A Geometrical Interpretation of the Symmetrical Invariant of Three Ternary Quadratics

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Introduction.

In the paper, "Sul sistema di tre forme ternarie quadratiche," Ciamberlini[1] has derived the complete irreducible system of concomitants for three ternary quadratics and has given a short treatment of their geometrical interpretations. Among the concomitants is the invariant $(abc)^2$ which is symmetrical and linear in the coefficients of each quadratic. The purpose of this note is to give a geometrical interpretation of the invariant, and to extend the result for symmetrical invariants of forms in higher dimensions.[2]

§ 1. *Notation.*

In symbolic form the point and line equations of the three conics are taken to be:—

$$f_1 = a_x^2 = a'^2_x = \dots, \qquad \phi_1 = u_\alpha^2 = u_{\alpha'}^2 = \dots.$$
$$f_2 = b_x^2 = b'^2_x = \dots, \qquad \phi_2 = u_\beta^2 = u_{\beta'}^2 = \dots.$$
$$f_3 = c_x^2 = c'^2_x = \dots, \qquad \phi_3 = u_\gamma^2 = u_{\gamma'}^2 = \dots.$$

where $\quad a_x = \sum_{i=1}^{3} a_i x_i, \quad u_\alpha = \sum_{i=1}^{3} u_i \alpha_i, \quad \alpha = (aa').$

The equation in symbolic form of the Φ-conic for f_1, f_2 is easily obtained from binary forms by the use of the Clebsch Transference Principle.

The invariant $(ab)^2 = 0$ signifies that the pairs of points represented by the binary equations $a_x^2 = 0$, $b_x^2 = 0$ form a harmonic range on a line, and so extending to ternary forms we have

$$(abu)^2 = 0$$

as the envelope of lines cutting the conics f_1, f_2 in harmonic point pairs—*i.e.* the Φ-conic for f_1, f_2.

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[1] *Giorn. di Mat., Napoli* **24** (1886), 141.

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Dually, we obtain $(\alpha\beta x)^2 = 0$ as the equation of the F-conic.

We shall use the notation

$$\Phi_{23} = (bcu)^2, \quad \Phi_{31} = (cau)^2, \quad \Phi_{12} = (abu)^2,$$

to denote the three Φ-conics associated with f_1, f_2, f_3; Φ_{ij} denoting the harmonic envelope of f_i, f_j.

From these results it is obvious that

$$(abc)^2 = 0$$

is the condition for the conic locus f_i to be outpolar to the conic envelope Φ_{jk}, where $i, j, k = 1, 2, 3$ and i, j, k are all different. This, in fact, is the interpretation ascribed to $(abc)^2$ by Ciamberlini. The following pages, however, set out a piece of geometry more in keeping with the symmetrical nature of the invariant.

§ 2. *The invariant* $(abc)^2$.

It will now be proved that, *if u is any line in the plane and u_i the polar with respect to f_i of the pole of u with respect to Φ_{jk}, then u_1, u_2, u_3 are concurrent when $(abc)^2 = 0$.*

Likewise, *if we take a point P in the plane, then the three points P_1, P_2, P_3 are collinear, where P_i is the pole with respect to Φ_{jk} of the polar of P with respect to f_i.*

Consider a line u. Then its pole with respect to Φ_{12} is

$$(abu)(abv) = 0, \quad (v, \text{ current coordinates}),$$

and the polar of this point with respect to f_3 is

$$(u_1), \quad (abu)(abc)\, c_x = 0, \quad (x, \text{ current}).$$

Thus, we get the three connexes

$$(u_1),\ (abc)(bcu)\, a_x = 0, \quad (u_2),\ (abc)(cau)\, b_x = 0, \quad (u_3),\ (abc)(abu)\, c_x = 0,$$

defining the three lines u_1, u_2, u_3 associated with the fixed line u. By the fundamental identity for determinantal permutations we have

$$(abc)(bcu)\, a_x + (abc)(cau)\, b_x + (abc)(abu)\, c_x \equiv (abc)^2 u_x.$$

Hence, if $\qquad (abc)^2 = 0,$

the three lines u_1, u_2, u_3 are concurrent in a point P. We may, however, start with a fixed point P and obtain three points P_1, P_2, P_3 collinear on u.

