and $\rho(p, p_{i(l)}{}^{n(l)}) \rightarrow 0$. In Y we have $x_l \in E_{i(l)}{}^{n(l)}$, and

$$d(x_l, p) \leqq d(x_l, p_{i(l)}{}^{n(l)}) + d(p, p_{i(l)}{}^{n(l)}).$$

Since d and ρ are compatible on K, we have $d(p, p_{i(l)}{}^{n(l)}) \rightarrow 0$, so from

$$d(x_l, p_{i(l)}{}^{n(l)}) \leqq \delta_{n(l)-1} + \epsilon_{n(l)} \rightarrow 0$$

we have $d(x_l, p) \rightarrow 0$.

For the converse, suppose that $d(x_l, p) \rightarrow 0$. Since $d(x_l, K) \leqq d(x_l, p) \rightarrow 0$, we have also $d(x_l, K) \rightarrow 0$, and hence $n(l) \rightarrow \infty$. Thus

$$d(p, p_{i(l)}{}^{n(l)}) \leqq d(x_l, p) + d(x_l, p_{i(l)}{}^{n(l)}) \leqq d(x_l, p) + \delta_{n(l)-1} + \epsilon_{n(l)} \rightarrow 0;$$

so from the definition

$$h(x_l, p) = 1/n(l) + \rho(p, p_{i(l)}{}^{n(l)})$$

and the fact that d and ρ are equivalent on K, it follows that $h(x_l, p) \rightarrow 0$.

The proof is complete in the case that $|E_i^n| = \mathfrak{m}_n$ for all $n < \omega$, $i \leqq k_n$. We turn now to the general case.

We define a function f from Y onto $H(\mathfrak{m}, K)$, with $f(p) = p$ for all $p \in K$.

We begin by defining f on the set $\cup\{E_i^1 : i \leqq k_1\}$. Since $\mathfrak{m}_1 \leqq |E_i^1| \leqq \mathfrak{m}_{r_1}$ for $i \leqq k_1$, there is $F_i^1 \subset E_i^1$ such that $|F_i^1| = \mathfrak{m}_1$; and there is $i' \leqq k_{r_1}$ such that $p_i^1 = p_{i'}{}^{r_1}$. We define f on E_i^1 so that

i) f is a one-to-one function,

ii) $f[F_i^1] = D_i^1$,

iii) $f[E_i^1 \backslash F_i^1] \subset D_{i'}{}^{r_1}$, and

iv) $|D_{i'}{}^{r_1} \backslash f[E_i^1]| = \mathfrak{m}_{r_1}$.

More generally, f having been defined on $\cup\{E_i^{n-1} : i \leqq k_{n-1}\}$, we note that since $\mathfrak{m}_n \leqq |E_i^n| \leqq \mathfrak{m}_{r_n}$ for $i \leqq k_n$, there is $F_i^n \subset E_i^n$ such that $|F_i^n| = \mathfrak{m}_n$; and there is $i' \leqq k_{r_n}$ such that $p_i^n = p_{i'}^{r_n}$.

We define f on E_i^n so that

i) f is a one-to-one function,

ii) $f[F_i^n] = D_i^n \backslash \cup_{l<n, j\leqq k_l} f[E_j^l]$,

iii) $f[E_i^n \backslash F_i^n] \subset D_{i'}^{r_n}$, and

iv) $|D_{i'}^{r_n} \backslash \cup_{l \leqq n, j \leqq k_l} f[E_j^l]| = \mathfrak{m}_{r_n}$.

It is clear that f is a one-to-one function from Y onto $H(\mathfrak{m}, K)$. The proof that f is a homeomorphism proceeds essentially as in the special case treated above: It is sufficient to verify for $x_l \notin K$ and $p \in K$ that $h(x_l, p) \to 0$ if and only if $d(x_l, p) \to 0$, and this follows from the compatibility of h and ρ on K and the fact that if $x_l \in E_{i(l)}^{n(l)} \subset Y$ and $f(x_l) \in D_{i'(l)}^{n'(l)} \subset H(\mathfrak{m}, K)$, then $n(l) \to \infty$ if and only if $n'(l) \to \infty$. We omit the details of the proof.

