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ON THE RANK OF THE FIRST RADICAL LAYER OF A P-CLASS GROUP OF AN ALGEBRAIC NUMBER FIELD

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Abstract. Let p be a prime number. Let M be a finite Galois extension of a finite algebraic number field k. Suppose that M contains a primitive pth root of unity and that the p-Sylow subgroup of the Galois group $G = \mathrm{Gal}(M/k)$ is normal. Let K be the intermediate field corresponding to the p-Sylow subgroup. Let $\mathfrak{g} = \mathrm{Gal}(K/k)$. The p-class group $\mathcal{C}$ of M is a module over the group ring $\mathbf{Z}_pG$, where $\mathbf{Z}_p$ is the ring of p-adic integers. Let J be the Jacobson radical of $\mathbf{Z}_pG$. $\mathcal{C}/J\mathcal{C}$ is a module over a semisimple artinian ring $\mathbf{F}_p\mathfrak{g}$. We study multiplicity of an irreducible representation Φ apperaring in $\mathcal{C}/J\mathcal{C}$ and prove a formula giving this multiplicity partially. As application to this formula, we study a cyclotomic field M such that the minus part of $\mathcal{C}$ is cyclic as a $\mathbf{Z}_pG$-module and a CM-field M such that the plus part of $\mathcal{C}$ vanishes for odd p.

To show the formula, we apply theory of central extensions of algebraic number field and study global and local Kummer duality between the genus group and the Kummer radical for the genus field with respect to M/K.

Introduction

Let k be a finite extension of $\mathbf{Q}$ and M be a finite Galois extension of k with a Galois group G. Let p be a prime number. Let H be the p-Hilbert class field of M. The Galois group $\mathrm{Gal}(H/M)$ is isomorphic to the p-Sylow subgroup $\mathcal{C}$ of the ideal class group of M. Let $\mathbf{F}_p$ be the field of p-elements and denote by $\mathbf{F}_pG$ the group ring of G over $\mathbf{F}_p$. Let J be the Jacobson radical of $\mathbf{F}_pG$. $\mathcal{C} \otimes \mathbf{F}_p$ is an $\mathbf{F}_pG$-module and $\mathcal{C} \otimes \mathbf{F}_p/J\mathcal{C} \otimes \mathbf{F}_p$ is called the first radical layer of $\mathcal{C} \otimes \mathbf{F}_p$.

We suppose that the p-Sylow group G_p of G is a normal subgroup. Let K be the intermediate field of M/k corresponding to G_p. K is a Galois extension of k such that $p \nmid [K : k]$. Denote by $\mathfrak{g}$ the Galois group of K/k. The abelian p-genus field of M/K is the compositum $H^{\mathrm{ab}}M$, where H^{ab} is the maximal abelian subfield of H/K. The central p-genus field H^{cent} is the intermediate field of H/M corresponding to $\prod_{\sigma \in G_p} \mathcal{C}^{\sigma-1}$. We shall

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show that J is generated by $\{g-1 : g \in G_p\}$. Therefore, $\mathcal{C} \otimes \mathbf{F}_p / J\mathcal{C} \otimes \mathbf{F}_p \cong \mathrm{Gal}(H^{\mathrm{cent}}/M) \otimes \mathbf{F}_p$ and $\mathbf{F}_p G/J \cong \mathbf{F}_p \mathfrak{g}$. This means that $\mathrm{Gal}(H^{\mathrm{cent}}/M) \otimes \mathbf{F}_p$ is an $\mathbf{F}_p\mathfrak{g}$-module which is isomorphic to the first radical layer.

$\mathbf{F}_p\mathfrak{g}$ is a semisimple artinian ring, because of $p \not| (\mathfrak{g} : 1)$, (*c.f.* Theorem 3.14, Curtis-Reiner [2]). By Theorem 1.3.5, Benson [1], we have an isomorphism

$$\mathbf{F}_p\mathfrak{g} \quad \cong \bigoplus_{i=1}^{r} M_{n_i}(\Delta_i)$$

of rings, where Δ_i is a division ring. Let L_i be the minimal left ideal of $M_{n_i}(\Delta_i)$. $\mathbf{F}_p\mathfrak{g}$ has exactly r isomorphism classes of irreducible modules L_i, $i = 1, \cdots, r$. Denote by Φ_i the irreducible character afforded with L_i. Put $\mathfrak{B}(\mathfrak{g}) = \{\Phi_i : 1 \le i \le r\}$. This set is called a basic set of irreducible $\mathbf{F}_p$-characters of $\mathfrak{g}$. These irreducible characters are linearly independent over $\mathbf{F}_p$, (*c.f.*, Lemma 3.3, Chapter 19, Karpilovsky [5]). Let 1_i be the inverse image of the identity matrix of $M_{n_i}(\Delta_i)$ with the above isomorphism. We see

$$1 \;=\; \sum_{i=1}^{r} 1_i, \qquad 1_i 1_j \;=\; \delta_{i,j} 1_i.$$

When an irreducible character Φ is given, there is Φ_i such that $\Phi_i = \Phi$. Hence, we write Δ_Φ, L_Φ, 1_Φ for Δ_i, L_i, 1_i. We also write d_Φ for n_i.

There is an integer $a_\Phi \ge 1$ such that

$$1_\Phi \,\mathrm{Gal}(H^{\mathrm{cent}}/M) \otimes \mathbf{F}_p \;\cong\; L_\Phi^{a_\Phi}.$$

We shall give a formula describing the value of a_Φ in the present paper. This will be done in Theorem 9 in §4. We are able to determine for what M and Φ the value of a_Φ equals 0 or 1. The number of generators of $\mathcal{C}$ over $\mathbf{Z}_pG$ is obtained from the values of a_Φ, where $\mathbf{Z}_pG$ denotes the group ring of G over the ring $\mathbf{Z}_p$ of p-adic integers. Suppose p is odd. When $k = \mathbf{Q}$ and M is a cyclotomic field and if $a_\Phi \le 1$ for every Φ such that $\tau 1_\Phi \ne 1_\Phi$, we have $\mathcal{C}^{\tau-1} \cong (\tau-1)\mathbf{Z}_pG/(\tau-1)S$, where τ is the complex conjugation and S is an ideal generated by the Stickelberger elements (*c.f.* Sinnott [9]). In general, it is equivalent to $\mathcal{C} = 0$ that a_Φ vanishes for every Φ. We are able to apply this to study the Greenberg conjecture (*c.f.* Greenberg [3]). We could obtain a criterion whether the conjecture holds in its "trivial case".

We denote by $\dim X$ the dimension over $\mathbf{F}_p$ for an $\mathbf{F}_p$-module X. The outline of our argument is as follows. Put $M^{\mathrm{ab}} = M \cap H^{\mathrm{ab}}$. There is a non-negative integer $\theta(\Phi)$ such that

$$a_\Phi = \frac{1}{\dim L_\Phi} \dim 1_\Phi \operatorname{Gal}(H^{\mathrm{ab}}/M^{\mathrm{ab}}) \otimes \mathbf{F}_p + \theta(\Phi).$$

We have the following inequality:

$$\begin{aligned} &\dim 1_\Phi \operatorname{Gal}(H^{\mathrm{ab}}/K) \otimes \mathbf{F}_p \geq \dim 1_\Phi \operatorname{Gal}(H^{\mathrm{ab}}/M^{\mathrm{ab}}) \otimes \mathbf{F}_p \\ \geq &\dim 1_\Phi \operatorname{Gal}(H^{\mathrm{ab}}/K) \otimes \mathbf{F}_p - \dim 1_\Phi \operatorname{Gal}(M^{\mathrm{ab}}/K) \otimes \mathbf{F}_p. \end{aligned}$$

Hence, if $1_\Phi \operatorname{Gal}(M^{\mathrm{ab}}/K) \otimes \mathbf{F}_p = 0$, we have

$$\dim 1_\Phi \operatorname{Gal}(H^{\mathrm{ab}}/K) \otimes \mathbf{F}_p = \dim 1_\Phi \operatorname{Gal}(H^{\mathrm{ab}}/M^{\mathrm{ab}}) \otimes \mathbf{F}_p.$$

We shall obtain a formula of the value of

$$\dim 1_\Phi \operatorname{Gal}(H^{\mathrm{ab}}/K) \otimes \mathbf{F}_p$$

by studying the representation of $\mathfrak{g}$ on $\operatorname{Gal}(H^{\mathrm{ab}}/K) \otimes \mathbf{F}_p$. In particular, when $G = G_p \times \mathfrak{g}$, we have $\theta(\Phi) = 0$ and $\dim 1_\Phi \operatorname{Gal}(M^{\mathrm{ab}}/K) \otimes \mathbf{F}_p = 0$ if Φ is not the unit character ε.

§1. The p-genus and central p-genus fields

The Jacobson radical J of $\mathbf{F}_pG$ is the intersection of every maximal left ideal, which is a two-sided ideal of $\mathbf{F}_pG$. Put $R = \mathbf{F}_pG/J$.

LEMMA 1. *J is generated by $\{g - 1 : g \in G_p\}$, and hence, $R \cong \mathbf{F}_p\mathfrak{g}$.*

Proof. Let $\mathbf{F}_pG \to \mathbf{F}_p\mathfrak{g}$ be a homomorphism induced from the canonical map $G \to \mathfrak{g}$. Let J' be the kernel of this homomorphism. J' is generated by $\{g - 1 : g \in G_p\}$ over $\mathbf{F}_p$. Since $\mathbf{F}_p$ is a local commutative ring, we are able to apply Proposition 5.26 in Curtis-Reiner [2] to this homomorphism. We have $J' \subset J$. J/J' is contained in the Jacobson radical of $\mathbf{F}_pG/J' = \mathbf{F}_p\mathfrak{g}$. Therefore, $J/J' = 0$. We obtain $J = J'$, and hence $\mathbf{F}_pG/J \cong \mathbf{F}_p\mathfrak{g}$. □

We identify $\mathbf{F}_pG/J$ with $\mathbf{F}_p\mathfrak{g}$ in the remainder part. Put $f_\Phi = \dim \Delta_\Phi$. f_Φ equals $\dim \operatorname{Hom}_R(L_\Phi, L_\Phi)$, (*c.f.* Theorem 1.3.5, Benson [1]). We have

$$\dim L_\Phi = f_\Phi d_\Phi. \tag{1.1}$$

Denote by A the group ring $\mathbf{Z}_p\mathfrak{g}$. We see $R = A \otimes \mathbf{F}_p$. Since $p \nmid (\mathfrak{g} : 1)$, the center $Z(A)$ (resp. $Z(R)$) of A (resp. R) is generated by $\{C(\sigma) =$

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$\frac{1}{(\mathfrak{g}:1)}\sum_{\tau\in\mathfrak{g}}\tau\sigma\tau^{-1} : \sigma \in \mathfrak{g}\}$. Hence, $Z(R)$ is the image of $Z(A)$ with the canonical map. We have the assumption of (iii) of Theorem 1.9.4, Benson [1] is satisfied for A and R. Thus, every 1_Φ is lifted on a primitive central idempotent $\tilde{1}_\Phi$ of A and

$$1 \quad = \quad \sum_{\Phi} \tilde{1}_\Phi.$$

Denote by R_Φ the simple ring $1_\Phi R$. Let Y be a finitely generated A-module. We have an isomorphism $1_\Phi(Y \otimes \mathbf{F}_p) \cong \tilde{1}_\Phi Y \otimes \mathbf{F}_p$. Since $Y \otimes \mathbf{F}_p \cong R \otimes_A Y$, we see $\tilde{1}_\Phi Y \otimes \mathbf{F}_p \cong R_\Phi \otimes_A Y$. Denote by $r_\Phi(Y)$ a non-negative integer such that

$$\tilde{1}_\Phi Y \otimes \mathbf{F}_p \;\cong\; R_\Phi \otimes_A Y \;\cong\; L_\Phi^{r_\Phi(Y)}.$$

We use $R_\Phi \otimes_A Y$ rather than $\tilde{1}_\Phi Y \otimes \mathbf{F}_p$ in the sequel. We have

$$(1.2) \quad r_\Phi(Y) \;=\; \frac{1}{f_\Phi d_\Phi} \dim R_\Phi \otimes_A Y, \quad \dim Y \otimes \mathbf{F}_p \;=\; \sum_{\Phi} r_\Phi(Y) f_\Phi d_\Phi.$$

LEMMA 2. *Let* $0 \to X \to Y \to Z \to 0$ *be an exact sequence of A-modules. Then, we have an exact sequence*

$$\mathrm{Tor}_1^{\mathbf{Z}_p}(\tilde{1}_\Phi Z, \mathbf{F}_p) \to R_\Phi \otimes_A X \xrightarrow{f} R_\Phi \otimes_A Y \to R_\Phi \otimes_A Z \to 0.$$

If $\tilde{1}_\Phi Z$ is $\mathbf{Z}_p$-torsion free or if $\tilde{1}_\Phi Y$ is an R-module, f is injective.

