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ERRATUM TO: CONNECTIVITY AND PURITY FOR LOGARITHMIC MOTIVES

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The proof of [1, Lemma 7.2] contains a gap: the equality $\omega_\sharp h_0(\Lambda_{\mathrm{ltr}}(\eta,\mathrm{triv})) = \omega_\sharp h_0(\omega^*\Lambda_{\mathrm{tr}}(\eta))$ is false. Indeed one can check that for $X \in \mathbf{Sm}(k)$ proper,

$$\mathrm{Hom}(\omega_\sharp h_0(\Lambda_{\mathrm{ltr}}(\eta_X,\mathrm{triv})),\mathbf{G}_a) \neq \mathrm{Hom}(\omega_\sharp h_0(\omega^*\Lambda_{\mathrm{tr}}(\eta_X)),\mathbf{G}_a),$$

as the left-hand side is $\mathbf{G}_a(\eta_X)$, whereas the right-hand side is $\mathbf{G}_a(X)$. For now, we can give a proof only of a weaker version of [1, Proposition 7.3]:

Proposition 0.1. *Let k be a perfect field. Then the compositions*

$$\mathbf{CI}^{\log}_{\mathrm{dNis}} \xrightarrow{i} \mathbf{Shv}^{\log}_{\mathrm{dNis}} \xrightarrow{\omega^{\log}_\sharp} \mathbf{Shv}_{\mathrm{Nis}}, \qquad \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}} \xrightarrow{i^{\mathrm{tr}}} \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}} \xrightarrow{\omega^{\mathrm{ltr}}_\sharp} \mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}$$

are faithful and exact. In particular, both functors are conservative.

Proof. Exactness follows from the exactness of i and $\omega^{\log}_\sharp$ (resp., i^{tr} and $\omega^{\mathrm{ltr}}_\sharp$). To show faithfulness, it is enough to show that for all $F \in \mathbf{CI}^{\log}_{\mathrm{dNis}}$ (resp., $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$), the unit map

$$F \to \omega^{\mathbf{CI}}_{\log}\omega^{\log}_\sharp F \quad (\text{resp., } F \to \omega^{\mathbf{CI}}_{\mathrm{ltr}}\omega^{\mathrm{ltr}}_\sharp F)$$

is injective. By [1, Theorem 5.10], we have that for all $X \in \mathbf{SmlSm}(k)$,

$$F(X) \hookrightarrow F(\underline{X} - |\partial X|) = \omega^*_{\log}\omega^{\log}_\sharp F \quad (\text{resp., } \omega^*_{\mathrm{ltr}}\omega^{\mathrm{ltr}}_\sharp F),$$

and hence $u\colon F \hookrightarrow \omega^*_{\log}\omega^{\log}_\sharp F$ (resp., $u^{\mathrm{tr}}\colon F \hookrightarrow \omega^*_{\mathrm{ltr}}\omega^{\mathrm{ltr}}_\sharp F$) is injective. Because F is $\overline{\square}$-local, the map u (resp., u^{tr}) factors through $\omega^{\mathbf{CI}}_{\log}\omega^{\log}_\sharp F$ (resp., $\omega^{\mathbf{CI}}_{\mathrm{ltr}}\omega^{\mathrm{ltr}}_\sharp F$), which concludes the proof. □

We believe that the full statement of a more general version of [1, Proposition 7.3] holds:

Conjecture 0.2. *The functors of Proposition 0.1 are full.*

We stress that the previous statement does not assume (RS), nor transfers. We cannot give a proof of [1, Lemma 7.2] at the moment, but we expect the statement to hold as a consequence of the following conjecture:

Conjecture 0.3. *The inclusion* $\iota^{\mathrm{tr}}\colon \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\Lambda) \subseteq \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\Lambda)$ *is Serre – that is, for all* $F \in \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\Lambda)$*, if* $G \subseteq F$ *is a subsheaf with log stransfers, then* G *is strictly* $\overline{\square}$*-invariant – that is,* G *lies in* $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\Lambda)$*.*

If Conjecture 0.3 holds, then the counit map $\iota^{\mathrm{tr}} h^0_{\mathrm{ltr}} G \to G$ is a monomorphism for all $G \in \mathbf{Shv}_{\mathrm{dNis}}(k,\Lambda)$. In particular, this would imply that the natural map

$$\omega^{\mathbf{CI}}_{\mathrm{ltr}} \omega^{\mathrm{ltr}}_{\mathbf{CI}} F = \iota^{\mathrm{tr}} h^0_{\mathrm{ltr}} \omega^*_{\mathrm{tr}} \omega^{\mathrm{ltr}}_{\sharp} \iota^{\mathrm{tr}} F \hookrightarrow \omega^*_{\mathrm{ltr}} \omega^{\mathrm{ltr}}_{\sharp} \iota^{\mathrm{tr}} F$$

is injective, so we could proceed as in [1, Proposition 7.3.] to prove Conjecture 0.2 in the case with transfers. On the other hand, we do not expect Conjecture 0.3 to hold for $\mathbf{CI}^{\log}_{\mathrm{dNis}}(k,\Lambda)$, as its counterpart is already false for the category of $\mathbf{A}^1$-local sheaves without transfers.

All the results of [1, §7] must be considered conjectural as well.

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Reference

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