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Polar Homology

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Abstract. For complex projective manifolds we introduce polar homology groups, which are holomorphic analogues of the homology groups in topology. The polar k-chains are subvarieties of complex dimension k with meromorphic forms on them, while the boundary operator is defined by taking the polar divisor and the Poincaré residue on it. One can also define the corresponding analogues for the intersection and linking numbers of complex submanifolds, which have the properties similar to those of the corresponding topological notions.

1 Introduction

In this paper we introduce certain homology groups defined for complex projective manifolds that can be regarded as a complex version of singular homology groups in topology. The idea of such a geometric analogue of topological homology comes from thinking of the Dolbeault (or $\bar{\partial}$) complex of $(0, k)$-forms on a complex manifold as an obvious analogue of the de Rham complex of k-forms on a smooth manifold. This poses an immediate question: "What is the analogue of the chain complex relevant to the context of complex manifolds?", which we address in detail below.

It should be mentioned that the correspondence between de Rham and Dolbeault complexes, or $d \leftrightarrow \bar{\partial}$, has the following natural extension.

$$\begin{aligned} d &\leftrightarrow \bar{\partial} \\ \text{de Rham complex} &\leftrightarrow \text{Dolbeault complex} \\ \text{smooth functions or sections} &\leftrightarrow \text{smooth functions or sections} \\ \text{flat bundles} &\leftrightarrow \text{holomorphic bundles} \\ \text{locally constant functions or sections} &\leftrightarrow \text{local holomorphic functions or sections} \\ \text{cohomology of locally constant sheaves} &\leftrightarrow \text{cohomology of sheaves of } \mathcal{O}_X\text{-modules} \end{aligned}$$

(Here $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on a complex (algebraic) manifold X.) Very informally, this table could be summarized in one line with

"Topology" versus "Complex Algebraic Geometry".

Our interest in this line of thinking is related to the ideas of Arnold on complex analytic analogues of differential geometric concepts (*cf.*, [A]). Some features of the above correspondence can also be found in the papers [FK, DT, KR]. In particular, the approach of Donaldson and Thomas [DT] of transferring differential geometric

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constructions into the context of complex analytic (or algebraic) geometry could lead one to a complexification of geometry in a sense similar to the complexification of topology pursued here.

There are also several motivations from mathematical physics: in particular, from considering any topological field theory of type B [ASL, LNS] and of BV type [AKSZ] or, *e.g.*, a complex analogue of the Chern-Simons gauge theory suggested in Reference [W]. The latter context leads us immediately to a search for a proper holomorphic analogue of the linking number (*cf.*, also [Ger, FT]).

1.1 Holomorphic Orientation

Let X be a compact complex manifold and u be a smooth $(0,k)$-form on it, $0 \leqslant k \leqslant n = \dim X$. We would like to treat such $(0,k)$-forms in the same manner as ordinary k-forms on a smooth manifold, but in the framework of complex geometry. In particular, we have to be able to integrate them over k-dimensional *complex* submanifolds in X. Recall that in the theory of differential forms, a form can be integrated over a real submanifold provided that the submanifold is endowed with an orientation. Thus, we need to find a holomorphic analogue of the orientation.

Obviously, if a k-dimensional submanifold $W \subset X$ is equipped with a holomorphic k-form ω, one can consider the following integral

$$\int_W \omega \wedge u \tag{1.1}$$

of the product of the $(k,0)$- and $(0,k)$-forms. Therefore we are going to regard a top degree holomorphic form on a complex manifold as an analogue of orientation. More generally, if the form ω is allowed to have first order poles on a smooth hypersurface in W, the above integral is still well-defined.

1.2 The Cauchy-Stokes Formula

The new feature brought by the presence of poles of ω shows up in the following relation. Consider the integral (1.1) with a meromorphic k-form ω having first order poles on a smooth hypersurface $V \subset W$. Let the smooth $(0,k)$-form u on X be $\bar{\partial}$-exact, that is $u = \bar{\partial} v$ for some $(0, k-1)$-form v on X. Then

$$\int_W \omega \wedge \bar{\partial} v = 2\pi i \int_V \operatorname{res} \omega \wedge v. \tag{1.2}$$

We shall exploit this straightforward generalization of the Cauchy formula as a complexified analogue of the Stokes theorem. Here $\operatorname{res} \omega$ denotes a $(k-1)$-form on V which is the Poincaré residue of ω (see Section 2.1).

1.3 Boundary Operator

The formula (1.2) prompts us to consider the pair (W, ω) consisting of a k-dimensional submanifold W equipped with a meromorphic form ω (with first order poles on V) as an analogue of a compact oriented submanifold with boundary. In the present paper we construct a homology theory in which the pairs (W, ω) will play the role of chains, while the boundary operator will take the form $\partial(W, \omega) = 2\pi i(V, \operatorname{res}\omega)$. Note that, in the situation under consideration, when the polar set V of the form ω is a smooth $(k-1)$-dimensional submanifold in a smooth k-dimensional W, the induced "orientation" on V is given by a regular $(k-1)$-form $\operatorname{res}\omega$. This means that $\partial(V, \operatorname{res}\omega) = 0$, or the boundary of a boundary is zero. The latter will be the source of the identity $\partial^2 = 0$ in the homology theory discussed below. We shall call it the *polar homology*.

1.4 Pairing to Smooth Forms

It is clear that the (would-be) polar homology groups of a complex manifold X should have a pairing to Dolbeault cohomology groups $H^{0,k}_{\bar\partial}(X)$. Indeed, for a polar k-chain (W, ω) and any $(0, k)$-form u such a pairing is given by the integral

$$\langle (W, \omega), u\rangle = \int_W \omega \wedge u.$$

In other words, the polar chain (W, ω) defines a current on X of degree $(n, n-k)$, where $n = \dim X$. This pairing descends to (co)homology classes by virtue of the Cauchy-Stokes formula (1.2), see Section 4.