Proof. Let U be an A-module. We have

$$\mathrm{Tor}_0^{\mathbf{Z}_p}(\tilde{1}_\Phi U, \mathbf{F}_p) \;=\; \tilde{1}_\Phi U \otimes_{\mathbf{Z}_p} \mathbf{F}_p \;\cong\; R_\Phi \otimes_A U.$$

The exact sequence follows from an exact sequence

$$0 \to \tilde{1}_\Phi X \;\to\; \tilde{1}_\Phi Y \;\to\; \tilde{1}_\Phi Z \;\to 0.$$

If $\tilde{1}_\Phi Z$ is $\mathbf{Z}_p$-torsion free, we have $\mathrm{Tor}_1^{\mathbf{Z}_p}(\tilde{1}_\Phi Z, \mathbf{F}_p) = 0$. If $\tilde{1}_\Phi Y$ is an R-module, we have $\tilde{1}_\Phi Y = \tilde{1}_\Phi X \oplus \tilde{1}_\Phi Z$, because R is semisimple. Hence, this short exact sequence splits. The lemma is proved.

By Lemma 1, we have

$$1_\Phi(\mathcal{C} \otimes \mathbf{F}_p / J\mathcal{C} \otimes \mathbf{F}_p) = R_\Phi \otimes_A (\mathcal{C}/\mathcal{C}^p \cup \cup_{\sigma \in G_p} \mathcal{C}^{\sigma-1}) \cong 1_\Phi \,\mathrm{Gal}(H^{\mathrm{cent}}/M) \otimes \mathbf{F}_p.$$

Let C_M be the idele class group of M. By cup product with the canonical class of $H^2(G_p, C_M)$, an isomorphism $H^{-3}(G_p, \mathbf{Z}) \to H^{-1}(G_p, C_M)$ is defined. This isomorphism is a $\mathfrak{g}$-isomorphism. Furthermore, there is a surjective homomorphism onto $\mathrm{Gal}(H^{\mathrm{cent}}/H^{\mathrm{ab}}M)$:

$$H^{-3}(G_p, \mathbf{Z}) \quad \xrightarrow{\cong} \quad H^{-1}(G_p, C_M) \quad \longrightarrow \quad \mathrm{Gal}(H^{\mathrm{cent}}/H^{\mathrm{ab}}M),$$

(*c.f.* Jehne [4], Miyake [7], Shirai [8]). Since the p-primary torsion subgroup of $H^{-3}(G_p, \mathbf{Z})$ is isomorphic to $H^{-3}(G_p, \mathbf{Z}_p)$, $\tilde{1}_\Phi \,\mathrm{Gal}(H^{\mathrm{cent}}/H^{\mathrm{ab}}M)$ is a homomorphic image of $\tilde{1}_\Phi H^{-3}(G_p, \mathbf{Z}_p)$. By towers of Galois extensions $H^{\mathrm{cent}} \supset H^{\mathrm{ab}}M \supset M$ and $H^{\mathrm{ab}} \supset M^{\mathrm{ab}} \supset K$, we have exact sequences

$$(1.3)\qquad \begin{gathered} 1 \to \mathrm{Gal}(H^{\mathrm{cent}}/H^{\mathrm{ab}}M) \to \mathrm{Gal}(H^{\mathrm{cent}}/M) \to \mathrm{Gal}(H^{\mathrm{ab}}/M^{\mathrm{ab}}) \to 1, \\ 1 \to \mathrm{Gal}(H^{\mathrm{ab}}/M^{\mathrm{ab}}) \to \mathrm{Gal}(H^{\mathrm{ab}}/K) \to \mathrm{Gal}(M^{\mathrm{ab}}/K) \to 1. \end{gathered}$$

Let Θ be the image of $H^{-3}(G_p, \mathbf{Z}_p)$ into $\mathrm{Gal}(H^{\mathrm{cent}}/M) \otimes \mathbf{F}_p$ of the homomorphism obtained by combining the surjection of $H^{-3}(G_p, \mathbf{Z}_p)$ onto $\mathrm{Gal}(H^{\mathrm{cent}}/H^{\mathrm{ab}}M)$ with the canonical map $\mathrm{Gal}(H^{\mathrm{cent}}/H^{\mathrm{ab}}M) \otimes \mathbf{F}_p \to \mathrm{Gal}(H^{\mathrm{cent}}/M) \otimes \mathbf{F}_p$. $\mathrm{Gal}(H^{\mathrm{ab}}/K)$ is an abelian p-group, because $\mathrm{Gal}(M^{\mathrm{ab}}/K)$ is an abelian p-group.

THEOREM 3. *Denote by $\theta(\Phi)$ the value of $r_\Phi(\Theta)$. We have*

$$\theta(\Phi) \;\le\; \frac{1}{f_\Phi d_\Phi} \dim \tilde{1}_\Phi H^{-3}(G_p, \mathbf{Z}_p) \otimes \mathbf{F}_p \;=\; \frac{1}{f_\Phi d_\Phi} \dim R_\Phi \otimes_A H^{-3}(G_p, \mathbf{Z}_p).$$

Further, the value $a_\Phi = r_\Phi(\mathcal{C} \otimes \mathbf{F}_p / J\mathcal{C} \otimes \mathbf{F}_p)$ satisfies an inequality

$$\begin{aligned} r_\Phi(\mathrm{Gal}(H^{\mathrm{ab}}/M^{\mathrm{ab}})) + \theta(\Phi) \;=\; a_\Phi \ge\; & r_\Phi(\mathrm{Gal}(H^{\mathrm{ab}}/K)) \\ & - r_\Phi(\mathrm{Gal}(M^{\mathrm{ab}}/K)) + \theta(\Phi). \end{aligned}$$

If one of $\tilde{1}_\Phi \,\mathrm{Gal}(M^{\mathrm{ab}}/K)$, $\tilde{1}_\Phi \,\mathrm{Gal}(H^{\mathrm{ab}}/K)^p$ and $\tilde{1}_\Phi \,\mathrm{Gal}(H^{\mathrm{ab}}/M^{\mathrm{ab}})$ vanishes, we have

$$\begin{aligned} r_\Phi(\mathrm{Gal}(H^{\mathrm{ab}}/K)) \;&=\; r_\Phi(\mathrm{Gal}(M^{\mathrm{ab}}/K)) + r_\Phi(\mathrm{Gal}(H^{\mathrm{ab}}/M^{\mathrm{ab}})), \\ a_\Phi \;&=\; r_\Phi(\mathrm{Gal}(H^{\mathrm{ab}}/K)) - r_\Phi(\mathrm{Gal}(M^{\mathrm{ab}}/K)) + \theta(\Phi). \end{aligned}$$

Proof. Since $H^{-3}(G_p, \mathbf{Z}_p)$ is an A-module, we have

$$r_\Phi(H^{-3}(G_p, \mathbf{Z}_p)) \;=\; \frac{1}{f_\Phi d_\Phi} \dim R_\Phi \otimes_A H^{-3}(G_p, \mathbf{Z}_p)$$

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from (1.2). Since Θ is a homomorphic image of $H^{-3}(G_p, \mathbf{Z}_p)$, we see $\theta(\Phi) \le r_\Phi(H^{-3}(G_p, \mathbf{Z}_p))$. By applying Lemma 2 to the sequence (1.3), we have the following exact sequences:

$$0 \to R_\Phi \otimes_A \Theta \to R_\Phi \otimes_A \operatorname{Gal}(H^{\mathrm{cent}}/M) \to R_\Phi \otimes_A \operatorname{Gal}(H^{\mathrm{ab}}/M^{\mathrm{ab}}) \to 0,$$
$$\to R_\Phi \otimes_A \operatorname{Gal}(H^{\mathrm{ab}}/M^{\mathrm{ab}}) \xrightarrow{f} R_\Phi \otimes_A \operatorname{Gal}(H^{\mathrm{ab}}/K)$$
$$\to R_\Phi \otimes_A \operatorname{Gal}(M^{\mathrm{ab}}/K) \to 0.$$

The inequality concerning a_Φ follows from these sequences. We also have f is injective if $\tilde{1}_\Phi \operatorname{Gal}(M^{\mathrm{ab}}/K) = 0$ or if $\tilde{1}_\Phi \operatorname{Gal}(H^{\mathrm{ab}}/K)$ is an R-module, or if $\tilde{1}_\Phi \operatorname{Gal}(H^{\mathrm{ab}}/M^{\mathrm{ab}}) = 0$. $\tilde{1}_\Phi \operatorname{Gal}(H^{\mathrm{ab}}/K)$ is an R-module if $\tilde{1}_\Phi \operatorname{Gal}(H^{\mathrm{ab}}/K)^p = 0$. If one of these conditions is satisfied, $r_\Phi(\operatorname{Gal}(H^{\mathrm{ab}}/M^{\mathrm{ab}}))$ is equal to difference of $r_\Phi(\operatorname{Gal}(H^{\mathrm{ab}}/K))$ and $r_\Phi(\operatorname{Gal}(M^{\mathrm{ab}}/K))$. This proves the theorem.

Let $\mathfrak{h}$ be a normal subgroup of $\mathfrak{g}$ and put $\mathfrak{g}' = \mathfrak{g}/\mathfrak{h}$. Denote by R' the group ring $\mathbf{F}_p\mathfrak{g}'$. Let $I_\mathfrak{h}$ be the ideal of R generated by $\{\sigma - 1 : \sigma \in \mathfrak{h}\}$. We have $R' \cong R/I_\mathfrak{h}$. Denote by π the canonical homomorphism $R \to R'$. Note $\operatorname{im} \pi = R'$ and $\ker \pi = I_\mathfrak{h}$. Since $\mathfrak{h}$ is normal, an element

$$1_\mathfrak{h} \quad = \quad \frac{1}{(\mathfrak{h} : 1)} \sum_{\sigma \in \mathfrak{h}} \sigma$$

of R is a central idempotent and $\ker \pi \ni 1 - 1_\mathfrak{h}$. Denote by $\tilde{\pi}$ restriction of π onto $1_\mathfrak{h} R$. $\tilde{\pi}$ is an isomorphism, because of $\pi(1_\mathfrak{h} R) = \pi(R)$ and $\dim 1_\mathfrak{h} R = \dim R'$. We have a decomposition

$$(1.4) \qquad R = 1_\mathfrak{h} R \oplus (1 - 1_\mathfrak{h})R, \quad 1_\mathfrak{h} R \cong R', \quad (1 - 1_\mathfrak{h})R = I_\mathfrak{h}.$$

Hence, $\ker \pi = (1 - 1_\mathfrak{h})R$. Let Ψ be an irreducible character of $\mathfrak{g}'$ and denote by 1_Ψ the corresponding primitive central idempotent of R'. We see $\tilde{\pi}^{-1}(1_\Psi)$ is also a primitive central idempotent. Let Φ be an irreducible character afforded with a minimal left ideal of $\tilde{\pi}^{-1}(1_\Psi)R$. We have $1_\Phi = \tilde{\pi}^{-1}(1_\Psi)$ and $\Phi = \Psi \circ \pi$. Conversely, if $\sigma 1_\Phi = 1_\Phi$ holds for every $\sigma \in \mathfrak{h}$, we have $1_\Phi 1_\mathfrak{h} = 1_\Phi$. Hence, $1_\Phi R \cong 1_\Phi R'$ by (1.4). Let Ψ be the character of $\mathfrak{g}'$ afforded with a minimal left ideal of $1_\Phi R'$. We have $\Phi = \Psi \circ \pi$. Therefore, we denote by the same symbol Φ this character Ψ.

LEMMA 4. *Suppose there are normal subfields K' of K/k and M' of M/k such that $M' \supset K'$, $M = M'K$ and $M' \cap K = K'$. Put* $\mathfrak{h} = \operatorname{Gal}(K/K')$.

Let H' be the p-Hilbert class field of M'. Denote by H'^{cent} the central p-genus field of M'/K' and by H'^{ab} the maximal abelian subfield of H'/K'. Let Θ' be the image of $H^{-3}(G_p, \mathbf{Z}_p)$ into $\mathrm{Gal}(H'^{\text{cent}}/M') \otimes \mathbf{F}_p$. Suppose $\sigma 1_\Phi = 1_\Phi$ holds for every $\sigma \in \mathfrak{h}$. Then,

$$r_\Phi(\mathrm{Gal}(H^{\text{ab}}/K)) \;=\; r_\Phi(\mathrm{Gal}(H'^{\text{ab}}/K')), \quad r_\Phi(\Theta) \;=\; r_\Phi(\Theta').$$

Moreover, we have $r_\Phi(\mathrm{Gal}(H^{\text{cent}}/M)) = r_\Phi(\mathrm{Gal}(H'^{\text{cent}}/M'))$.