Remark. It follows from the Atsuji-Rainwater theorem mentioned after the proof of Proposition 3.4 that the metric d used above for Y makes the spaces Y and $H(\mathfrak{m}, K)$ uniformly isomorphic.

3.8 COROLLARY. *The space $H(\mathfrak{m}, K, A)$ of 2.2 is homeomorphic to $H(\mathfrak{m}, K)$ if and only if $|A| < \mathfrak{m}$. Thus, if the continuum hypothesis is assumed, then all spaces $H(\mathfrak{m}, K, A)$ are homeomorphic to $H(\mathfrak{m}, K)$.*

4. Stone-Čech compactifications. We consider the problem of computing certain cardinal functions on Stone-Čech compactifications of metric spaces, in terms of cardinal properties of the metric spaces. The functions which we treat are the density, the power, and the weight.

As to density, the result is simple.

4.1 PROPOSITION. *If X is metrizable, then $\delta\beta X = \delta X$.*

Proof. In general, if X is dense in Y, then any dense subset of X is dense in Y, and hence $\delta X \geqq \delta Y$. Thus, $\delta X \geqq \delta\beta X$.

For the opposite inequality, we use the (obvious) fact that for any space Y, if $\mathscr{U}$ is a family of disjoint open sets and D is dense, then $|\mathscr{U}| \leqq |D|$. By a theorem of Haratomi [**9**] (see also [**3**]), *metrizable* X contains a family $\mathscr{U}$ of δX disjoint open sets. For each $U \in \mathscr{U}$, let U' be an open subset of βX with $U' \cap X = U$. Then $\{U' | U \in \mathscr{U}\}$ is disjoint. It follows that $\delta\beta X \geqq |\mathscr{U}| = \delta X$.

4.2. *Remarks.* (a) The proof of Proposition 4.1 works for any compactification bX of (metrizable) X, showing directly that $\delta bX = \delta X$. We note, however, that $\delta bX = \delta\beta X$ for every compactification bX of every Tychonoff space X. This follows easily from the facts that the Stone extension $\beta X \to bX$ of the injection of X into bX is "irreducible" (i.e., it carries proper closed sets to proper subsets), and that an irreducible map preserves density.

(b) For appropriate (non-metrizable) Tychonoff spaces X, the inequality $\delta\beta X < \delta X$ can occur; see for example [**2**].

We turn now to the cardinals $\mathbf{w}\beta X$ and $|\beta X|$, where the situation is more involved. A good starting point is the following important result of Hausdorff and Pospíšil (see [**8**]).

4.3. THEOREM. *If X is an infinite discrete space, then* $\mathbf{w}\beta X = \exp|X|$ *and* $|\beta X| = \exp\exp|X|$.

We record the following familiar lemma (whose proof is available, for example, in [**3**]).

4.4. LEMMA. *If Y is a regular Hausdorff space, then* $\mathbf{w}Y \leqq \exp \delta Y$ *and* $|Y| \leqq \exp \mathbf{w}Y \leqq \exp\exp\delta Y$.

Theorem 4.3 now extends easily, as follows:

4.5. THEOREM. *Let X be metrizable, and suppose that X has a closed discrete set of power* $\mathbf{w}X$. *Then* $\mathbf{w}\beta X = \exp \mathbf{w}X$ *and* $|\beta X| = \exp\exp \mathbf{w}X$.

Proof. The inequalities $\leqq$ are given by Lemma 4.4. If D is a closed discrete subset of X with $|D| = \mathbf{w}X$ then (since X is normal) the set D is "C^*-embedded" in X, hence in βX; it follows that the closure of D in βX is βD [**8**]. Thus, βX contains a copy of βD, so that $\mathbf{w}\beta X \geqq \mathbf{w}\beta D = \exp \mathbf{w}X$ and $|\beta X| \geqq \exp\exp \mathbf{w}X$ by Theorem 4.3.