Example 1.5 Now we are already able to find out what could be the polar homology groups HP_k of a complex projective curve X. In this (and in any) case, all the 0-chains are cycles. Let (P, a) and (Q, b) be two 0-cycles, where P, Q are points on X and $a, b \in \mathbb{C}$. They are polar homologically equivalent iff $a = b$. Indeed, $a = b$ is necessary and sufficient for the existence of a meromorphic 1-form α on X, such that $\operatorname{div}_\infty \alpha = P + Q$ and $\operatorname{res}_P \alpha = 2\pi i a$, $\operatorname{res}_Q \alpha = -2\pi i b$. (The sum of all residues of a meromorphic differential on a projective curve is zero by the Cauchy theorem.) Then we can write in terms of polar chain complex (to be defined in detail in Section 3) that $(P, a) - (Q, a) = \partial(X, \alpha)$. Thus, $\mathrm{HP}_0(X) = \mathbb{C}$.

As to polar 1-cycles, these correspond to all possible holomorphic 1-forms on X. On the other hand, there are no 1-boundaries, since there are no polar 2-chains in X. Hence $\mathrm{HP}_1(X) \cong \mathbb{C}^g$, where g is the genus of the curve X. (In particular, the polar Euler characteristic of X equals $1 - g$ and coincides with its holomorphic Euler characteristic.)

Similar considerations show that for any n-dimensional X we have $\mathrm{HP}_n(X) = H^0(X, \Omega^n_X)$ and, if X is connected, also $\mathrm{HP}_0(X) = \mathbb{C}$.

1.6 Polar Intersections

One can define a complex (polar) analogue of the intersection number in topology. For instance, let (X, μ) be a complex manifold equipped with a meromorphic volume form μ without zeros (its "polar orientation"). Consider two polar cycles (A, α) and (B, β) of complimentary dimensions that intersect transversely in X (here α and β are

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volume forms, or "polar orientations," on the corresponding submanifolds). Then the polar intersection number is defined by the formula

$$\langle (A,\alpha)\cdot(B,\beta)\rangle = \sum_{P\in A\cap B} \frac{\alpha(P)\wedge\beta(P)}{\mu(P)}.$$

(For explanations, see Section 5.9.) At every intersection point P, the ratio in the right-hand-side is the "comparison" of the orientations of the polar cycles at that point with the orientation of the ambient manifold. This is a straightforward analogue of the use of mutual orientation of cycles in the definition of the topological intersection number. Note, that in the polar case the intersection number does not have to be an integer. (Rather, it is a holomorphic function of the "parameters" (A,α), (B,β) and (X,μ).)

Similarly, there is a polar analogue of the intersection product of cycles when they intersect over a manifold of positive dimension (see Section 5).

1.7 Polar Links

By developing this analogy further we come to a polar analogue of the linking number. For instance, in complex dimension three we start with two smooth polar 1-cycles (C_1,α_1) and (C_2,α_2), *i.e.* C_1 and C_2 are smooth complex curves equipped with holomorphic 1-forms in a three dimensional X. Let us take the 1-cycles which are polar boundaries. This means, in particular, that there exists such a 2-chain (S_2,β_2) that $(C_2,\alpha_2)=\partial(S_2,\beta_2)$. Suppose, the curves C_1 and C_2 have no common points and S_2 is a smooth surface which intersects transversely with the curve C_1. Then, in analogy with the topological linking number of two curves in a three-fold, we define the polar linking number of the 1-cycles above as the polar intersection number of the 2-chain (S_2,β_2) with the 1-cycle (C_1,α_1):

$$\ell k_{\text{polar}}\big((C_1,\alpha_1),(C_2,\alpha_2)\big) := \sum_{P\in C_1\cap S_2} \frac{\alpha_1(P)\wedge\beta_2(P)}{\mu(P)}.$$

One can show that the expression above does not depend on the choice of (S_2,β_2), and has certain invariance properties mimicking those of the topological linking number in "polar" language. We are going to discuss the properties of ℓk_{polar} in more detail in a future publication.

Remark 1.8 Most of the above discussion extends to polar chains (A,α) where the meromorphic p-form α is not necessarily of top degree, that is $0\leqslant p\leqslant k$, where $k=\dim_{\mathbb{C}} A$. To define the boundary operator we have to restrict ourselves to the meromorphic forms with logarithmic singularities. The corresponding polar homology groups enumerated by two indices k and p $(0\leqslant p\leqslant k)$. The relations of this homology groups with the groups of algebraic cycles, as well as the relation of the polar linking to the Weil pairing and Parshin symbols, will be discussed elsewhere [iKR] (see, though, some remarks in Section 4(B) below).

2 Preliminaries

(A) Polar Divisors and Residues

The Poincaré residue is a higher-dimensional generalization of the classical Cauchy residue, where the residue at a point in a domain of one complex variable is generalized to the residue at a hypersurface.

2.1 Let M be an n-dimensional complex manifold and ω be a meromorphic n-form on M which is allowed to have first order poles on a smooth hypersurface V. Then, the form ω can be locally expressed as

$$\omega = \frac{\rho \wedge dz}{z} + \varepsilon, \tag{2.1}$$

where $z = 0$ is a local equation of V and ρ (respectively, ε) is a holomorphic $(n-1)$-form (resp., n-form). Then the restriction $\rho|_V$ is an unambiguously defined holomorphic $(n-1)$-form on V.

Definition 2.2 The Poincaré residue of the n-form ω in (2.1) is the following $(n-1)$-form on V

$$\operatorname{res} \omega := \rho|_V .$$

2.3 It is straightforward to extend this to the case of normal crossing divisors. Suppose that the meromorphic n-form ω has the first order poles on a normal crossing divisor $V = \bigcup_i V_i$ in M. [Normal crossing divisor means that V has only smooth components V_i (each entering with multiplicity one) that intersect generically.] Analogously to the Definition 2.2 one can define a residue at each component V_i. The resulting $(n-1)$-forms $\operatorname{res}_{V_j} \omega$ are then meromorphic and have first order poles at the pairwise intersections $V_{ij} = V_i \cap V_j$. One can now consider the repeated Poincaré residue at V_{ij}. Representing ω as $\omega = \varrho \wedge \frac{dz_i}{z_i} \wedge \frac{dz_j}{z_j}$, where $z_i = 0$ and $z_j = 0$ are local equations of the components V_i and V_j respectively one finds that

$$\operatorname{res}_{i,j} \omega := \operatorname{res}_{V_{ij}}(\operatorname{res}_{V_j} \omega) = \operatorname{res}_{z_i=0}\Big(\operatorname{res}_{z_j=0} \varrho \wedge \frac{dz_i}{z_i} \wedge \frac{dz_j}{z_j}\Big) = \varrho|_{V_{ij}}.$$

Note that

$$\operatorname{res}_{i,j} \omega = -\operatorname{res}_{j,i} \omega.$$

Notation Let us denote by $\operatorname{res} \omega$ the collection of $(k-1)$-forms $\operatorname{res}_{V_j} \omega$, the residues of ω at the components of the normal crossing divisor $\operatorname{div}_\infty \omega = \bigcup_i V_i$.