Proof. We have $\mathrm{Gal}(M'/K') \cong G_p$ and $\mathrm{Gal}(M/M') \cong \mathfrak{h}$. Let H_1 be the maximal abelian subfield of H/K'. We see $H^{\text{ab}} \supset H_1$. $\mathrm{Gal}(H^{\text{ab}}/K')$ is a semidirect product of the p-Sylow subgroup $\mathrm{Gal}(H^{\text{ab}}/K)$ and $\mathfrak{h}$, because K/K' is a Galois extension and $p \not| [K : K']$. Hence, $\mathrm{Gal}(H_1K/K) \cong \mathrm{Gal}(H^{\text{ab}}/K)/\prod_{\sigma\in\mathfrak{h}} \mathrm{Gal}(H^{\text{ab}}/K)^{\sigma-1}$. We have the following isomorphism of R-modules:

$$\mathrm{Gal}(H_1K/K) \otimes \mathbf{F}_p \;\cong\; \mathrm{Gal}(H^{\text{ab}}/K) \otimes \mathbf{F}_p / \sum_{\sigma\in\mathfrak{h}} (\mathrm{Gal}(H^{\text{ab}}/K) \otimes \mathbf{F}_p)^{\sigma-1}.$$

By (1.4), we see $\mathrm{Gal}(H^{\text{ab}}/K) \otimes \mathbf{F}_p = 1_\mathfrak{h} \mathrm{Gal}(H^{\text{ab}}/K) \otimes \mathbf{F}_p \oplus I_\mathfrak{h} \mathrm{Gal}(H^{\text{ab}}/K) \otimes \mathbf{F}_p$. Thus, we have $1_\Phi \mathrm{Gal}(H_1K/K) \otimes \mathbf{F}_p \cong 1_\Phi \mathrm{Gal}(H^{\text{ab}}/K) \otimes \mathbf{F}_p$. Namely,

$$r_\Phi(\mathrm{Gal}(H_1K/K)) = r_\Phi(\mathrm{Gal}(H^{\text{ab}}/K)).$$

Let H_2 be the intermediate field corresponding to $\prod_{\sigma\in\mathfrak{h}} \mathcal{C}^{\sigma-1}$ in H/M. Since $\mathfrak{h}$ acts on $\mathrm{Gal}(H_1M/M)$ trivially, we have $H_2 \supset H_1$. Moreover, since $p \not| \sharp H^{-3}(\mathfrak{h}, \mathbf{Z})$, the central p-genus field of M/M' coincides with the abelian p-genus field of M/M'. We have $H_2 = H'M$. Note $H'M = H'K$, because of $H' \supset M'$ and $M = M'K$. Let H_2^{ab} be the maximal abelian subfield of H_2/K. Since $H_2 = H'K$, we have $\mathrm{Gal}(H_2/K) \cong \mathrm{Gal}(H'/K')$. Taking the maximal abelian quotients, we obtain $H_2^{\text{ab}} = {H'}^{\text{ab}}K$. This implies $H_1K = H'^{\text{ab}}K$, because of $H_2^{\text{ab}} \supset H_1 \supset H'^{\text{ab}}$. Therefore, we have $\mathrm{Gal}(H_1K/K) \cong \mathrm{Gal}(H'^{\text{ab}}/K')$ and

$$r_\Phi(\mathrm{Gal}(H_1K/K)) = r_\Phi(\mathrm{Gal}(H'^{\text{ab}}/K')).$$

This proves $r_\Phi(\mathrm{Gal}(H^{\text{ab}}/K)) = r_\Phi(\mathrm{Gal}(H'^{\text{ab}}/K'))$.

Let $N_{M/M'}$ be the norm map. By extending ideals of M' onto M, we have a natural map $j_{M/M'} : \mathcal{C}' \to \mathcal{C}$. Since $N_{M/M'} \circ j_{M/M'}$ is $[M : M']$-th power map, $j_{M/M'}$ is injective and $N_{M/M'}$ is surjective. Note

$$\frac{1}{(\mathfrak{h} : 1)} j_{M/M'} \circ N_{M/M'} \;=\; \tilde{1}_\mathfrak{h}.$$

We have $\tilde{1}_{\mathfrak{h}}\mathcal{C} \cong \mathcal{C}'$. Hence, $1_{\mathfrak{h}}\mathcal{C} \otimes \mathbf{F}_p \cong \mathcal{C}' \otimes \mathbf{F}_p$. Further, since $1_{\mathfrak{h}}\sigma = \sigma 1_{\mathfrak{h}}$ for every $\sigma \in G_p$, we have $\tilde{1}_{\mathfrak{h}} J\mathcal{C} \otimes \mathbf{F}_p \cong J\tilde{1}_{\mathfrak{h}}\mathcal{C} \otimes \mathbf{F}_p$. Therefore,

$$1_{\mathfrak{h}}(\mathcal{C} \otimes \mathbf{F}_p / J\mathcal{C} \otimes \mathbf{F}_p) \cong \mathcal{C}' \otimes \mathbf{F}_p / J\mathcal{C}' \otimes \mathbf{F}_p.$$

This proves $r_\Phi(\mathcal{C} \otimes \mathbf{F}_p / J\mathcal{C} \otimes \mathbf{F}_p) \cong r_\Phi(\mathcal{C}' \otimes \mathbf{F}_p / J\mathcal{C}' \otimes \mathbf{F}_p)$.

We have a commutative diagram

$$\begin{array}{ccc} & \nearrow & \tilde{1}_\Phi \operatorname{Gal}(H^{\mathrm{cent}}/M) \otimes \mathbf{F}_p \\ \tilde{1}_\Phi H^{-3}(G_p, \mathbf{Z}_p) & & \downarrow\cong \\ & \searrow & \tilde{1}_\Phi \operatorname{Gal}(H'^{\mathrm{cent}}/M') \otimes \mathbf{F}_p \end{array}$$

Thus, $1_\Phi\Theta \cong 1_\Phi\Theta'$. The lemma is proved.

§2. The Kummer group

We denote by ζ_m a primitive m-th root of unity. We are able to calculate the value of $r_\Phi(\mathcal{C})$ in $M(\zeta_p)$ by virtue of Lemma 4 if $M \not\ni \zeta_p$. We may suppose $\zeta_p \in M$. Let H^* be the maximal elementary $(p, \cdots, p)$-abelian subfield of H/K. We see $\operatorname{Gal}(H^*/K) = \operatorname{Gal}(H^{\mathrm{ab}}/K) \otimes \mathbf{F}_p$. Let $B = H^{\times p} \cap K^\times / K^{\times p}$ be the Kummer group of H^*/K. We consider B a submodule of $K^\times \otimes \mathbf{F}_p = K^\times / K^{\times p}$. Denote by $\mathbf{T}$ an R-module $< \zeta_p > \otimes \mathbf{F}_p$. The Kummer pairing is a non-degenerate pairing having values in $\mathbf{T}$:

$$< b, g > = \sqrt[p]{b}^{g-1} \otimes 1, \qquad b \in B, \quad g \in \operatorname{Gal}(H^*/K).$$

Since H^* is Galois over k, B and $\operatorname{Gal}(H^*/K)$ are R-modules. Let ω be an irreducible character afforded with $\mathbf{T}$. The action of $\sigma \in \mathfrak{g}$ satisfies the following relation on the pairing

$$< \sigma b, \sigma g > \quad = \quad \omega(\sigma) < b, g > .$$

Let $\hat{\Phi}$ be a character defined by $\hat{\Phi}(\sigma) = \Phi(\sigma^{-1})$. The reflection of Φ is defined to be a character $\hat{\Phi}\omega$. Denote by Φ^* the reflection of Φ. Let $\hat{L}_\Phi = \operatorname{Hom}(L_\Phi, \mathbf{F}_p)$ be an R-module where $\sigma \in \mathfrak{g}$ acts by $\sigma f = f \circ \sigma^{-1}$ for $f \in \hat{L}_\Phi$. Let $L^*_\Phi = \hat{L}_\Phi \otimes \mathbf{T}$ be an R-module where $\mathfrak{g}$ acts diagonally. $\hat{\Phi}$ (resp. Φ^*) is afforded with $\hat{L}_\Phi$ (resp. L^*_Φ).

Lemma 5. *We have*

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(1) $\hat{\hat{L}}_\Phi \cong L_\Phi$ *and* $L_\Phi^{**} \cong L_\Phi$,

(2) Φ^* *is irreducible,*

(3) $f_\Phi = f_{\Phi^*}$ *and* $d_\Phi = d_{\Phi^*}$.

Proof. By (iii) of Lemma 10.26, Curtis-Reiner [2], we have $\hat{\hat{L}}_\Phi \cong L_\Phi$. When finitely generated R-modules L_1 and L_2 are given, we consider $\mathrm{Hom}(L_1, L_2)$ (resp. $L_1 \otimes L_2$) be an R-module by $\sigma f = \sigma \circ f \circ \sigma^{-1}$ (resp. $\sigma(x \otimes y) = \sigma x \otimes \sigma y$) for $f \in \mathrm{Hom}(L_1, L_2)$ (resp. $x \otimes y \in L_1 \otimes L_2$). By Proposition 10.30, Curtis-Reiner [2], we have

$$\hat{L}_1 \otimes L_2 \cong \mathrm{Hom}(L_1, L_2),$$

where $\hat{L}_1 = \mathrm{Hom}(L_1, \mathbf{F}_p)$. Since $L_\Phi^{**} \cong \mathrm{Hom}(L_\Phi^*, \mathbf{F}_p) \otimes \mathbf{T}$ and $\hat{\mathbf{T}} \otimes \mathbf{T} \cong \mathbf{F}_p$, an isomorphism $L_\Phi^{**} \cong L_\Phi$ follows from

$$\begin{aligned} \mathrm{Hom}(\hat{L}_\Phi \otimes \mathbf{T}, \mathbf{F}_p) \otimes \mathbf{T} \;&\cong\; \mathrm{Hom}(\mathbf{T}, \mathrm{Hom}(\hat{L}_\Phi, \mathbf{F}_p)) \otimes \mathbf{T} \\ &\cong\; \mathrm{Hom}(\mathbf{T}, L_\Phi) \otimes \mathbf{T} \;\cong\; \hat{\mathbf{T}} \otimes L_\Phi \otimes \mathbf{T} \;\cong\; L_\Phi. \end{aligned}$$

Suppose $L_\Phi^* = M_1 \oplus M_2$ for non-trivial submodules M_i. We have $L_\Phi^{**} \cong M_1^* \oplus M_2^*$. However, this contradicts to that L_Φ is simple. Thus, L_Φ^* is simple and Φ^* is irreducible. Since $\mathrm{Hom}(L_\Phi, L_\Phi)^{\mathfrak{g}} = \mathrm{Hom}_R(L_\Phi, L_\Phi)$ and $\hat{L}_\Phi \otimes L_\Phi \cong \mathrm{Hom}(L_\Phi, L_\Phi)$, we have $(\hat{L}_\Phi \otimes L_\Phi)^{\mathfrak{g}} \cong \mathrm{Hom}_R(L_\Phi, L_\Phi)$. By (10.32), Curtis-Reiner [2], we also have $\mathrm{Hom}_R(L_\Phi, L_\Phi) \cong \mathrm{Hom}_R(\hat{L}_\Phi, \hat{L}_\Phi)$. This implies $(\hat{L}_\Phi \otimes L_\Phi)^{\mathfrak{g}} \cong (L_\Phi \otimes \hat{L}_\Phi)^{\mathfrak{g}}$. Since $\hat{L}_{\Phi^*} \otimes L_{\Phi^*} = \mathrm{Hom}(\hat{L}_\Phi \otimes \mathbf{T}, \mathbf{F}_p) \otimes (\hat{L}_\Phi \otimes \mathbf{T}) \cong (\hat{\mathbf{T}} \otimes L_\Phi) \otimes (\hat{L}_\Phi \otimes \mathbf{T}) \cong L_\Phi \otimes \hat{L}_\Phi$, we obtain $(\hat{L}_{\Phi^*} \otimes L_{\Phi^*})^{\mathfrak{g}} \cong (L_\Phi \otimes \hat{L}_\Phi)^{\mathfrak{g}}$. Therefore,

$$\begin{aligned} f_\Phi \;&=\; \dim \mathrm{Hom}_R(L_\Phi, L_\Phi) \;=\; \dim(\hat{L}_\Phi \otimes L_\Phi)^{\mathfrak{g}} \\ &=\; \dim(\hat{L}_{\Phi^*} \otimes L_{\Phi^*})^{\mathfrak{g}} \;=\; \dim \mathrm{Hom}_R(L_{\Phi^*}, L_{\Phi^*}) \;=\; f_{\Phi^*}. \end{aligned}$$

Since $\dim L_\Phi = \dim L_{\Phi^*}$, we have $d_\Phi = d_{\Phi^*}$ from (1.1).