We set out now to complete the picture.

4.6. PROPOSITION. *Let X be a metrizable space with no closed discrete subset of power* $\mathbf{w}X$, *and* (*by Proposition* 1.3) *let*

$$\mathbf{w}X = \mathfrak{m} = \textstyle\sum_{n<\omega}\mathfrak{m}_n \quad \textit{with} \quad \mathfrak{m}_n < \mathfrak{m}_{n+1} < \mathfrak{m} \textit{ for } n < \omega.$$

Then

$$\textstyle\sum_{n<\omega} \exp \mathfrak{m}_n \leqq \mathbf{w}\beta X \leqq \exp \mathfrak{m}, \textit{ and } \sum_{n<\omega} \exp\exp \mathfrak{m}_n \leqq |\beta X| \leqq \exp\exp \mathfrak{m}.$$

Proof. The upper bounds follow from 4.4. For the lower bounds, we use 1.1 to find in X, for each n, a closed discrete set D_n of power $\mathfrak{m}_n$, and then (as in 4.5) a copy of βD_n in βX. Thus $\exp \mathfrak{m}_n \leqq \mathbf{w}\beta X$ and $\exp\exp \mathfrak{m}_n \leqq |\beta X|$ from 4.3, and the results follow.

Whether or not the upper and lower bounds in 4.6 coincide depends on the axioms of set theory. If the limit cardinal $\mathfrak{m}$ is a strong limit cardinal (i.e., if $\exp \mathfrak{k} < \mathfrak{m}$ for all $\mathfrak{k} < \mathfrak{m}$)—and hence surely if the generalized continuum hypothesis holds—then

$$\textstyle\sum_{n<\omega} \exp \mathfrak{m}_n = \mathfrak{m} < \exp \mathfrak{m} \text{ and } \sum_{n<\omega} \exp\exp \mathfrak{m}_n = \mathfrak{m} < \exp\exp \mathfrak{m}.$$

On the other hand it is, according to the results of Easton [**5**], consistent to have, say,

$$\exp \mathfrak{m}_0 = \exp \mathfrak{m} \text{ (and hence } \exp\exp \mathfrak{m}_0 = \exp\exp \mathfrak{m}).$$

But whatever the axioms adopted for set theory, the lower bounds of Proposition 4.6 are exact. We prove this by use of Theorem 3.6 and explicit computations concerning $\beta H(\mathfrak{m}, K)$.

If $\{X_a : a \in A\}$ is a family of topological spaces, we denote by $\sum_{a\in A} X_a$ their "topological sum"; this is the disjoint union of the sets X_a, topologized so that a set G is open if and only if $G \cap X_a$ is open in X_a for each $a \in A$.

Now, in $H(\mathfrak{m}, K)$, with $\omega < \mathfrak{m} = \sum_{n<\omega} \mathfrak{m}_n$ as before, set

$$D(\mathfrak{m}_1) = \{x: d(x, K) \geqq 1\};$$

and for $1 < n < \omega$ set

$$D(\mathfrak{m}_n) = \{x: 1/n \leqq D(x, K) < 1/(n-1)\}.$$

Then $|D(\mathfrak{m}_n)| = \mathfrak{m}_n$, and $D(\mathfrak{m}_n)$ is closed and discrete.

Thus $H(\mathfrak{m}, K)\backslash K = \sum D(\mathfrak{m}_n)$, and $H(\mathfrak{m}, K)$ can be written as $K \cup \sum D(\mathfrak{m}_n)$.

Since $D(\mathfrak{m}_n)$ is closed and discrete, it follows as in the proof of Proposition 4.6 that the closure in $\beta H(\mathfrak{m}, K)$ of $D(\mathfrak{m}_n)$ is a copy of $\beta D(\mathfrak{m}_n)$.