We obtain a dual interpretation in terms of the F-conics for $(\alpha\beta\gamma)^2 = 0$ by using the identity

$$(\alpha\beta\gamma)(\beta\gamma x)\, u_\alpha + (\alpha\beta\gamma)(\gamma\alpha x)\, u_\beta + (\alpha\beta\gamma)(\alpha\beta x)\, u_\gamma \equiv (\alpha\beta\gamma)^2 u_x.$$

§ 3. *Extension to Three Dimensions.*

For four quaternary quadratic forms

$$f_1 = a_x^2, \quad f_2 = b_x^2, \quad f_3 = c_x^2, \quad f_4 = d_x^2,$$

$(abcu)^2 = 0$ represents the quadric envelope of planes cutting the quadrics f_1, f_2, f_3 in three conics having the symmetrical property considered above.

This follows by application of the Clebsch Transference Principle to the ternary invariant $(abc)^2$.

Hence, we obtain four quadric envelopes which we can specify by

$$\Phi_{123} = (abcu)^2, \quad \Phi_{234} = (bcdu)^2, \quad \Phi_{341} = (cdau)^2, \quad \Phi_{412} = (dabu)^2.$$

As in § 2, the vanishing of $(abcd)^2$, the symmetrical invariant of four quadrics, is the condition for any f_i to be outpolar with respect to Φ_{jkl}, where $i, j, k, l = 1, 2, 3, 4$, and i, j, k, l are all different.

If, however, we consider the four connexes

$$(u_1), \quad a_x(abcd)(bcdu) = 0, \qquad (u_2), \quad b_x(abcd)(cdau) = 0,$$
$$(u_3), \quad c_x(abcd)(dabu) = 0, \qquad (u_4), \quad d_x(abcd)(abcu) = 0,$$

then, since

$$(abcd)(bcdu)\,a_x + (abcd)(cadu)\,b_x + (abcd)(abdu)\,c_x + (abcd)(acbu)\,d_x \equiv (abcd)^2 u_x$$

it follows that $(abcd)^2 = 0$ *is the condition for the four planes* u_1, u_2, u_3, u_4 *to meet in a point, where* u_i *is the polar with respect to* f_i *of the pole of* u *with respect to* Φ_{jkl}.

Dually, we can interpret $(\alpha\beta\gamma\delta)^2 = 0$ by using the quadric loci $(\alpha\beta\gamma x)^2 = 0$ etc. It is obvious that by repeated application of the Clebsch Transference Principle it is possible to interpret $(abc\,..\,p)^2 = 0$ for p quadrics in $[p-1]$.

§ 4. $(abc)^2$ *in relation to the F-conics of the* Φ*-conics.*

In this paragraph we suppose $(abc)^2 = 0$. The F-conic of Φ_{12}, Φ_{23}, is

$$F_{(12)(23)} \equiv (ab\,.\,b'\,c\,.\,x)^2 = 0.$$

Thus
$$[(\dot{a}b'\,c)\,\dot{b}_x]^2 = 0,$$

and
$$(abc)^2 f_2 + c_\beta^2 f_1 - 2\,(ab'\,c)\,b_x\,(bb'\,c)\,a_x = 0,$$

so that
$$(abc)^2 f_2 + {}_x a_\beta\, c_x = 0, \tag{1}$$

and ${}_x a_\beta\, c_x = 0$ represents the conic which is the locus of a point x whose polars with respect to f_1, f_3 are conjugate lines with respect to ϕ_2.

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From the result (1) we see that the conic ${}_x a_\beta c_x = 0$ meets f_2 in four points which are vertices of harmonic pencils of tangents to Φ_{12}, Φ_{23}. For, since $(abc)^2 = 0$, it follows that

$$F_{(12)(23)} \equiv {}_x a_\beta c_x = 0,$$
$$F_{(23)(31)} \equiv {}_x a_\gamma b_x = 0,$$
$$\text{and} \quad F_{(31)(12)} \equiv {}_x b_\alpha c_x = 0.$$

The condition for $d_x^2 \equiv {}_x a_\beta c_x$ to be outpolar with respect to Φ_{31} is $(cad)^2 = 0$,

i.e., $$(caa')\, a'_\beta c'_\beta (cac') = 0.$$

Hence $$\tfrac{1}{2} c_\alpha^2 c_\beta^2 - \tfrac{1}{2} c_\alpha c'_\alpha c_\beta c'_\beta = 0,$$

and therefore $$(\alpha\beta\gamma)^2 = 0.$$

Thus, for $(abc)^2 = 0$ and $(\alpha\beta\gamma)^2 = 0$ simultaneously, the symmetrical property holds for the Φ-conics and their F-conics.