(B) The Push-Forward Map (See [Gr])

For a finite covering $f\colon X \to Y$ and a function φ on X one can define its push-forward, or the trace, $f_*\varphi$, as a function on Y whose value at a point is calculated by summing over the preimages taken with multiplicities. The operation f_* can be generalized to p-forms and to the maps f which are only generically finite.

2.4 Suppose that $f\colon X \to Y$ is a proper, surjective holomorphic mapping where both X and Y are smooth complex manifolds of the same dimension n. The *push-forward map* is a mapping

$$f_*\colon \Gamma(X, \Omega_X^p) \to \Gamma(Y, \Omega_Y^p).$$

The push-forward map is also defined for meromorphic forms, $f_*\colon \Gamma(X, \mathcal{M}^p) \to \Gamma(Y, \mathcal{M}^p)$.

Its construction is as follows. First note that f is generically finite, *i.e.*, there is an analytic hypersurface $D \subset Y$ such that f is finite unramified covering away from this hypersurface D. Hence, for sufficiently small open neighborhood U of any point in $Y^* := Y \setminus D$, the inverse image $f^{-1}(U) = U_1 \sqcup \cdots \sqcup U_d$ is a disjoint union of d open sets U_j, such that $f|_{U_j}$ is an isomorphism with the inverse $s_j\colon U \to U_j$. Given a form ω on X, one defines its push-forward

$$f_*\omega := s_1^*\omega + \cdots + s_d^*\omega$$

in U, and therefore, in Y^*. One can check that the form $f_*\omega$ extends across the smooth points of D and, hence, to the whole of the manifold Y, since the remaining part of D has codimension greater than one. The resulting form $f_*\omega$ is holomorphic (resp. meromorphic) on Y provided the form ω was holomorphic (resp. meromorphic) on X.

The operations of push-forward and residue are related in the following way.

Proposition 2.5 *Let $\pi\colon X \to Y$ be a proper, surjective holomorphic map of complex manifolds of the same dimension n. Let ω be a meromorphic form with only first order poles on a smooth hypersurface V in X. Suppose $V_o := \pi(V)$ is a smooth hypersurface in Y. Then $\pi_*\omega$ has first order poles on V_o and*

$$\operatorname{res} \pi_*\omega = \bar{\pi}_* \operatorname{res} \omega,$$

where $\bar{\pi}\colon V \to V_o$ is the restriction of π.

3 Polar Homology of Projective Varieties

Here we define a homological complex based on the notion of the polar boundary. The construction is analogous to the definition of homology of a topological space with replacement of continuous maps by complex analytic ones. The notion of the boundary (of a simplex or a cell) is replaced by the Poincaré residue of a meromorphic differential form. There are however important distinctions. First, we shall only have an analogue of the non-torsion part of homology. Second, unlike the topological homology, where in each dimension k one uses all continuous maps of one standard object (the standard k-simplex or the standard k-cell) to a given topological space, in polar homology we deal with complex analytic maps of a large class of k-dimensional varieties to a given one.

3.1 Polar Chains

In this section we deal with complex projective varieties, *i.e.*, subvarieties of a complex projective space. (In this setting the complex analytic considerations are equivalent to algebraic ones.) By a smooth projective variety we always understand a smooth and connected one. For a smooth variety M, we denote by Ω_M^p the sheaf of regular p-forms on M. The sheaf $\Omega_M^{\dim M}$ of forms of the top degree on M will sometimes be denoted by K_M.

The space of polar k-chains for a complex projective variety X, $\dim X = n$, will be defined as a $\mathbb{C}$-vector space with certain generators and relations.

Definition 3.2 The space of *polar k-chains* $\mathcal{C}_k(X)$ is a vector space over $\mathbb{C}$ defined as the quotient $\mathcal{C}_k(X) = \hat{\mathcal{C}}_k(X)/\mathcal{R}_k$, where the vector space $\hat{\mathcal{C}}_k(X)$ is freely generated by the triples (A, f, α) described in (i), (ii), (iii) and $\mathcal{R}_k$ is defined as relations (R1), (R2), (R3) imposed on the triples.

(i) A is a smooth complex projective variety, $\dim A = k$;
(ii) $f\colon A \to X$ is a holomorphic map of projective varieties;
(iii) α is a rational k-form on A with first order poles on $V \subset A$, where V is a normal crossing divisor in A, *i.e.*, $\alpha \in \Gamma\big(A, \Omega_A^k(V)\big)$.

The relations are:

(R1) $\lambda(A, f, \alpha) = (A, f, \lambda\alpha)$
(R2) $\sum_i (A_i, f_i, \alpha_i) = 0$ provided that $\sum_i f_{i*}\alpha_i \equiv 0$, where $\dim f_i(A_i) = k$ for all i and the push-forwards $f_{i*}\alpha_i$ are considered on the smooth part of $\bigcup_i f_i(A_i)$;
(R3) $(A, f, \alpha) = 0$ if $\dim f(A) < k$.

Remarks on the Definition

3.3 By definition, $\mathcal{C}_k(X) = 0$ for $k < 0$ and $k > \dim X$.

3.4 In what follows we sometimes will make no distinction between a triple (A, f, α) and the equivalence class defined by it in $\mathcal{C}_k(X)$. An arbitrary polar chain can thus be written as a sum of triples of the form $\sum_i (A_i, f_i, \alpha_i)$. A chain equivalent to a single triple will be called *prime*. For a chain $a = \sum_i (A_i, f_i, \alpha_i)$, let us call the subvariety $\bigcup_i f_i(A_i)$ in X the *support* of a. If the support of a chain is a smooth subvariety in X, such a chain will be called *smooth*. One can show that smooth chains are prime, since we suppose that "smooth" implies "connected" (see 3.1).