We have $1_\Phi B$ and $1_{\Phi^*}\, \mathrm{Gal}(H^*/K)$ are dual to each other by means of the Kummer pairing. Hence, a formula

$$\dim 1_\Phi B \;=\; \dim 1_{\Phi^*}\, \mathrm{Gal}(H^*/K) \tag{2.1}$$

is obtained. Since $f_\Phi d_\Phi = f_{\Phi^*} d_{\Phi^*}$, we have

$$r_\Phi(B) \;=\; r_{\Phi^*}(\mathrm{Gal}(H^{\mathrm{ab}}/K)) \tag{2.2}$$

from (1.2) and (2.1).

In accordance with Leopoldt [6], we are able to obtain a formula for $r_\Phi(B)$ by modifying the argument developed there. Denote by K_v the completion of K at a place v. Let U_v be the unit group of K_v when v is a finite place. Let μ_v be the p-primary torsion subgroup of $K_v^\times$. We have a tower of Kummer extensions:

$$K_v^* \;=\; K_v(\sqrt[p]{x} : x \in K_v^\times) \;\supset\; \tilde{K}_v^* \;=\; K_v(\sqrt[p]{x} : x \in U_v) \;\supset\; K_v.$$

Denote by v_0 a place of k. Let v (resp. w) denote a prolongation of v_0 (resp. v) onto K (resp. M). Suppose $v_0|p$. Let p^f be the absolute degree of the valuation ideal $\mathfrak{p}_v$. $K_v(\zeta_{p^{pf}-1})$ is the unramified abelian extension of degree p over K_v. Put

$$\tilde{\xi}_v \;=\; \sum_{i=0}^{p-1} \zeta_{p^{pf}-1}^{p^{fi}} \zeta_p^i.$$

We have $\tilde{\xi}_v^p \in K_v$ and $K_v(\zeta_{p^{pf}-1}) = K_v(\tilde{\xi}_v)$. Let ξ_v be an element of U_v such that $\tilde{\xi}_v^p \in \xi_v K_v^{\times p}$. We have $K_v(\sqrt[p]{\xi_v}) = K_v(\zeta_{p^{pf}-1})$. Let K_v' be the inertia field in M_w/K_v. Similarly, there is an unit ξ_v' of K_v' such that $K_v'(\sqrt[p]{\xi_v'})$ is the unramified abelian extension of degree p over K_v'.

Lemma 6. *Let v_0 be a finite place of k. Let a be an element of $K_v^\times$.*

(1) *Suppose $v_0 \nmid p$. $M_w(\sqrt[p]{a})/M_w$ is unramified if and only if the ramification index of $K_v(\sqrt[p]{a})/K_v$ divides that of M_w/K_v.*

(2) *Suppose $v_0|p$. Let M_w^{ab} be the maximal abelian subfield of M_w/K_v. We have $M_w(\sqrt[p]{a})$ is unramified if and only if $\sqrt[p]{a} \in M_w^{\mathrm{ab}}(\sqrt[p]{\xi_v'})$.*

Proof. Since $M_w(\sqrt[p]{a})/M_w$ is tamely ramified for $v_0 \nmid p$, (1) is obvious. $M_w(\sqrt[p]{\xi_v'})$ is unramified and of degree p over M_w for $v_0|p$. Let G_w (resp. G_w') be the Galois group of M_w/K_v (resp. $M_w(\sqrt[p]{\xi_v'})/K_v$)). Denote by G_w^c and $G_w'^c$ the subgroups generated by commutators. Since G_w is a homomorphic image of G_w', we have $\sharp G_w^c \le \sharp G_w'^c$. We see $\sharp\, G_w' = p\sharp\, G_w$ and $(G_w' : G_w'^c) \ge p(G_w : G_w^c)$. Hence, $\sharp G_w'^c = \sharp G_w^c$. This implies that the maximal abelian subfield of $M_w(\sqrt[p]{\xi_v'})/K_v$ is $M_w^{\mathrm{ab}}(\sqrt[p]{\xi_v'})$. This proves (2), because $K_v(\sqrt[p]{a})$ is an abelian extension of K_v.

Let P be the set consisting of every place of k_0 lying above p. Denote by $P(K)$ the set of every prolongation onto K of every place contained in

P. We use this convention for an arbitrary set of places of k. Denote by $e(w/v)$ the ramification index of M_w/K_v for a finite place v_0. Let T and S be sets of places of k defined by

$$\begin{aligned} T &= \{v_0 \notin P : v_0 \not| \infty,\ p|e(w/v)\} \cup \{v_0 \in P : M_w^{\mathrm{ab}}(\sqrt[p]{\xi_v'}) \cap K_v^* \not\subset \tilde{K}_v^*\}, \\ S &= \{v_0 : w \text{ is real and } K_v^{\times p} \neq K_v^\times\}, \end{aligned}$$

respectively. Let V_v be a closed subgroup of $K_v^\times$ such that

$$(2.3) \qquad V_v = \begin{cases} M_w^{\mathrm{ab}}(\sqrt[p]{\xi_v'})^{\times p} \cap U_v & \text{if} \quad v \in P(K)\backslash T(K), \\ M_w^{\mathrm{ab}}(\sqrt[p]{\xi_v'})^{\times p} \cap K_v^\times & \text{if} \quad v \in P(K) \cap T(K), \\ K_v^{\times p} & \text{if} \quad v \in S(K). \end{cases}$$

Note V_v contains U_v^p (resp. $K_v^{\times p}$) if $v \in P(K)\backslash T(K)$ (resp. $v \in P(K) \cap T(K)$). We observe that $M(\sqrt[p]{a})/M$ is unramified for $a \in K^\times$ if and only if $a \in V_v K_v^{\times p}$ for every $v \in P(K) \cup S(K)$ and $v(a) \equiv 0 \bmod p$ for every finite places v not contained in $T(K)$, where we abuse notation and denote by the same symbol v the normalized additive valuation belonging to v. B is a subgroup of $K^\times \otimes \mathbf{F}_p$ consisting of $a \otimes 1$ for every $a \in K^\times$ satisfying this condition.

Let E_T be the group of T-units of K: $E_T = \{a \in K^\times : v(a) = 0 \text{ for every finite place } v \notin T(K)\}$. Let $P_T = K^\times/E_T$ and $D_T = \oplus_{v \not| \infty, v \notin T(K)} K_v^\times/U_v$. P_T is considered a subgroup of D_T by a diagonal map. We have an exact sequence

$$0 \to E_T \otimes \mathbf{F}_p \xrightarrow{i} K^\times \otimes \mathbf{F}_p \xrightarrow{j} P_T \otimes \mathbf{F}_p \to 0,$$

because P_T is torsion free. Put $B_1 = i^{-1}(\ker j \cap B)$. $j(B)$ consists of $(a) \otimes 1$ such that $a \otimes 1 \in B$. Let C_T be the p-torsion subgroup of D_T/P_T. Let $(a) \otimes 1$ be an element of $j(B)$ with $a \otimes 1 \in B$. Since $v(a) \equiv 0 \bmod p$ for every finite place $v \notin T(K)$, there is $\mathfrak{a} \in D_T$ such that $\mathfrak{a}^p = (a)$. We observe $\mathfrak{a}$ is principal if and only if $a \in K^{\times p} E_T$. Since $a \in K^{\times p} E_T$ implies $a \otimes 1 \in \ker j$, a correspondence $(a) \otimes 1 \to cl(\mathfrak{a})$ defines an injective homomorphism of $j(B)$ into C_T. Denote by B_0 the image of this homomorphism. Observe that $cl(\mathfrak{a}) \in B_0$ if and only if there is $a \in K^\times$ such that $\mathfrak{a}^p = (a)$ and $a \in V_v K_v^{\times p}$ for every $v \in P(K) \cup S(K)$. We have Leopoldt's decomposition of the Kummer group B:

$$0 \to B_1 \to B \to B_0 \to 0,$$

c.f. Leopoldt [6]. Hence, we obtain a formula

$$r_\Phi(B) \quad = \quad r_\Phi(B_1) + r_\Phi(B_0). \tag{2.4}$$

Let V be a direct product of V_v for $v \in P(K) \cup S(K)$. Put

$$U \quad = \quad \prod_{v \in P(K) \backslash T(K)} U_v \times \prod_{v \in P(K) \cap T(K)} K_v^\times \times \prod_{v \in S(K)} K_v^\times. \tag{2.5}$$

V is an open subgroup of U. Let $\iota : E_T \to U$ be the diagonal map. $\iota(E_T)V$ is also an open subgroup. Let U' be a closed subgroup of the idele group of K such that

$$U' \quad = \quad \prod_{v \nmid \infty, v \notin P(K) \cup T(K)} U_v \times \prod_{v \in T(K) \backslash P(K)} K_v^\times \times \prod_{v \mid \infty, v \notin S(K)} K_v^\times.$$

Let K_0 be the class field of K corresponding to an open subgroup $K^\times UU'$ of the idele group of K. Let K_1 be a class field of K such that $K_1 \supset K_0$ and $\mathrm{Gal}(K_1/K_0) \cong K^\times UU'/K^\times VU'$. Since $K^\times \cap UU' = E_T$, we have $\mathrm{Gal}(K_1/K_0) \cong UU'/E_T VU'$. Further, by projection onto U, we have

$$\mathrm{Gal}(K_1/K_0) \quad \cong \quad U/\iota(E_T)V.$$

Since $V \supset U^p$, we have a homomorphism $E_T \otimes \mathbf{F}_p \to U/V$. Observe that the cokernel is $U/\iota(E_T)V$ and that the kernel is B_1. We have an exact sequence

$$0 \to B_1 \to E_T \otimes \mathbf{F}_p \to U/V \to \mathrm{Gal}(K_1/K_0) \to 0. \tag{2.6}$$

Let $K_{P\backslash T}^\times$ be a subgroup of $K^\times$ consisting of a such that $v(a) = 0$ for every $v \in P(K) \backslash T(K)$. We extend ι onto $K_{P\backslash T}^\times$. Note $\iota(K_{P\backslash T}^\times)^p \subset V$. Each element $c \in C_T$ contains $\mathfrak{a} \in D_T$ such that $\mathfrak{a}^p$ is generated by an element a of $K_{P\backslash T}^\times$. If there is $\mathfrak{a}' \in c$ such that $\mathfrak{a}'^p = (a')$ for $a' \in K_{P\backslash T}^\times$, we have $b \in K^\times$ and $x \in E_T$ such that $a = xb^p a'$. We see $\iota(b^p) \in V$, because $b^p \in K_{P\backslash T}^\times$ means $b \in K_{P\backslash T}^\times$. Hence, $\iota(a)\iota(E_T)V = \iota(a')\iota(E_T)V$. Therefore, a homomorphism

$$\rho : C_T \quad \to \quad \mathrm{Gal}(K_1/K_0) \tag{2.7}$$

is well-defined by $\rho(c) = \iota(aE_T)V$. Let $(a) \otimes 1 \in j(B)$. We may assume $a \in V_v K_v^{\times p}$ for every $v \in P(K) \cup S(K)$. Note $V_v K_v^{\times p} = V_v$ for $v \in S(K)$. By approximation theorem of valuations, there exists $b_v \in K^\times$ for each $v \in P(K) \backslash T(K)$ such that $v(b_v) = 1$ and $u(b_v) = 0$ for every $u \in P(K) \backslash (T(K) \cup$