4.7. THEOREM. $\beta H(\mathfrak{m}, K) = K \cup \sum \beta D(\mathfrak{m}_n)$.

That is, there are *no* points in $\beta H(\mathfrak{m}, K)\backslash H(\mathfrak{m}, K)$ other than those in the closure of the sets $D(\mathfrak{m}_n)$.

In particular, $\beta H(\mathfrak{m}) = \{p\} \cup \sum \beta D(\mathfrak{m}_n)$; that is, $\beta H(\mathfrak{m})$ is the one-point compactification of $\sum \beta D(\mathfrak{m}_n)$. Theorem 4.7 says that $\beta H(\mathfrak{m}, K)$ is obtained by compactifying $\sum \beta D(\mathfrak{m}_n)$ with K. In particular, then, any compact metric space can be adjoined to $\sum \beta D(\mathfrak{m}_n)$ to create a Stone-Čech compactification.

We now prove Theorem 4.7.

Since $\sum \beta D(\mathfrak{m}_n)$ is dense in $\beta H(\mathfrak{m}, K)$, it is enough to show that every accumulation point p of $\sum \beta D(\mathfrak{m}_n)$ in $\beta H(\mathfrak{m}, K)$ lies in K. If not, then there is a closed neighborhood U of p such that $U \cap K = \emptyset$. For infinitely many n, there is $x_n \in U \cap D(\mathfrak{m}_n)$, and since $d(x_n, K) \to 0$ and K is compact, the set $\{x_n : n < \omega\}$ has an accumulation point $q \in U$, a contradiction.

4.8. COROLLARY. $\mathbf{w}\beta H(\mathfrak{m}, K) = \sum_{n<\omega} \exp \mathfrak{m}_n$, *and* $|\beta H(\mathfrak{m}, K)| = \sum_{n<\omega} \exp \exp \mathfrak{m}_n$.

Proof. That $|\beta H(\mathfrak{m}, K)| = \sum_{n<\omega} \exp \exp \mathfrak{m}_n$ is immediate from Theorem 4.7 (and the hypothesis $\mathfrak{m} > \omega$). Now the identity function

$$K + \sum \beta D(\mathfrak{m}_n) \to K \cup \sum \beta D(\mathfrak{m}_n) = \beta H(\mathfrak{m}, K)$$

is continuous, and the domain clearly has weight $\omega + \sum_{n<\omega} \exp \mathfrak{m}_n = \sum_{n<\omega} \exp \mathfrak{m}_n$. But a continuous function with compact range cannot increase weight (see for example [**6**, Corollary 2 to Theorem 3.1.11]. The proof is complete.

4.9. COROLLARY. *Let X be a metrizable space with no closed discrete subset of power* $\mathbf{w}X > \omega$, *let* $\mathbf{w}X = \mathfrak{m} = \sum_{n<\omega} \mathfrak{m}_n$ *as in Proposition* 1.3, *and let K be the compact set of Theorem* 3.2. *Then*

(a) *βX is a continuous image of $\beta H(\mathfrak{m}, K)$;*

(b) $\mathbf{w}\beta X = \sum_{n<\omega} \exp \mathfrak{m}_n$ *and* $|\beta X| = \sum_{n<\omega} \exp \exp \mathfrak{m}_n$; *and*

(c) *every point of $\beta X \backslash X$ is in the closure (in βX) of a subset of X of power $< \mathfrak{m}$—indeed there is a sequence $\{E_n : n < \omega\}$ of subsets of X, each of power $< \mathfrak{m}$, such that $\beta X = K \cup \bigcup_{n<\omega} \mathrm{cl}_{\beta X} E_n$.*