3.5 The relation (R2) allows us, in particular, to refer to prime polar chains as pairs replacing a triple (A, f, α) by a pair $(\hat{A}, \hat{\alpha})$, where $\hat{A} = f(A) \subset X$, $\hat{\alpha}$ is defined only on the smooth part of $\hat{A}$ and $\hat{\alpha} = f_*\alpha$ there. Due to the relation (R2), such a pair $(\hat{A}, \hat{\alpha})$

carries precisely the same information as (A, f, α).[1] (The only point to worry about is that such pairs cannot be arbitrary. In fact, by the Hironaka theorem on resolution of singularities, any subvariety $\hat{A} \subset X$ can be the image of some regular A, but the form $\hat{\alpha}$ on the smooth part of $\hat{A}$ cannot be arbitrary.)

3.6 The relation (R2) also represents additivity with respect to α, that is

$$(A, f, \alpha_1) + (A, f, \alpha_2) = (A, f, \alpha_1 + \alpha_2).$$

Formally speaking, the right hand side makes sense only if $\alpha_1 + \alpha_2$ is an admissible form on A, that is if its polar divisor $\operatorname{div}_\infty(\alpha_1 + \alpha_2)$ has normal crossings. However, one can always replace A with a variety $\tilde{A}$ obtained from A by a blow-up, $\pi \colon \tilde{A} \to A$, in such a way that $\pi^*(\alpha_1 + \alpha_2)$ is admissible on $\tilde{A}$, *i.e.*, $\operatorname{div}_\infty(\alpha_1 + \alpha_2)$ is already a normal crossing divisor. (This is again the Hironaka theorem.) Then (R2) says that $(A, f, \alpha_1) + (A, f, \alpha_2) = \bigl(\tilde{A}, f \circ \pi, \pi^*(\alpha_1 + \alpha_2)\bigr)$.

Definition 3.7 The *boundary operator* $\partial \colon \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(X)$ is defined by

$$\partial(A, f, \alpha) = 2\pi i \sum_i (V_i, f_i, \operatorname{res}_{V_i} \alpha)$$

(and by linearity), where V_i are the components of the polar divisor of α, $\operatorname{div}_\infty \alpha = \bigcup_i V_i$, and the maps $f_i = f|_{V_i}$ are restrictions of the map f to each component of the divisor.

Theorem 3.8 *The boundary operator ∂ is well defined, i.e. it is compatible with the relations (R1), (R2), (R3).*

Proof We have to show that ∂ respects the relations (R1), (R2), (R3), in other words, ∂ maps equivalent sums of triples to equivalent ones. It is trivial with (R1). To check (R2), let us recall Proposition 2.5. Consider a sum of triples $\sum_i (A_i, f_i, \alpha_i)$ belonging to (R2), that is $\dim A_i = \dim f_i(A_i) = k, \forall i$, and $\sum_i f_{i*}\alpha_i = 0$ on the smooth part of $\bigcup_i f_i(A_i)$. Since the irreducible components of $\bigcup_i f_i(A_i)$ can be treated separately it is natural to consider only the case when all the triples have the same support, $f_i(A_i) = \hat{A} \subset X, \forall i$. Let $V_i \subset A_i$ be the divisor of poles of α_i and let $\hat{V} := \bigcup_i f_i(V_i) \subset \hat{A}$. We want to prove that

$$\sum_i \bar{f}_{i*} \operatorname{res} \alpha_i = 0$$

on the smooth part of $\hat{V}$, where $\bar{f}_{i*} \colon V_i \to \hat{V}$ for each i is the restriction of the map f_i.

Suppose first that there exists a smooth point of $\hat{V}$ which is smooth also in $\hat{A}$. Then the Proposition 2.5 applied in a neighborhood of that point gives us the desired vanishing $\sum_i \bar{f}_{i*} \operatorname{res} \alpha_i = 0$, as a consequence of the equality $\sum_i f_{i*}\alpha_i = 0$. This

[1] Note that the consideration of triples (A, f, α) instead of pairs $(\hat{A}, \hat{\alpha})$ which we used in Section 1 is similar to the definition of chains in the singular homology theory; in the latter case, one considers the mappings of abstract simplices into the manifold, but morally it is only "images of simplices" that matter.

is however not enough for our proof since some components of $\hat{V}$ may lie entirely in the set of singular points of $\hat{A}$. To overcome this problem we apply the Hironaka theorem replacing $\hat{A}$ with a smooth variety $\tilde{A}$, a blow-up of $\hat{A}$, and correspondingly blowing up all A_i, so that the following diagram is commutative:

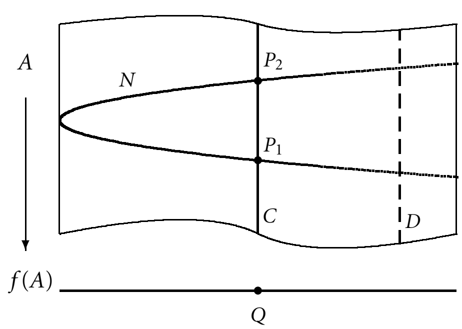
$$\begin{tikzcd} A_i \arrow[d, "f_i"'] & \tilde{A}_i \arrow[l, "\pi_i"'] \arrow[d, "\tilde{f}_i"] \\ \hat{A} & \tilde{A} \arrow[l, "\pi"'] \end{tikzcd}$$

Then we apply Proposition 2.5 on the blown up side.

We must recall now that the divisor $V_i = \operatorname{div}_\infty \alpha_i$ could have components that were mapped by f_i to subvarieties of dimension less than $k-1$; hence, we conclude that we have just proved the following statement (symbolically): if $a \in (R2)$ then $\partial a \in (R2) + (R3)$.

Now, it remains to prove the compatibility of ∂ with (R3). Let $a = (A, f, \alpha)$ be a degenerate triple described in (R3), *i.e.*, $\dim f(A) < k = \dim A$. We shall show that $\partial a \in$ (R2) + (R3) in this case. The polar divisor $V = \operatorname{div}_\infty \alpha$, $\dim V = k-1$, is, by assumptions of Definition 3.2, a normal crossing divisor in A. Let us split the components of V into two parts: non-degenerate and degenerate ones. That is $V = N \cup D$ where $\dim f(N) = k-1$ and $\dim f(D) < k-1$. According to this splitting, ∂a is represented as a sum of two terms corresponding to $\operatorname{res}_N \alpha$ and $\operatorname{res}_D \alpha$. The second term belongs to (R3) and we have to show only that the first one belongs to (R2), *i.e.*, that $\bar{f}_* \operatorname{res}_N \alpha = 0$, where $\bar{f} = f|_N$. Recall that we suppose that $\dim f(A) < k$. If it happens that $\dim f(A) < k-1$, we have $N = \varnothing$ and there is nothing to prove. Therefore we may assume that $\dim f(A) = k-1$ and, by irreducibility of A, $f(A) = f(N)$.