$\{v\})$. Since $v(ab_v^{-v(a)}) = 0$ and $v(a) \equiv 0 \bmod p$, there is $b \in K^\times$ such that $ab^p \in K_{P\backslash T}^\times$. We have $\iota(ab^p) \in V$. This implies $B_0 \subset \ker \rho$. Conversely, let $cl(\mathfrak{a}) \in \ker \rho$. There are $a \in K_{P\backslash T}^\times$, $b \in K^\times$ such that $(a) = (b^p)\mathfrak{a}^p$, and there are $x \in E_T$, $w \in V$ such that $\iota(a) = \iota(x)w$. Since $\iota(ax^{-1}) \in V$, we have $ax^{-1} \otimes 1 \in B$. This means $cl(\mathfrak{a}) \in B_0$. We have $\ker \rho \subset B_0$. Therefore, another exact sequence

$$(2.8) \qquad 0 \to B_0 \to C_T \xrightarrow{\rho} \mathrm{Gal}(K_1/K_0)$$

is obtained. Let $\alpha(\Phi) = r_\Phi(\mathrm{coker}\,\rho)$

THEOREM 7. *Put* $\beta_T(\Phi) = r_\Phi(E_T \otimes \mathbf{F}_p)$, $\gamma_T(\Phi) = r_\Phi(C_T)$, $\kappa_1(\Phi) = r_\Phi(U \otimes \mathbf{F}_p)$ *and* $\kappa_2(\Phi) = r_\Phi(V/U^p)$. *We have* $r_{\Phi^*}(\mathrm{Gal}(H^{\mathrm{ab}}/K)) = r_\Phi(B_1) + r_\Phi(B_0)$ *and*

$$\beta_T(\Phi)+\gamma_T(\Phi) \;\geq\; r_\Phi(B_1)+r_\Phi(B_0) \;=\; \alpha(\Phi)+\beta_T(\Phi)+\gamma_T(\Phi)-\kappa_1(\Phi)+\kappa_2(\Phi).$$

Proof. By (2.2) and (2.4), the value of $r_{\Phi^*}(\mathrm{Gal}(H^{\mathrm{ab}}/K))$ equals the sum of $r_\Phi(B_1)$ and $r_\Phi(B_0)$. We see $\beta_T(\Phi) \geq r_\Phi(B_1)$ and $\gamma_T(\Phi) \geq r_\Phi(B_0)$. Hence $\beta_T(\Phi) + \gamma_T(\Phi) \geq r_{\Phi^*}(\mathrm{Gal}(H^{\mathrm{ab}}/K))$. Since $U/V \cong (U/U^p)/(V/U^p)$, we have $r_\Phi(U/V) = r_\Phi(U/U^p) - r_\Phi(V/U^p)$. The values of $r_\Phi(B_1)$ and $r_\Phi(B_0)$ are described with $\alpha(\Phi)$, $\beta_T(\Phi)$, $\gamma_T(\Phi)$ and $\kappa_i(\Phi)$ by means of the sequences (2.6) and (2.8).

§3. The character of the representation on $\mathrm{Gal}(H^*/K)$

The character afforded with a finitely generated R-module Y may have no significance. We have

$$Y \;=\; \bigoplus_\Phi 1_\Phi Y \;\cong\; L_\Phi^{r_\Phi(Y)}.$$

However, if $r_\Phi(Y) \equiv 0 \bmod p$ for every Φ, the character is 0. By this reason, we introduce a free abelian group $ch(R)$ on $\mathfrak{B}(\mathfrak{g})$. $\sum m_\Phi \Phi > \sum n_\Phi \Phi$ means $m_\Phi \geq n_\Phi$ holds for every Φ and there is at least one Φ such that $m_\Phi > n_\Phi$. In the sequel, if we say *a character afforded with* Y or *a character of the representation on* Y, we mean an element $\sum_\Phi r_\Phi(Y)\Phi$ of $ch(R)$.

Let α (resp. β_T, γ_T) be the character of the representation of $\mathfrak{g}$ on an R-module coker ρ (resp. $E_T \otimes \mathbf{F}_p$, C_T). Let κ_1 (resp. κ_2) be the character afforded with an R-module $U \otimes \mathbf{F}_p$ (resp. V/U^p). Let φ_B be the character afforded with B. By virtue of Theorem 7, we have

$$(3.1) \qquad \beta_T + \gamma_T \;\geq\; \varphi_B \;=\; \alpha + \beta_T + \gamma_T - \kappa_1 + \kappa_2.$$

LEMMA 8. *Let P and Q be finitely generated $\mathbf{Z}_p$-torsion free A-modules. If $P \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \cong Q \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, we have $P \otimes_{\mathbf{Z}_p} \mathbf{F}_p \cong Q \otimes_{\mathbf{Z}_p} \mathbf{F}_p$.*

Proof. By virtue of Corollary 18.16, Curtis-Reiner [2], we have $P \cong Q$ if $P \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \cong Q \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Hence, $P \otimes_{\mathbf{Z}_p} \mathbf{F}_p \cong Q \otimes_{\mathbf{Z}_p} \mathbf{F}_p$.

Let T_∞ be union of T and all of infinite places $\{v_{0,1}, \cdots, v_{0,r}\}$ of k. For each $v_0 \in T_\infty$, we choose a prolongation v onto K and fix it once for all. Denote by v_i the prolongation of $v_{0,i}$. Let $\mathfrak{g}_v \subset \mathfrak{g}$ be the decomposition group of v. Denote by $\mathbf{Z}\mathfrak{g}/\mathfrak{g}_v$ a left $\mathbf{Z}\mathfrak{g}$-module

$$\bigoplus_{\bar{\sigma} \in \mathfrak{g}/\mathfrak{g}_v} \mathbf{Z}\bar{\sigma},$$

where $\bar{\sigma}$ denotes a coset $\sigma\mathfrak{g}_v$. Similarly, we denote by $\mathbf{F}_p\mathfrak{g}/\mathfrak{g}_v$ a left R-module generated by $\bar{\sigma}$. We see $\mathbf{Z}\mathfrak{g}/\mathfrak{g}_v \otimes \mathbf{F}_p = \mathbf{F}_p\mathfrak{g}/\mathfrak{g}_v$.

We have a system of Minkowsky's units of the relative Galois field K/k. Namely, there exist units H_i of K such that

$$\begin{aligned} |H_i|_{v_i} &> 1, \qquad |H_i|_{\sigma v_j} < 1 \quad \text{if} \quad i \neq j \ \text{or} \ \ \sigma \notin \mathfrak{g}_{v_i}, \\ \sigma H_i &= H_i \qquad \text{for} \ \sigma \in \mathfrak{g}_{v_i}, \end{aligned}$$

where $|\cdot|_{v_i}$ is the multiplicative valuation belonging to v_i. Let F be a $\mathfrak{g}$-submodule of the unit group $E_\emptyset$ generated by $\{H_i\}_{i=1}^r$. Let μ_K be the torsion submodule of $E_\emptyset$. We have $(E_\emptyset : F\mu_K) < \infty$. Put $W_\infty = \oplus_{i=1}^r \mathbf{Z}\mathfrak{g}/\mathfrak{g}_{v_i}$. Denote by $\bar{H}_i$ the image of H_i into $F\mu_K/\mu_K$. We have a surjective $\mathfrak{g}$-homomorphism of W_∞ onto $F\mu_K/\mu_K$:

$$\ell_\infty : \left(\sum_{\bar{\sigma} \in \mathfrak{g}/\mathfrak{g}_{v_i}} m_{i,\bar{\sigma}}\bar{\sigma} : 1 \leq i \leq r \right) \longrightarrow \prod_{i=1}^r \prod_{\bar{\sigma} \in \mathfrak{g}/\mathfrak{g}_{v_i}} \sigma\bar{H}_i^{m_{i,\bar{\sigma}}}.$$

Put $e_i = \sum_{\bar{\sigma} \in \mathfrak{g}/\mathfrak{g}_{v_i}} \tilde{\sigma} \in \mathbf{Z}\mathfrak{g}$, where $\tilde{\sigma}$ denotes a representative of $\bar{\sigma}$. $h_i = e_i H_i$ is a unit of k and there is a non-trivial relation

$$1 = \prod_{i=1}^r h_i^{m_i}.$$

Put $e = (m_i e_i : 1 \leq i \leq r) \in W_\infty$. We see $\ker \ell_\infty \supset \mathbf{Z}\mathfrak{g}e \neq 0$. Since the $\mathbf{Z}$-rank of $E_\emptyset$ equals $\sum_{i=1}^r (\mathfrak{g} : \mathfrak{g}_{v_i}) - 1$ and since the action of $\mathfrak{g}$ on $\mathbf{Z}\mathfrak{g}e$ is trivial, we have $\ker \ell_\infty$ is a trivial $\mathfrak{g}$-module of rank 1. Set $P = E_\emptyset/\mu_K \otimes \mathbf{Z}_p$

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and $Q = (W_\infty \otimes \mathbf{Z}_p)/(\ker \ell_\infty \otimes \mathbf{Z}_p)$ in Lemma 8. Since $W_\infty/\ker \ell_\infty$ is torsion free, we have

$$E_\emptyset/\mu_K \otimes \mathbf{F}_p \;\cong\; W_\infty \otimes \mathbf{F}_p/\ker \ell_\infty \otimes \mathbf{F}_p \;\cong\; \left(\bigoplus_{i=1}^{r} \mathbf{F}_p\mathfrak{g}/\mathfrak{g}_{v_i}\right)/\mathbf{F}_p.$$

Similarly, E_T contains an element H_v for each $v_0 \in T$ such that $v(H_v) > 0$ and $u(\sigma H_v) = 0$ if $v \neq u$ or if $\sigma \notin \mathfrak{g}_v$. Let F_T be a $\mathfrak{g}$-submodule of E_T generated by $\{H_v\colon v_0 \in T\}$ and $E_\emptyset$. We have $(E_T : F_T) < \infty$ and the $\mathbf{Z}$-rank of $F_T/E_\emptyset$ equals $\sum_{v_0\in T}(\mathfrak{g} : \mathfrak{g}_v)$. Put $W_T = \oplus_{v_0\in T}\mathbf{Z}\mathfrak{g}/\mathfrak{g}_v$. By considering a surjective homomorphism $W_T \to F_T/E_\emptyset$, we obtain $W_T \cong F_T/E_\emptyset$. Set $P = E_T/E_\emptyset \otimes \mathbf{Z}_p$ and $Q = W_T \otimes \mathbf{Z}_p$ in Lemma 8. We have

$$E_T/E_\emptyset \otimes \mathbf{F}_p \quad\cong\quad \bigoplus_{v_0\in T} \mathbf{F}_p\mathfrak{g}/\mathfrak{g}_v. \tag{3.2}$$

Denote by R_v the group ring $\mathbf{F}_p\mathfrak{g}_v$. Let $\mathrm{ind}_v\mathbf{F}_p$ be the induced module $\mathrm{ind}_{\mathfrak{g}_v}^{\mathfrak{g}}\mathbf{F}_p = R \otimes_{R_v} \mathbf{F}_p$ of the trivial R_v-module $\mathbf{F}_p$. Note $\mathrm{ind}_v\mathbf{F}_p = \mathbf{F}_p\mathfrak{g}/\mathfrak{g}_v$. Let ε_{v_0} be a character afforded with $\mathrm{ind}_v\mathbf{F}_p$. Since the sequences

$$\begin{aligned} 0 \;\to\; \mu_K \otimes \mathbf{F}_p \;\to\; E_\emptyset \otimes \mathbf{F}_p \;\to\; E_\emptyset/\mu_K \otimes \mathbf{F}_p \;\to\; 0,\\ 0 \;\to\; E_\emptyset \otimes \mathbf{F}_p \;\to\; E_T \otimes \mathbf{F}_p \;\to\; E_T/E_\emptyset \otimes \mathbf{F}_p \;\to\; 0\end{aligned}$$

are exact, we have

$$\beta_T \quad=\quad \omega + \sum_{v_0\in T_\infty} \varepsilon_{v_0} - \varepsilon \tag{3.3}$$

from the above isomorphisms.