Proof. There is a continuous function g from $H(\mathfrak{m}, K)$ onto a dense subset of X. (In the notation developed above in § 3, we have $g = h \circ f^{-1} : H(\mathfrak{m}, K) \to X$, where Y is the space defined in the proof of Proposition 3.4, f is the function of 3.4, and h is the homeomorphism from Y onto $H(\mathfrak{m}, K)$ defined in the proof of Theorem 3.6.) The Stone extension $\bar{g}$ of g takes $\beta H(\mathfrak{m}, K)$ onto βX, so (a) is proved. The equalities of (b) follow from 4.6, 4.8 and the fact (as in the proof of 4.8) that

$$\mathbf{w}\beta X = \mathbf{w}(\bar{g}[\beta H(\mathfrak{m}, K)]) \leqq \mathbf{w}\beta H(\mathfrak{m}, K).$$

To prove (c) we let $D(\mathfrak{m}_n)$ be as in Theorem 4.7 and the paragraphs preceding it, and we set $E_n = g[D(\mathfrak{m}_n)]$. For every $p \in \beta X \backslash X$ there is $q \in \beta H(\mathfrak{m}, K)$ such that $\bar{g}(q) = p$, and since $\bar{g}[H(\mathfrak{m}, K)] = g[H(\mathfrak{m}, K)] \subset X$ we have $q \in \beta H(\mathfrak{m}, K) \backslash H(\mathfrak{m}, K)$. It follows from Theorem 4.7 that there is $n < \omega$ such that $q \in \mathrm{cl}_{\beta H(\mathfrak{m},K)} D(\mathfrak{m}_n)$, and from the continuity of $\bar{g}$ we have $p \in \mathrm{cl}_{\beta X} E_n$, as required. The proof is complete.

We recall, finally, that the space βX may be identified with the set of "z-ultrafilters" on X suitably topologized (see [**8**]), and that for X discrete there are many $p \in \beta X$ which are "uniform", i.e., for which $|E| = |X|$ whenever $E \subset X$ and $p \in \mathrm{cl}_{\beta X} E$. In this respect, as with Theorem 4.7, we find the result of Corollary 4.9 (c) unexpected and even bizarre: despite the presence of the huge dense subset $E = \bigcup_{n<\omega} E_n$ of X, no ultrafilter $p \in \beta X$ is uniform over E.

References

1. Masahiko Atsuji, *Uniform continuity of continuous functions of metric spaces*, Pacific J. Math. *8* (1958), 11–16.

2. W. W. Comfort, *An example in density character*, Archiv der Math. *14* (1963), 422–423.

3. ——— *A survey of cardinal invariants*, General Topology and Its Applications *1* (1971), 163–199.

4. W. W. Comfort and S. Negrepontis, *The theory of ultrafilters*, Grundlehren der Math. Wissenschaften Band *211* (Springer-Verlag. Heidelberg. 1974).

5. William B. Easton, *Powers of regular cardinals*, Annals of Math. Logic *1* (1970), 139–178.

6. R. Engelking, *Outline of general topology* (North-Holland Publishing Co. Amsterdam. 1968).

7. Ronald C. Freiwald, *Cardinalities of metric completions*, Fundamenta Math. *78* (1973), 275–280.

8. Leonard Gillman and Meyer Jerison, *Rings of continuous functions* (D. Van Nostrand Co., Inc. Princeton, New Jersey. 1960).

9. K. Haratomi, *Über höherstufige Separabilität und Kompaktheit*, Jap. J. Math. *8* (1931), 113–141.

10. John Rainwater, *Spaces whose finest uniformity is metric*, Pacific J. Math. *9* (1959), 567–570.

11. Friedrich Karl Schmidt, *Über die Dichte metrischer Räume*, Math. Annalen *106* (1932), 457–472.

12. A. H. Stone, *Cardinals of closed sets*, Mathematika *6* (1959), 99–107.
13. Paul Urysohn, *Sur un espace métrique universel*, Bull. des Sci. Math. 2me Série *51* (1927), 43–64.
14. Stephen Willard, *General topology* (Addison-Wesley Publishing Co. Reading, Massachusetts. 1970).

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