Then, for a generic smooth point $Q \in f(A)$, its preimage in A, $C := f^{-1}(Q) \subset A$, is a smooth projective curve. This curve intersects with N over the set $\bar{f}^{-1}(Q)$ and we may suppose that the latter consists of a finite number of points P_i which are smooth in N and that the intersections are transverse there.



Let $\beta(P_i)$ denote the values of $\operatorname{res}_N \alpha$ at the points $P_i \in N \cap C$ and pick up a non-vanishing $(k-1)$-form β_o at Q (recall that $Q = f(P_i)$, $\forall i$). Let us show that

$$\sum_i \frac{\beta(P_i)}{f^* \beta_o(P_i)} = 0$$

(this would mean that $\bar{f}_* \operatorname{res}_N \alpha = 0$ on the smooth part of $\bar{f}(N) = f(A)$ — the required result). To prove this, let us notice that there exists a meromorphic 1-differential ω on C such that

$$\omega(P) \otimes f^*\beta_o(P) = \alpha(P), \quad P \in C.$$

(ω is obtained by dividing α by the non-vanishing form $f^*\beta_o$.) This equality is understood in the sense of the natural isomorphism

$$K_C \otimes (f^* K_{f(A)^*})|_C = K_A|_C,$$

where $f(A)^*$ is the smooth part of $f(A)$. It is easy to see now that for $\beta(P_i) = \operatorname{res}_N \alpha(P_i)$ we have

$$\sum_i \frac{\beta(P_i)}{f^*\beta_o(P_i)} = \sum_i \operatorname{res}_{P_i} \omega = 0.$$

The latter equality follows from the observation that P_i are the only points on C where ω has poles. Indeed, the poles of ω are located on $\operatorname{div}_\infty \alpha \cap C = (N \cap C) \cup (D \cap C)$. One part of this gives us the points P_i, $\{P_i\} = N \cap C$, while the rest, $D \cap C$, corresponding to the "degenerate" part D of $\operatorname{div}_\infty \alpha$ can be assumed to be empty, $D \cap C = \varnothing$. Indeed, we could have assumed from the very beginning that $C = f^{-1}(Q)$ does not meet D because $\dim f(D) < k - 1$ and we might suppose that $Q \notin f(D)$. ■

Theorem 3.9 $\partial^2 = 0$.

Proof We need to prove this for triples $(A, f, \alpha) \in \mathcal{C}_k(X)$, *i.e.*, for forms α with normal crossing divisors of poles. The repeated residue at pairwise intersections differs by a sign according to the order in which the residues are taken, see 2.3. Thus the contributions to the repeated residue from different components cancel out (or, the residue of a residue is zero).[2] ■

Definition 3.10 For a smooth complex projective variety X, $\dim X = n$, the chain complex

$$0 \to \mathcal{C}_n(X) \xrightarrow{\partial} \mathcal{C}_{n-1}(X) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{C}_0(X) \to 0$$

is called the *polar chain complex* of X. Its homology groups, $\mathrm{HP}_k(X)$, $k = 0, \dots, n$, are called the *polar homology groups* of X.

Example 3.11 For a projective curve of genus g the polar homology groups are as follows: $\mathrm{HP}_0 = \mathbb{C}$, $\mathrm{HP}_1 = \mathbb{C}^g$, and $\mathrm{HP}_k = 0$ for $k \geq 2$. One can readily see that the approach with triples coincides with the consideration of Introduction.

[2]One can note that an example of the polar divisor $\{xy = 0\}$ for the form $dx \wedge dy/xy$ in $\mathbb{C}^2$ should be viewed as a complexification of a polygon vertex in $\mathbb{R}^2$. Indeed, the cancellation of the repeated residues on different components of the divisor is mimicking the calculation of the boundary of a boundary of a polygon: every polygon vertex appears twice with different signs as a boundary point of two sides.

Remark 3.12 The functorial properties of polar homology are straightforward. A regular morphism of projective varieties $h\colon X \to Y$ defines a homomorphism $h_*\colon \mathrm{PH}_k(X) \to \mathrm{PH}_k(Y)$.

Remark 3.13 The definitions of polar chains can be generalized to the case of p-forms on k-manifolds, *i.e.*, for the forms of not necessarily top degree, $p \le k$. Instead of meromorphic k-forms with poles of the first order we have to restrict ourselves to p-forms with logarithmic singularities. The definition of the boundary operator ∂, the property $\partial^2 = 0$, and the definition of the polar homology groups can be carried over to this, more general, situation. The polar homology groups are then enumerated by two indices: $\mathrm{HP}_{k,k-p}(M)$. The definition above corresponds to the $p = k$ case. We will discuss the more general polar homology groups elsewhere [iKR].

3.14 Relative Polar Homology

Let Z be a projective subvariety in a projective X. Analogously to the topological relative homology we can define the polar relative homology of the pair $Z \subset X$.