Let $U_v^{(1)}$ be the group of principal units of K_v for $v \in P(K)$ and μ_v be the p-primary torsion subgroup of $U_v^{(1)}$. $U_v \otimes \mathbf{F}_p \cong U_v/U_v^p$ is isomorphic to $U_v^{(1)} \otimes \mathbf{F}_p$ and $U_v^{(1)}/\mu_v$ is a torsion free $\mathbf{Z}_p$-module. Thus, the sequence

$$0 \;\to\; \mu_v \otimes \mathbf{F}_p \;\to\; U_v \otimes \mathbf{F}_p \;\to\; U_v^{(1)}/\mu_v \otimes \mathbf{F}_p \;\to\; 0$$

is exact. The p-adic logarithm maps $U_v^{(1)}$ into K_v, which is a $\mathfrak{g}_v$-homomorphism. Let $\mathfrak{p}_v$ be the valuation ideal of K_v. There is $m \geq 1$ such that the p-adic exponential function converges on $\mathfrak{p}_v^m$. Let b be a normal basis of K_v/k_{v_0} and $\{b_1, \cdots, b_{m_{v_0}}\}$ be a $\mathbf{Q}_p$-basis of k_{v_0}, where $m_{v_0} = [k_{v_0} : \mathbf{Q}_p]$. We have

$$K_v \quad=\quad \bigoplus_{i=1}^{m_{v_0}} \mathbf{Q}_p\mathfrak{g}_v(bb_i).$$

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We may assume $bb_i \in \mathfrak{p}_v^m$. Let F'_v be a $\mathbf{Z}_p\mathfrak{g}_v$-submodule of $\mathfrak{p}_v^m$ generated by $bb_1, \cdots, bb_{m_{v_0}}$. We have $F'_v \cong (\mathbf{Z}_p\mathfrak{g}_v)^{m_{v_0}}$. Let F_v be the inverse image of F'_v in $U_v^{(1)}$ by the p-adic logarithm. We have

$$F_v\mu_v/\mu_v \quad \cong \quad F'_v \quad \cong (\mathbf{Z}_p\mathfrak{g}_v)^{m_{v_0}}.$$

Since $(U_v^{(1)} : F_v) < \infty$, we obtain $U_v^{(1)}/\mu_v \otimes_{\mathbf{Z}_p} \mathbf{F}_p \cong R_v^{m_{v_0}}$ by putting $P = U_v^{(1)}/\mu_v$ and $Q = F_v\mu_v/\mu_v$ in Lemma 8. Note $U_v^{(1)}/\mu_v \otimes \mathbf{F}_p \cong U_v^{(1)}/\mu_v U_v^{(1)p} \cong U_v/\mu_v \otimes_{\mathbf{Z}} \mathbf{F}_p$. Hence, we have

$$U_v \otimes \mathbf{F}_p \quad \cong \quad (\mu_v \otimes \mathbf{F}_p) \oplus R_v^{m_{v_0}}. \tag{3.4}$$

Denote by ω_{v_0} a character afforded with $\mathrm{ind}_v\mu_v \otimes \mathbf{F}_p \cong \mathrm{ind}_v\mathbf{T}$. Let $\varphi_{\mathfrak{g}}$ denote the character of the left regular representation on R. Since $\mathrm{ind}_v R_v = R$ and $\prod_{u|v_0} U_u \otimes \mathbf{F}_p \cong \mathrm{ind}_v U_v \otimes \mathbf{F}_p$, the character of representation on $\prod_{u|v_0} U_u \otimes \mathbf{F}_p$ is $\omega_{v_0} + m_{v_0}\varphi_{\mathfrak{g}}$. Since $K_v^\times/U_v \cong \mathbf{Z}$, we have $\mathrm{ind}_v(K_v^\times/U_v) \otimes \mathbf{F}_p$ affords ε_{v_0}. Thus, the character of the representation on $\prod_{u|v_0} K_v^\times \otimes \mathbf{F}_p$ is $\omega_{v_0} + m_{v_0}\varphi_{\mathfrak{g}} + \varepsilon_{v_0}$ for $v_0 \in P \cap T$. If $v_0 \in S$, we denote by μ_u the torsion subgroup of $K_u^\times$ for $u|v_0$ and have $\prod_{u|v_0} \mu_u^\times \otimes \mathbf{F}_p$ affords ω_{v_0}. Note $\omega_{v_0} = 0$ if $p \neq 2$. We obtain

$$\kappa_1 \quad = \quad [k : \mathbf{Q}]\varphi_{\mathfrak{g}} + \sum_{v_0 \in P} \omega_{v_0} + \sum_{v_0 \in P\cap T} \varepsilon_{v_0} + \sum_{v_0 \in S} \omega_{v_0} \tag{3.5}$$

because of $\sum_{v_0 \in P} m_{v_0} = [k : \mathbf{Q}]$.

We have an isomorphism of R-modules:

$$V/U^p \quad \cong \quad \bigoplus_{v_0 \in P\backslash T} \mathrm{ind}_v V_v/U_v^p \oplus \bigoplus_{v_0 \in P\cap T} \mathrm{ind}_v V_v/K_v^{\times p}.$$

Let W_1 (resp. W_2) be the submodule generated by V_v (resp. $V_v \cap U_v$) in $K_v^\times \otimes \mathbf{F}_p$ (resp. $U_v \otimes \mathbf{F}_p$). By inclusion $U_v \otimes \mathbf{F}_p \subset K_v^\times \otimes \mathbf{F}_p$, we consider W_2 a submodule of W_1. Since $K_v^\times/U_v \cong \mathbf{Z}$, we have $W_1/W_2 \cong \mathbf{F}_p$ if $v_0 \in P \cap T$. Note $W_1 = W_2$ if $v_0 \in P\backslash T$. Let W_3 be a submodule of $U_v \otimes \mathbf{F}_p$ generated by $\xi_v \otimes 1$ over R_v. Since $K_v(\sqrt[p]{\xi_v})$ is compositum of K_v and the unramified abelian extension of degree p over k_{v_0}, we see $\mathrm{Gal}(K_v(\sqrt[p]{\xi_v})/K_v) \cong \mathbf{F}_p$ as R_v-modules. By the Kummer duality, this means $W_3 \cong \mathbf{T}$. Put $W_v = K_v^\times$ if $v_0 \in P \cap T$, and $W_v = U_v$ for $v_0 \in P\backslash T$. We have $\dim W_3 = 1$ and $(V_v/W_v^p) \cong (W_1/W_2) \oplus (W_2/W_3) \oplus W_3$. By (3.4), we have

$$R_v^{m_{v_0}} \;\cong\; (U_v \otimes \mathbf{F}_p)/W_3 \;\supset\; W_2/W_3.$$

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Let δ_{v_0} be the character afforded with $\mathrm{ind}_{v_0} W_2/W_3$. We have

$$(3.6)\quad \kappa_2 \;=\; \sum_{v_0 \in P} (\omega_{v_0} + \delta_{v_0}) + \sum_{v_0 \in P \cap T} \varepsilon_{v_0}, \qquad \delta_{v_0} \;\le\; m_{v_0}\varphi_{\mathfrak{g}}.$$

We have the following formula from (3.5), (3.6) and (2.6):

$$(3.7)\quad \begin{aligned} \kappa_1 - \kappa_2 &= [k:\mathbf{Q}]\varphi_{\mathfrak{g}} - \textstyle\sum_{v_0 \in P} \delta_{v_0} + \sum_{v_0 \in S} \omega_{v_0}, \\ \kappa_1 - \kappa_2 - \beta_T + \varphi_{B_1} &\ge \alpha \end{aligned}$$

where φ_{B_1} is the character of the representation onto B_1.

§4. The value of a_{Φ^*}

We denote by $\varphi(\Phi)$ the coefficient of Φ for an element φ of $ch(R)$. If φ is a character afforded with an R-module Y, we have

$$(4.1)\quad \varphi(\Phi) \;=\; r_\Phi(Y) \;=\; \frac{1}{f_\Phi} \dim \mathrm{Hom}_R(L_\Phi, Y) \;=\; \frac{1}{f_\Phi} \dim \mathrm{Hom}_R(Y, L_\Phi)$$

because of $f_\Phi = \dim \mathrm{Hom}_R(L_\Phi, L_\Phi)$. We recall characters:

characters	modules
α	coker $(C_T \xrightarrow{\rho} \mathrm{Gal}(K_1/K_0) = U/\iota(E_T)V)$
β_T	$E_T \otimes \mathbf{F}_p$
γ_T	C_T
δ_{v_0}	$\mathrm{ind}_v(V_v \cap U_v/U_v^p < \xi_v >)$ for $v_0 \in P$
ε_{v_0}	$\mathrm{ind}_v \mathbf{F}_p$ for $v \mid v_0$

Let θ be the character afforded with Θ.

THEOREM 9. *The value of* $a_{\Phi^*} = r_{\Phi^*}(\mathcal{C} \otimes \mathbf{F}_p / J\mathcal{C} \otimes \mathbf{F}_p)$ *satisfies an inequality*

$$\begin{aligned} &\beta_T(\Phi) + \gamma_T(\Phi) + \theta(\Phi^*) \\ \ge\; & a_{\Phi^*} \\ \ge\; & \alpha(\Phi) + \beta_T(\Phi) + \gamma_T(\Phi) + \theta(\Phi^*) + \sum_{v_0 \in P} \delta_{v_0}(\Phi) \\ & \qquad - \sum_{v_0 \in S} \omega_{v_0}(\Phi) - [k:\mathbf{Q}] d_\Phi - r_{\Phi^*}(\mathrm{Gal}(M^{\mathrm{ab}}/K)). \end{aligned}$$

Moreover, we have the following statements:

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(1) *If* $\tilde{1}_{\Phi^*}\operatorname{Gal}(M^{\mathrm{ab}}/K) = 0$ *or if* $\tilde{1}_{\Phi^*}\operatorname{Gal}(H^{\mathrm{ab}}/K)^p = 0$*, the value of* a_{Φ^*} *equals*

$$\alpha(\Phi) + \sum_{v_0\in T_\infty} \varepsilon_{v_0}(\Phi) + \omega(\Phi) - \varepsilon(\Phi) + \gamma_T(\Phi) + \theta(\Phi^*) + \sum_{v_0\in P} \delta_{v_0}(\Phi) \\ - \sum_{v_0\in S} \omega_{v_0}(\Phi) - [k:\mathbf{Q}]d_\Phi - r_{\Phi^*}(\operatorname{Gal}(M^{\mathrm{ab}}/K)).$$

(2) *If* $G = \mathfrak{g}\times G_p$*, we have* $\tilde{1}_{\Phi^*}\operatorname{Gal}(M^{\mathrm{ab}}/K)$ *and* $\tilde{1}_{\Phi^*}H^{-3}(G_p,\mathbf{Z}_p)$ *vanish for* $\Phi^*\neq\varepsilon$.

(3) *Suppose* $p>2$*. If* $\delta_{v_0}(\Phi) = m_{v_0}d_\Phi$ *for every* $v_0\in P$*, we have* $\alpha(\Phi)=0$.

Proof. The inequality and the statement (1) follow from Theorem 3, 7 and formulas (3.7), (3.3). If $G=\mathfrak{g}\times G_p$, we have $\mathfrak{g}$ acts trivially on G_p with conjugation. Hence, $\operatorname{Gal}(M^{\mathrm{ab}}/K)$ and $H^{-3}(G_p,\mathbf{Z}_p)$ are trivial $\mathfrak{g}$-modules. Hence,

$$\tilde{1}_{\Phi^*}\operatorname{Gal}(M^{\mathrm{ab}}/K) = 0, \qquad \tilde{1}_{\Phi^*}H^{-3}(G_p,\mathbf{Z}_p) = 0$$

whenever $\Phi^*\neq\varepsilon$. This proves (2). Suppose p is odd. We see $\omega_{v_0}=0$ for $v_0\in S$. If $\delta_{v_0}(\Phi)=m_{v_0}d_\Phi$ for every $v_0\in P$, we have $\sum_{v_0\in P}\delta_{v_0}(\Phi)=[k:\mathbf{Q}]d_\Phi$. By (3.7), we have $\alpha(\Phi)\le\kappa_1(\Phi)-\kappa_2(\Phi)=0$, because of $\beta_T-\varphi_{B_1}\ge 0$.

We denote by U_M, V_M the closed subgroup of the idele group of K defined by (2.3) and (2.5) with adding subscript to specify the field M. We also denote by ρ_M the map defined in (2.7). We write B_M, $B_{1,M}$, $B_{0,M}$ for B, B_1, B_0.

LEMMA 10. *Let* p *be an odd prime. If* $k_{v_0}\not\ni\zeta_p$ *and* $\mathfrak{g}_v$ *acts on* $\operatorname{Gal}(M_w^{\mathrm{ab}}/K_v)$ *trivially for every* $v_0\in P$*, we have* $T\cap P=\emptyset$*. Moreover, we have an* R_v*-isomorphism* $V_v/U_v^p\cong\mathbf{T}^c$ *for* $c=\dim\operatorname{Gal}(M_w^{\mathrm{ab}}(\sqrt[p]{\xi'_v})\cap K_v^*/K_v)$.