Definition 3.15 The relative polar homology $\mathrm{HP}_k(X, Z)$ is the homology of the following quotient complex of chains:

$$\mathcal{C}_k(X, Z) = \mathcal{C}_k(X)/\mathcal{C}_k(Z).$$

Here we use the natural embedding of the chain groups $\mathcal{C}_k(Z) \hookrightarrow \mathcal{C}_k(X)$. This leads to the long exact sequence in polar homology:

$$\cdots \to \mathrm{HP}_k(X) \to \mathrm{HP}_k(X, Z) \xrightarrow{\partial} \mathrm{HP}_{k-1}(Z) \to \mathrm{HP}_{k-1}(X) \to \cdots$$

3.16 Systems of Coefficients

One can introduce the notion of a homological system of coefficients appropriate for the polar complex. The most geometrical example would be, perhaps, to supply projective varieties A, $f\colon A \to X$, with coherent sheaves $\mathcal{F}_{A,f}$ obeying certain relations between $\mathcal{F}_{A_1,f_1}$ and $\mathcal{F}_{A_2,f_2}$ when $f_1(A_1) = f_2(A_2)$ and related by some homomorphisms playing the role of the residue. We do not study this in the present paper, but let us mention that the homology groups appearing in Sections 3.13 above and 4.8 below can be viewed as an example. On the other hand, the simplest case of a polar homological system of coefficients corresponds to $\mathcal{F}_{A,f} = f^*\mathcal{F} \otimes K_A$, where $\mathcal{F}$ is a locally free sheaf on X and α in the triple (A, f, α) is understood as a global section of $f^*\mathcal{F} \otimes K_A(V)$. Let us denote the corresponding homology as $\mathrm{HP}_k(X, \mathcal{F})$. This case is mentioned in Sections 4.4, 4.5.

4 Polar Chains and Differential Forms

(A) Dolbeault Cohomology as Polar de Rham Cohomology

As we discussed in the Introduction, the Dolbeault complex of $(0, k)$-forms should be related to the polar homology in the same way as the de Rham complex of smooth forms is related to the topological homology (*e.g.*, singular homology). Now, after the definitions of Section 3 are given, we are able to make this point more explicit.

4.1 In a smooth projective variety X, consider a polar k-chain, for instance, a prime one, *i.e.* (an equivalence class of) a triple $a = (A, f, \alpha)$. Such a triple can be regarded as a linear functional on the space of smooth $(0, k)$-forms on X. Let u be a smooth $(0, k)$-form on X, then the pairing is given by the following integral:

$$\langle a, u\rangle := \int_A \alpha \wedge f^* u. \tag{4.1}$$

The integral is well defined since α has only first order poles on a normal crossing divisor. It is now straightforward to show that the pairing $\langle\ ,\ \rangle$ descends to the space of equivalence classes of triples $\mathcal{C}_k(X)$, *i.e.*, that it is compatible with the relations (R1), (R2), (R3) of Definition 3.2. Indeed, (R1) is obvious, compatibility with (R3) follows from noticing that $f^*u = 0$ if $\dim f(A) < k$, and the compatibility with (R2) follows from the relation $\int_A \alpha \wedge f^* u = \int_{f(A)} f_*\alpha \wedge u$ if $\dim f(A) = k$, where the last integral is taken over the smooth part of $f(A)$.

Remark 4.2 Let us notice that the last considerations can be used[3] as an alternative definition of the polar chain complex on a smooth projective variety X (or any smooth compact complex manifold). The pairing above can be thought of as a map $\hat{\varphi}\colon \hat{\mathcal{C}}_k(X) \to \mathcal{D}^{n,n-k}(X)$, where $\hat{\mathcal{C}}_k(X)$ is the vector space freely generated by the triples (A, f, α) (see Definition 3.2) and $\mathcal{D}^{n,n-k}(X)$ is the space of currents of degree $(n, n-k)$ on X which is defined as a space of certain linear functionals on the smooth $(0, k)$-forms (see [GH]). Then the relations (R1), (R2), (R3) in the Definition 3.2 correspond to the kernel of the map $\hat{\varphi}$. In other words, the space of polar chains $\mathcal{C}_k(X)$ can be defined as a subspace of currents — the image of $\hat{\varphi}$. We have thus an embedding

$$\varphi\colon \mathcal{C}_k(X) \hookrightarrow \mathcal{D}^{n,n-k}(X).$$

Moreover, the Cauchy-Stokes formula (1.2) shows that $\mathcal{C}_k(X)$ is a subcomplex of the $\bar{\partial}$-complex of currents $\mathcal{D}^{n,n-k}(X)$, *i.e.* for $a \in \mathcal{C}_k(X)$ we have $\varphi(\partial a) = \bar{\partial}\varphi(a)$. (This is in fact shown also in the proof of 4.3 below.)

Proposition 4.3 *The pairing (4.1) defines the following homomorphism in (co)homology:*

$$\rho\colon\ \mathrm{HP}_k(X) \to H^{n,n-k}_{\bar{\partial}}(X), \tag{4.3}$$

where $n = \dim X$.

[3]We thank A. Levin for emphasizing this point.

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Proof By the Serre duality, ρ is the map $\mathrm{HP}_k(X) \to \left(H^{0,k}_{\bar\partial}(X)\right)^*$ and it is sufficient to verify that the pairing (4.1) vanishes if $\partial a = 0$ and $u = \bar\partial v$, or if $\bar\partial u = 0$ and $a = \partial b$. This follows immediately from the Cauchy-Stokes formula (1.2):

$$\int_A \alpha \wedge f^*(\bar\partial u) = 2\pi i \int_{\mathrm{div}_\infty \alpha} (\mathrm{res}\, \alpha) \wedge f^*(u),$$

that is $\langle a, \bar\partial u\rangle = \langle \partial a, u\rangle$. ■

A number of examples suggests that, for projective manifolds, the homomorphism (4.3) should be in fact an isomorphism.

Conjecture 4.4 **(Polar de Rham Theorem)** *For a smooth projective manifold X the mapping ρ: $\mathrm{HP}_k(X) \to H^{n,n-k}_{\bar\partial}(X)$ is an isomorphism of the polar homology and Dolbeault cohomology groups. Equivalently, in terms of dual cohomology groups,*

$$\mathrm{HP}^k(X) \cong H^{0,k}_{\bar\partial}(X).$$

An analogous conjecture that $\mathrm{HP}_k(X, \mathcal{F}) \cong H^{n-k}(X, K_X \otimes \mathcal{F})$ sounds reasonable also for polar (co)homology with coefficients in locally free sheaves on X (see Section 3.16).

Example 4.5 If X is a complex curve of genus g, one has $\mathrm{HP}_0(X) \cong \mathbb{C} \cong H^{1,1}_{\bar\partial}(X)$ and $\mathrm{HP}_1(X) \cong \mathbb{C}^g \cong H^{1,0}_{\bar\partial}(X)$ (see Example 1.5). Considering also $\mathrm{HP}_k(X, \mathcal{F})$ in case of $\mathcal{F} = \mathcal{T}_X$ (the tangent sheaf of X) one can check that $\mathrm{HP}_0(X, \mathcal{T}_X) \cong H^{0,1}_{\bar\partial}(X)$ and $\mathrm{HP}_1(X, \mathcal{T}_X) \cong H^{0,0}_{\bar\partial}(X)$.

Example 4.6 As an other example of a direct computation of polar homology, let us consider the manifold $X = \mathbb{C}P^1 \times \mathbb{C}P^1$. We are going to show that $\mathrm{HP}_0(X) \cong \mathbb{C}$ and $\mathrm{HP}_1(X) = \mathrm{HP}_2(X) = 0$ in this case. Note that this result will agree with Conjecture 4.4. First of all, $\mathrm{HP}_0(X) \cong \mathbb{C}$ follows from the connectedness of X as in Example 1.5. $\mathrm{HP}_2(X) = 0$ follows from the fact that there are no holomorphic 2-forms on $X = \mathbb{C}P^1 \times \mathbb{C}P^1$.

It remains to prove that $\mathrm{HP}_1(X)$ vanishes. In other words, we have to show that if (A, f, α) is a polar 1-cocycle then there exists a meromorphic 2-form β on X with the first order poles on the curve $C := f(A) \subset X$ and such that $\mathrm{res}\, \beta = \gamma$ where $\gamma = f_*\alpha$ is a 1-form on C defined in smooth points of C. Let $\pi \colon X \to \mathbb{C}P^1$ denote the projection on the first factor in $X = \mathbb{C}P^1 \times \mathbb{C}P^1$ and suppose C has no components lying in a fiber of π (the opposite case can be considered separately without further complications). Let us now consider a fiber F, which is also, of course, a copy of $\mathbb{C}P^1$, and suppose F intersects our curve C transversely (this holds for a generic fiber). We should construct a meromorphic section β of $\Omega^2(X)$ along $F \subset X$ in such a way that it would have a first order pole at each point of the intersection $P \in F \cap C$ with a residue equal to the value γ_P of γ at P. This is equivalent to defining a 1-form on F with prescribed residues at each $P \in F \cap C$. (Indeed, the 2-form β can be considered as a 1-form on F with coefficients in the conormal bundle to F, the latter being canonically trivial.) Moreover, the sum of the residues, $\sum \gamma_P$, over the set of intersection points $F \cap C$ is zero. The latter follows from the equality $\pi_* \gamma \equiv \pi_* f_* \alpha = 0$. Indeed, $\pi_* f_* \alpha$ would have to be a 1-form globally holomorphic on $\mathbb{C}P^1$ and therefore it has to vanish. Thus, the section β with the desired properties (in particular, $\operatorname{res} \beta = \gamma$) exists and it is unique on a generic F. Hence we have constructed a 2-form β on an open subset in X.

Now we can show that β extends to the whole of X, or, better, we just repeat the above construction for an arbitrary fiber with only one modification as follows. If the intersection at the point $P \in F \cap C$ is not transverse (in particular, if C is singular at P), we cannot use the residue γ_P there. Therefore, let us replace γ_P by $\tilde{\gamma}_P = \big(\pi^*(\pi \circ \tilde{f})_* \alpha\big)(P)$, where $\tilde{f}$ is the restriction of the map $f\colon A \to X$ to a neighborhood $U \ni f^{-1}(P)$, such that $f(U)$ contains no intersection points of $F \cap C$ other than P. Now, $\tilde{\gamma}_P$ defines an element of the conormal bundle to F at P for any point $P \in F \cap C$. Such an element coincides with γ_P when the intersection is transverse at P. This makes the construction of β obeying $\operatorname{res} \beta = \gamma$ global over X. The latter shows that (A, f, α) is a polar 1-boundary, and hence $\mathrm{HP}_1(X) = 0$.

Remark 4.7 Consider the *polar Euler characteristic*,

$$\chi_{\mathrm{pol}}(X) := \sum_{k=0}^{n} (-1)^k \dim \mathrm{HP}_k(X),$$

of an n-dimensional variety X. Then, if the conjecture (4.4) is true, for a smooth projective X one obtains the equality $\chi_{\mathrm{pol}}(X) = \chi_{\mathrm{hol}}(X)$ of the polar and holomorphic Euler characteristics, where

$$\chi_{\mathrm{hol}}(X) = \sum_{k=0}^{n} (-1)^k \dim H^{0,k}_{\bar{\partial}}(X).$$

(B) Forms of Any Degree

4.8 So far we considered polar chains with complex volume forms. More generally, one could consider polar (k, p)-chains (A, f, α), where α is a meromorphic $(k-p)$-form of not necessarily maximal degree, $0 \leqslant p \leqslant k$, on A that can have only logarithmic singularities on a normal crossing divisor.[4] The requirement of log-singularities is needed to have a convenient definition of the residue and, hence, the boundary operator.

The Cauchy-Stokes formula (1.2) extends to this case as well. As a consequence, the natural pairing between polar (k, p)-chains and smooth (p, k)-forms on X gives us as before the homomorphism (*cf.* (4.3))

$$\rho\colon\ \mathrm{HP}_{k,p}(X) \to H^{n-p,n-k}_{\bar{\partial}}(X).$$

[4] An important property of such forms on projective varieties is that they are closed, see [De].

However, unlike the case $p = 0$, the map ρ is not, in general, an isomorphism for other values of p, $0 < p \leqslant k$. For instance, at least in the case of $p = k$, this is easy to see for the following reason.

4.9 Polar Chains With $p = k$

In this case the triples (A, f, α) involve 0-forms (*i.e.*, just functions) α on projective varieties A. Then the requirement of log-singularities amounts here to saying that these functions are holomorphic on A. Since A is compact, these functions must be constant. In particular, we conclude that all polar (k, k)-chains are polar cycles.