Proof. By (2.3), we see $V_v\,K_v^{\times p}/K_v^{\times p}$ is the Kummer radical of a Kummer extension $M_w^{\mathrm{ab}}(\sqrt[p]{\xi'_v})\cap K_v^*/K_v$. Let $\mathbf{Q}_p^{ur}$ be the maximal unramified abelian p-extension of $\mathbf{Q}_p$. Since $M_w^{\mathrm{ab}}(\sqrt[p]{\xi'_v})$ is a subfield $\mathbf{Q}_p^{ur}M_w^{\mathrm{ab}}$ and since $\mathfrak{g}_v$ acts trivially on $\operatorname{Gal}(\mathbf{Q}_p^{ur}M_w^{\mathrm{ab}}/K_v)$ with conjugation, we have $\operatorname{Gal}(M_w^{\mathrm{ab}}(\sqrt[p]{\xi'_v})/K_v)$ is a trivial $\mathfrak{g}_v$-module. Thus, considering the Kummer pairing, the Kummer radical is isomorphic to $\mathbf{T}^c$ for $c=\dim\operatorname{Gal}(M_w^{\mathrm{ab}}(\sqrt[p]{\xi'_v})\cap K_v^*/K_v)$. Note $\mathbf{T}\not\cong\mathbf{F}_p$ as $\mathfrak{g}_v$-modules, because of $k_{v_0}\not\ni$

ζ_p. Since $K_v^\times/U_vK_v^{\times p}$ is a trivial $\mathfrak{g}_v$-module, the image of $V_vK_v^{\times p}/K_v^{\times p} \to K_v^\times/U_vK_v^{\times p}$ is 0 in an exact sequence

$$0 \to V_v \cap U_v/U_v^p \to V_vK_v^{\times p}/K_v^{\times p} \to K_v^\times/U_vK_v^{\times p}.$$

Hence, $V_v \subset U_vK_v^{\times p}$. This proves $T \cap P = \emptyset$.

LEMMA 11. *Let p be an odd prime. Suppose that M is an abelian extension of $\mathbf{Q}$ containing ζ_{p^t} for $t \geq 2$ such that $K_v(\zeta_{p^t})$ is ramified over K_v for every $v_0 \in P$. Furthermore, we also suppose $k_{v_0} \not\ni \zeta_p$ for every $v_0 \in P$. Put $N = M \cap \cup_{n\geq 1}K(\zeta_{p^n})$. We have a commutative diagram*

$$\begin{array}{ccccccc}
0 & \longrightarrow & 1_\Phi B_{0,N} & \longrightarrow & 1_\Phi C_\emptyset & \xrightarrow{\rho_{N,\Phi}} & 1_\Phi(U_N/\iota(E_\emptyset)V_N) \\
 & & {\scriptstyle g'_\Phi}\downarrow & & {\scriptstyle g_\Phi}\downarrow & & {\scriptstyle h_\Phi}\downarrow \\
0 & \longrightarrow & 1_\Phi B_{0,M} & \longrightarrow & 1_\Phi C_T & \xrightarrow{\rho_{M,\Phi}} & 1_\Phi(U_M/\iota(E_T)V_M)
\end{array}$$

where $\rho_{M,\Phi}$ and $\rho_{N,\Phi}$ denote restriction onto the 1_Φ-components, respectively. Moreover, we have

$$\sum_{v_0\in T} \varepsilon_{v_0}(\Phi) \geq r_\Phi(\ker h_\Phi) = \sum_{v_0\in T} \varepsilon_{v_0}(\Phi) - r_\Phi(B_{1,M}) + r_\Phi(B_{1,N}).$$

If $\sum_{v_0\in T} \varepsilon_{v_0}(\Phi) = 0$, we have $r_\Phi(B_M) = r_\Phi(B_N)$.

Proof. By Lemma 10, we have $T \cap P = \emptyset$. Hence, $U_M = U_N$. To prove the lemma, we need to show $V_M = V_N$. Denote by $V_{M,v}$ be the v-component of the direct product $V_M = \prod_{v\in P(K)} V_v$. We see $V_{M,v} \subset U_v$. Since $\mathbf{Q}_p^{ur}M_w$ contains $M_w(\sqrt[p]{\xi'_v})$ and is abelian over $\mathbf{Q}_p$, there are $m \geq t$, $a > 0$ and $b > 0$ such that a is prime to p and so that $\mathbf{Q}_p(\zeta_{p^m}, \zeta_{p^{ap^b}-1})$ contains $M_w(\sqrt[p]{\xi'_v})$. The p-Sylow subgroup of $\mathrm{Gal}(\mathbf{Q}_p(\zeta_{p^m}, \zeta_{p^{ap^b}-1})/\mathbf{Q}_p)$ is isomorphic to $\mathbf{Z}/p^{m-1}\mathbf{Z} \times \mathbf{Z}/p^b\mathbf{Z}$. Hence, $\mathrm{Gal}(M_w(\sqrt[p]{\xi'_v})/K_v)$ is an abelian p-group generated by one or two elements. Let p^e be the ramification index of $K_v(\zeta_{p^t})/K_v$. We see

$$\mathrm{Gal}(K'_v(\zeta_{p^t}, \sqrt[p]{\xi'_v})/K'_v) \cong \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p^e\mathbf{Z},$$

because of $K'_v(\zeta_{p^t}) \cap K'_v(\sqrt[p]{\xi'_v}) = K'_v$, where K'_v is the inertia field of M_w/K_v. This means $\mathrm{Gal}(M_w(\sqrt[p]{\xi'_v})/K_v)$ is not cyclic. We have an R_v-isomorphism

$$\mathrm{Gal}(M_w(\sqrt[p]{\xi'_v}) \cap K_v^*/K_v) \cong \mathbf{F}_p^2.$$

Therefore, by Lemma 10, we have

$$V_{M,v}/U_v^p \cong \mathbf{T}^2. \tag{4.2}$$

In particular, we have $V_{N,v}/U_v^p \cong \mathbf{T}^2$ if $M = N$. Suppose $M \neq N$. Let u be restriction of w onto N and K_v'' be the inertia field of N_u/K_v. Let $K_v''(\sqrt[p]{\xi_v''})$ be an unramified cyclic extension of degree p over K_v''. Since $M_w(\sqrt[p]{\xi_v'}) \supset N_u(\sqrt[p]{\xi_v''})$, we have $V_{M,v} \supset V_{N,v}$. $V_{M,v} = V_{N,v}$ follows from isomorphisms $V_{M,v}/U_v^p \cong V_{N,v}/U_v^p \cong \mathbf{T}^2$. This proves $V_M = V_N$.

Denote by U (resp. V) the group $U_M = U_N$ (resp. $V_M = V_N$) with omitting the subscript. Let Y be a submodule of $D_\emptyset$ generated by $\{\sigma\mathfrak{p}_v : \sigma \in \mathfrak{g}, v_0 \in T\}$. We have $D_\emptyset = D_T \oplus Y$. Denote by $\mathfrak{a} = \mathfrak{a}' + \mathfrak{a}''$ the decomposition of $\mathfrak{a} \in D_\emptyset$ into a sum of $\mathfrak{a}' \in D_\emptyset$ and $\mathfrak{a}'' \in Y$. A homomorphism $D_\emptyset/P_\emptyset \to D_T/P_T$ is induced from $\mathfrak{a} \to \mathfrak{a}'$. Let $g : C_\emptyset \to C_T$ be restriction of this homomorphism onto the p-torsion submodules. Denote by h a canonical map $U/\iota(E_\emptyset)V \to U/\iota(E_T)V$. Let g_Φ (resp. h_Φ) be restriction of g (resp. h) onto the Φ-component. Put $g_\Phi' = g_{\Phi|1_\Phi B_{0,N}}$. Since $\mathfrak{a}^p = (a)$ in $D_\emptyset$ for $a \in K^\times$ implies $\mathfrak{a}'^p = (a)$ in D_T and since every element $c \in C_\emptyset$ contains $\mathfrak{a} \in D_\emptyset$ such that $(\mathfrak{a}, p) = 1$, the commutativity of the diagram follows from the definition (2.7) of the maps ρ_M and ρ_N.

We have $B_{1,N} \cong E_\emptyset \cap \iota^{-1}(V)/E_\emptyset^p$ and $B_{1,M} \cong E_T \cap \iota^{-1}(V)/E_T^p$. Observe

$$\begin{aligned}\ker h = \iota(E_T)V/\iota(E_\emptyset)V &\cong \iota(E_T)/(\iota(E_T) \cap V)\iota(E_\emptyset)\\ &\cong E_T/(E_T \cap \iota^{-1}(V))E_\emptyset,\end{aligned}$$

because of $\ker \iota \cap E_T \subset E_T \cap \iota^{-1}(V)$. Therefore, we obtain exact sequences

$$0 \to B_{1,N} \to B_{1,M} \to (E_T \cap \iota^{-1}(V))/(E_\emptyset \cap \iota^{-1}(V))E_T^p \to 0,$$

$$0 \to (E_T \cap \iota^{-1}(V))/(E_\emptyset \cap \iota^{-1}(V))E_T^p \to E_T/E_\emptyset E_T^p \to \ker h \to 0.$$

Thus, $r_\Phi(\ker h) \leq \sum_{v_0 \in T} \varepsilon_{v_0}(\Phi)$ follows from (3.2). Furthermore, we also have the formula of $r_\Phi(\ker h)$ from these sequences.

Suppose $\sum_{v_0 \in T} \varepsilon_{v_0}(\Phi) = 0$. By virtue of the formula, we obtain $\ker h_\Phi = 0$ and $r_\Phi(B_{1,N}) = r_\Phi(B_{1,M})$. Thus, h_Φ is an isomorphism. Since $D_T/P_T \cong D_\emptyset/P_\emptyset Y$, we have an exact sequence

$$\tilde{1}_\Phi Y \otimes \mathbf{Z}_p \to \tilde{1}_\Phi D_\emptyset/P_\emptyset \otimes \mathbf{Z}_p \to \tilde{1}_\Phi D_T/P_T \otimes \mathbf{Z}_p \to 0.$$

Denote by $\tilde{\mathfrak{p}}_v$ an element $\mathfrak{p}_v \otimes 1$ of $Y \otimes \mathbf{Z}_p$. We see $Y \otimes \mathbf{Z}_p = \oplus_{v_0 \in T} A \cdot \tilde{\mathfrak{p}}_v$. Since $A \cdot \tilde{\mathfrak{p}}_v \cong A \otimes_{\mathbf{Z}_p \mathfrak{g}_v} \mathbf{Z}_p$ and $1_\Phi(R \otimes_{R_v} \mathbf{F}_p) \cong L_\Phi^{\varepsilon_{v_0}(\Phi)}$, we have $1_\Phi(A \cdot \tilde{\mathfrak{p}}_v \otimes \mathbf{F}_p) = 0$.

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This implies $\tilde{1}_\Phi(A\cdot\tilde{\mathfrak{p}}_v)=0$. Hence, $\tilde{1}_\Phi Y\otimes\mathbf{Z}_p=0$. We see $\tilde{1}_\Phi D_\emptyset/P_\emptyset\otimes\mathbf{Z}_p\cong\tilde{1}_\Phi D_T/P_T\otimes\mathbf{Z}_p$. Since $1_\Phi C_\emptyset$ (resp. $1_\Phi C_T$) is the p-torsion submodule of $\tilde{1}_\Phi D_\emptyset/P_\emptyset\otimes\mathbf{Z}_p$ (resp. $\tilde{1}_\Phi D_T/P_T\otimes\mathbf{Z}_p$), we have $1_\Phi C_\emptyset\cong 1_\Phi C_T$. Therefore, g_Φ is an isomorphism. By the commutative diagram, g'_Φ is also an isomorphism. We obtain $r_\Phi(B_{0,N})=r_\Phi(B_{0,M})$. Consequently, $r_\Phi(B_M)=r_\Phi(B_N)$ holds by virtue of the formula (2.4).

Proposition 12. *Let notations and assumptions be same as those in Lemma 11. In addition, we suppose $k=\mathbf{Q}$ and that Φ satisfies $\tau 1_\Phi=1_\Phi$ for the complex conjugation τ. Then, we have $a_{\Phi^*}\leq 1$ if and only if Φ satisfies one of the following conditions:*

(1) $\gamma_T(\Phi)=\sum_{v_0\in T}\varepsilon_{v_0}(\Phi)=0$ *if* $\omega_p(\Phi)=1$,

(2) $r_\Phi(B_{1,M})\leq 1$ *and* $\gamma_T(\Phi)=0$ *if* $\omega_p(\Phi)=0$, $\Phi\neq\varepsilon$ *and* $\sum_{v_0\in T}\varepsilon_{v_0}(\Phi)=1$,

(3) $\gamma_P(\Phi^*)\leq 1$ *if* $\omega_p(\Phi)=0$, $\Phi\neq\varepsilon$ *and* $\sum_{v_0\in T}\varepsilon_{v_0}(\Phi)=0$,

(4) $\sharp T+\alpha(\varepsilon)\leq 2$ *if* $\Phi=\varepsilon$.

Proof. We may take $t=2$ in Lemma 11. Suppose $\tau 1_\Phi=1_\Phi$. Note $\tau 1_\Phi=1_\Phi$ is equivalent to $\tau 1_{\Phi^*}\neq 1_{\Phi^*}$. $\tau 1_{\Phi^*}\neq 1_{\Phi^*}$ means $\Phi^*\neq\varepsilon$. Since $M/\mathbf{Q}$ is abelian, we have $\theta(\Phi^*)=r_{\Phi^*}(\mathrm{Gal}(M^{\mathrm{ab}}/\mathbf{Q}))=0$ from (2) of Theorem 9. $\mathfrak{g}_v$ is a normal subgroup generated by τ for $v_0=\infty$. ε_∞ is afforded with $\mathbf{F}_p\mathfrak{g}/<\tau>$. Hence, $\varepsilon_\infty(\Phi)=1$. By (3.3), we have

$$\beta_T(\Phi)=1+\sum_{v_0\in T}\varepsilon_{v_0}(\Phi)-\varepsilon(\Phi).$$

Let $v_0=p$. By (3.6) and (4.2), we have $\kappa_2=\delta_p+\omega_p=2\omega_p$. Hence, $\delta_p=\omega_p$.

Suppose $\omega_p(\Phi)\neq 0$. We have $\delta_p(\Phi)=1$. Hence, $\Phi\neq\varepsilon$. Note $1=m_p=d_\Phi$. By (1) and (3) of Theorem 9, we observe a_{Φ^*} equals $1+\sum_{v_0\in T}\varepsilon_{v_0}(\Phi)+\gamma_T(\Phi)$. Thus, $a_{\Phi^*}\leq 1$ if and only if $\sum_{v_0\in T}\varepsilon_{v_0}(\Phi)=\gamma_T(\Phi)=0$.

Suppose $\omega_p(\Phi)=0$ and $\Phi\neq\varepsilon$. We have

$$a_{\Phi^*}\;=\;\alpha(\Phi)+\sum_{v_0\in T}\varepsilon_{v_0}(\Phi)+\gamma_T(\Phi).$$

We observe that $\sum_{v_0\in T}\varepsilon_{v_0}(\Phi)\leq 1$ if $a_{\Phi^*}\leq 1$. Firstly, let Φ satisfy $\sum_{v_0\in T}\varepsilon_{v_0}(\Phi)=1$. If $a_{\Phi^*}\leq 1$, we have $\alpha(\Phi)=\gamma_T(\Phi)=0$. Since $\gamma_T(\Phi)=0$

implies $r_\Phi(B_{0,M}) = 0$, we have $r_\Phi(B_{1,M}) \le 1$ from (2.4). Conversely, if $r_\Phi(B_{1,M}) \le 1$ and $\gamma_T(\Phi) = 0$, we also have $a_{\Phi^*} \le 1$ from (2.4). Secondly, let Φ satisfy $\sum_{v_0 \in T} \varepsilon_{v_0}(\Phi) = 0$. By virtue of Lemma 11, we are able to reduce to N. We may suppose $M = N$. Since $T \subset P$, we see $T = \emptyset$ from Lemma 10. Since $\kappa_2 = 2\omega_p$, we have $1_\Phi(V/U^p) = 0$. Put $W = \prod_{v \in P(K)} K_v^\times$. We also have $1_\Phi(VW^p/W^p) = 0$. Hence, $1_\Phi(W/VW^p) \cong 1_\Phi(W/W^p)$. Observe that $a \otimes 1 \in K^\times \otimes \mathbf{F}_p$ for $(a) \in D_\emptyset^p$ is an element of B_M if and only if the image of a into W/VW^p is equal to 0. When $a \otimes 1 \in 1_\Phi B_M$, we have $\iota(a) \in VW^p$ is equivalent to $\iota(a) \in W^p$, because of $1_\Phi(W/VW^p) \cong 1_\Phi(W/W^p)$. Namely, $K(\sqrt[p]{a})/K$ is unramified p-decomposed, where we call $K(\sqrt[p]{a})/K$ p-decomposed if every place lying above p is completely decomposed there. Let K_2 be the maximal unramified p-decomposed elementary $(p, \cdots, p)$-abelian extension of K. Let B' be the Kummer radical of K_2/K. By the Kummer duality, B' is dual to $D_P/P_P \otimes \mathbf{F}_p$. Hence, $1_\Phi B'$ and $1_{\Phi^*} D_P/P_P \otimes \mathbf{F}_p$ are dual to each other. We have $\gamma_P(\Phi^*) = r_\Phi(B')$. $1_\Phi B' \supset 1_\Phi B_M$ follows from the above argument. Since K_2 is a subfield of the maximal unramified abelian p-extension H of M, we have $B_M \supset B'$. We have $1_\Phi B_M = 1_\Phi B'$. Therefore, we obtain $a_{\Phi^*} = \gamma_P(\Phi^*)$.

Suppose $\Phi = \varepsilon$. We see $\Phi^* = \omega$. Let M_0 be the maximal p-extension of $\mathbf{Q}$ contained in M. We have $M = M_0 K$. By Lemma 4, we are able to compute the value a_ω in $M_0(\zeta_p)$. Hence, suppose $M = M_0(\zeta_p)$ and $K = \mathbf{Q}(\zeta_p)$. We have

$$a_\omega \;=\; \alpha(\varepsilon) + \sum_{v_0 \in T} \varepsilon_{v_0}(\varepsilon) - 1 + \gamma_T(\varepsilon).$$

Since the class number of $\mathbf{Q}$ is one, we have $\gamma_T(\varepsilon) = 0$. Therefore, $a_\omega = \alpha(\varepsilon) + \sharp T - 1$, because of $\varepsilon_{v_0}(\varepsilon) = \dim \mathrm{Hom}_R(R \otimes_{R_v} \mathbf{F}_p, \mathbf{F}_p) = \dim \mathrm{Hom}_{R_v}(\mathbf{F}_p, \mathbf{F}_p) = 1$.

PROPOSITION 13. *Let p be odd and k be a totally real field. Let M be a CM-field containing ζ_{p^n} for $n \ge 2$. Suppose $M = K(\zeta_{p^n})$ and $M \ne K$. Let h_M^+ be the class number of the maximal totally real subfield of M and h_K^- be the relative class number of K. Let τ be the complex conjugation of M.*

(1) *We have $p \nmid h_M^+$ if $p \nmid h_K^-$ and if $a_\varepsilon = 0$.*

(2) *Suppose $k = \mathbf{Q}$ and $M/\mathbf{Q}$ is abelian. We have $p \nmid h_M^+$, if $\gamma_\emptyset(\Phi^*) = 1$ and $\alpha(\Phi^*) = \omega_p(\Phi^*) = 0$ for every Φ^* such that $\Phi^* \ne \omega$, $\tau 1_{\Phi^*} \ne 1_{\Phi^*}$ and $\gamma_\emptyset(\Phi^*) > 0$.*

Proof. Let M_0 be the maximal p-extension over $\mathbf{Q}$ contained in $\mathbf{Q}(\zeta_{p^n})$. Since $M = M_0K$, we see $G = G_p \times \mathfrak{g}$. Hence, $\mathcal{C}$ is a $\mathbf{Z}_p\mathfrak{g}$-module. Since $\mathcal{C} = \oplus_\Phi \tilde{1}_\Phi \mathcal{C}$, we have $\mathcal{C}^{\tau+1} = 0$ if and only if $\tilde{1}_\Phi \mathcal{C} = 0$ for every Φ such that $\tau 1_\Phi = 1_\Phi$. Further, $\tilde{1}_\Phi \mathcal{C} = 0$ is equivalent to $a_\Phi = 0$. Thus, $p \not| h_M^+$ if and only if $a_\Phi = 0$ for every Φ such that $\tau 1_\Phi = 1_\Phi$.

By (2) of Theorem 9, we have $\theta(\Phi) = 0$ and $\tilde{1}_\Phi \operatorname{Gal}(M^{\mathrm{ab}}/K) = 0$ for $\Phi \neq \varepsilon$. Note $P \supset T$. We can show $T = \emptyset$. In fact, for $v \in P(K)$, if M_w/K_v is unramified, we see $M_w(\sqrt[p]{\xi'_v}) \cap K_v^* = K_v(\sqrt[p]{\xi_v})$, because $M_w(\sqrt[p]{\xi'_v})/K_v$ is unramified. Contrarily, if M_w/K_v is ramified, we have $M_w(\sqrt[p]{\xi'_v})/K_v$ is not cyclic, because $M_w(\sqrt[p]{\xi'_v})/M_w$ is an unramified extension of degree p. Hence $\operatorname{Gal}(M_w(\sqrt[p]{\xi'_v})/K_v) \cong \mathbf{Z}/p^a\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$, where $p^a = [M_w : K_v]$. Let $K_v(\sqrt[p]{x})/K_v$ be an extension of degree p for $x \in K_v^\times$ such that $M_wK_v(\sqrt[p]{x}) = M_w(\sqrt[p]{\xi'_v})$. Since $K'_v \subset M_w = K_v(\zeta_{p^n})$, there is $1 \leq e < n$ such that $K'_v = K_v(\zeta_{p^e})$. Considering the structure of the Galois group, we have $K_v(\zeta_{p^{e+1}}, \sqrt[p]{x})$ contains $K'_v(\sqrt[p]{\xi'_v})$. Since $\{\zeta_{p^{e+1}}^p, \sqrt[p]{x}^p, \sqrt[p]{\xi'_v}^p\} \subset K'_v$, there are integers $0 \leq t, s < p$ such that $\xi'_v = \zeta_{p^e}^t x^s y^p$ for $y \in K_v'^\times$. We see $x^s = \xi'_v \zeta_{p^e}^{-t} y^{-p} \in K_v$. Note $s \neq 0$, because $K'_v(\zeta_{p^{e+1}})/K'_v$ is ramified. Since K'_v/K_v is unramified, this shows that $x \in U_v K_v^{\times p}$. Thus, we have $T = \emptyset$.

Since τ generates a normal subgroup, we have $\beta_\emptyset(\Phi^*) = 0$ for $\Phi \neq \varepsilon$ such that $\tau 1_\Phi = 1_\Phi$. Therefore,

$$(4.3)\quad \gamma_\emptyset(\Phi^*) \geq a_\Phi = \alpha(\Phi^*) + \gamma_\emptyset(\Phi^*) + \sum_{v_0 \in P} \delta_{v_0}(\Phi^*) - d_{\Phi^*}[k : \mathbf{Q}].$$

Since $C_\emptyset^{\tau-1} = \bigoplus_{\tau 1_\Phi = 1_\Phi} 1_{\Phi^*} C_\emptyset$, we have $\gamma_\emptyset(\Phi^*) = 0$ for every $\Phi \neq \varepsilon$ such that $\tau 1_\Phi = 1_\Phi$, if $p \not| h_K^-$. This proves (1).

Suppose $k = \mathbf{Q}$ and that $M/\mathbf{Q}$ is abelian. By Lemma 4, the computation of the value of a_ε is reduced to M_0. Hence, we have $a_\varepsilon = 0$, because of $p \not| h_{M_0}^+$. Let Φ be an irreducible character such that $\gamma_\emptyset(\Phi^*) > 0$, $\Phi \neq \varepsilon$ and $\tau 1_\Phi = 1_\Phi$. Note $\delta_p = \omega_p$ and $d_{\Phi^*} = 1$. We observe $a_\Phi > 0$ from (4.3) if $\gamma_\emptyset(\Phi^*) \geq 2$. If $\gamma_\emptyset(\Phi^*) = 1$, we also observe $a_\Phi = 0$ if and only if $\alpha(\Phi^*) = \omega_p(\Phi^*) = 0$. We have (2).

EXAMPLE 1. Example Let M be the cyclotomic field $\mathbf{Q}(\zeta_m, \zeta_{p^n})$, $p \not| m$, $n \geq 2$ and k be the maximal p-subfield of $\mathbf{Q}(\zeta_m)/\mathbf{Q}$. We have $K = \mathbf{Q}(\zeta_m, \zeta_p)$. If p is not decomposed in $k/\mathbf{Q}$, the class number of M_0k is prime to p. By Lemma 4, this means $a_\varepsilon = 0$. For the following pairs (m, p),

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we have $p \nmid h_K^-$ and that p is not decomposed in k:

$$\begin{array}{lllllll} (13,3), & (3,13), & (4,13), & (4,17), & (3,19), & (5,11), & (35,3) \\ (15,7), & (16,7), & (28,5), & (20,7), & (4,29), & (7,11), & (3,31), \\ (8,17), & (12,17), & (19,5), & (4,41), & (20,11), & (3,43), & (47,3), \\ (65,3), & (39,5), & (52,5), & (84,5), & (60,7). & & \end{array}$$

These are additional examples to Yamashita [10] for which the Greenberg conjecture are valid for p in the maximal real subfields of $\mathbf{Q}(\zeta_m, \zeta_p)$.

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