Thus, the space of polar (k, k)-cycles in a projective manifold X is the same as the vector space generated over $\mathbb{C}$ by all k-dimensional algebraic cycles in X. (Note that the replacement of the triples (A, f, α) by the pairs (A, α) with $A \subset X$ is especially convenient when αs are 0-forms.) In this case one can show that the homomorphism ρ maps $\mathrm{HP}_{k,k}(X)$ to the algebraic part of $H^{r,r}(X)$, where $r = n - k$, or more precisely, to

$$H^{r,r}_{\mathrm{alg}}(X, \mathbb{C}) := \left(H^{r,r}_{\bar\partial}(X) \cap H^{2r}(X, \mathbb{Q})\right) \otimes \mathbb{C} \subset H^{r,r}_{\bar\partial}(X).$$

This allows us to conclude that ρ is not surjective, in general. Indeed, there are examples where $H^{r,r}_{\mathrm{alg}}(X, \mathbb{C})$ is strictly smaller than $H^{r,r}_{\bar\partial}(X)$. For instance, for a generic algebraic $K3$ surface one has that $\dim H^{1,1}(X) = 20$, while $\dim H^{1,1}_{\mathrm{alg}}(X, \mathbb{C}) = 1$, see [Tju]. We also note that by the Hodge conjecture, the image of ρ coincides with $H^{r,r}_{\mathrm{alg}}(X, \mathbb{C})$.

Remark 4.10 It would be, certainly, very interesting to describe the polar homology groups $\mathrm{HP}_{k,p}(X)$ for all values of p. In particular, it is not clear whether the groups $\mathrm{HP}_{k,p}(X)$ are finite-dimensional.[5]

5 Intersection in Polar Homology

We define here a polar analogue of the topological intersection product. In particular, for polar cycles of complimentary dimensions one obtains a complex number, called the *polar intersection number*.

Recall that in topology, one considers a smooth oriented closed manifold M and two oriented closed submanifolds $A, B \subset M$ of complementary dimensions, *i.e.*, $\dim A + \dim B = \dim M$. Suppose, A and B intersect transversely at a finite set of points. Then to each intersection point P one assigns ± 1 (local intersection index) by comparing the mutual orientations of the tangent vector spaces $T_P A$, $T_P B$, and $T_P M$.

5.1 Polar Oriented Manifolds

Let now M be a compact complex manifold of dimension n, on which we would like to define a polar intersection theory. It has to be polar oriented, *i.e.*, equipped with a

[5]It should be mentioned that there is a map of the complex of polar chains to the complex considered in [BO]. The corresponding (co)homology groups are, however, quite different, as already the simplest example of a complex curve shows. We are grateful to S. Bloch for illuminating discussions on this relation.

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complex volume form. As the discussion below shows, the n-form μ defining its polar orientation has to have no zeros on M, since we are going to consider expressions in which μ, the orientation of the ambient manifold, enters a denominator. Therefore we adopt the following terminology.

Definition 5.2

(i) A compact complex manifold M, endowed with a nowhere vanishing holomorphic volume form μ, is said to be a *polar oriented closed* manifold.
(ii)If the volume form μ on a compact complex manifold M is nonvanishing and meromorphic with only first order poles on a normal crossing divisor $N \subset M$, then M is called a *polar oriented* manifold *with boundary*. The hypersurface N is then endowed with a polar orientation $\nu := \operatorname{res}\mu \neq 0$ and (N, ν) is called the *polar boundary* of (M, μ).

Remark 5.3 By definition, polar oriented closed manifolds are complex manifolds whose canonical bundle is trivial (Calabi-Yau, Abelian manifolds or, for example, any complex tori, if we do not restrict ourselves to algebraic manifolds). We have just defined the notion of the polar orientation in a more restrictive sense than before, when we considered the definition of chains. In fact, polar chains with their orientations are to be compared to oriented piece-wise smooth submanifolds in differential topology, while the ambient space on which we want to have Poincaré duality has to be everywhere smooth and oriented. Zeros of a volume form could be regarded as a complex analogue of singularities of a real manifold.[6]

(A) Polar Intersection Number

5.4 Let (M, μ) be a polar oriented closed manifold of dimension n. In such a case we define the following natural pairing between its polar homology groups $\mathrm{HP}_p(M)$ and $\mathrm{HP}_{n-p}(M)$ of complimentary dimension.

According to Proposition 4.3, the above groups can be mapped to the Dolbeault cohomology groups $H^{n,n-p}_{\bar\partial}(M)$ and $H^{n,p}_{\bar\partial}(M)$, respectively. On a polar oriented closed manifold we are given a nowhere vanishing section μ of the line bundle Ω^n_M. Hence, we have the isomorphism $H^{n,n-p}_{\bar\partial}(M) \overset{\mu^{-1}}{\to} H^{0,n-p}_{\bar\partial}(M)$. Using this and the product in Dolbeault cohomology we obtain the following pairing:

$$H^{n,p}_{\bar\partial}(M) \otimes H^{n,n-p}_{\bar\partial}(M) \xrightarrow{\mathrm{id}\otimes\mu^{-1}} H^{n,p}_{\bar\partial}(M) \otimes H^{0,n-p}_{\bar\partial}(M) \longrightarrow H^{n,n}_{\bar\partial}(M) \xrightarrow{\sim} \mathbb{C}.$$

Together with the homomorphism $\rho\colon\ \mathrm{HP}_k(X) \to H^{n,n-k}_{\bar\partial}(X)$, this yields the pairing

$$\langle\ \cdot\ \rangle\colon\ \mathrm{HP}_p(M) \otimes \mathrm{HP}_{n-p}(M) \to \mathbb{C}.$$

[6]For instance, on a complex curve X of genus g one has $\mathrm{HP}_1(X) = \mathbb{C}^g$, and a holomorphic 1-differential representing a generic element in $\mathrm{HP}_1(X)$ has $2g-2$ zeros. From this point of view the complex genus g curve is like a graph which has g loops joined by $g-1$ edges and having $2g-2$ trivalent (*i.e.*, "non-smooth") points. The "smooth cases" are $\mathbb{CP}^1$, which corresponds to a real segment, and an elliptic curve, which is a complex counterpart of the circle in this precise sense.

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Here, in fact, we interchanged the order of factors (see the explicit formula (5.5) below).

5.5 Consider two polar cycles, $[a] \in \mathrm{HP}_p(M)$ and $[b] \in \mathrm{HP}_{n-p}(M)$. Let t_a and t_b be the Dolbeault forms representing $\rho([a])$ and $\rho([b])$ respectively. Then, the above product, which we denoted by $\langle a \cdot b \rangle$, can be written as

$$\langle a \cdot b \rangle = \int_M t_b \wedge \frac{t_a}{\mu}.$$

Note that t_a is an $(n, n-p)$-form and thus, t_a/μ is a $(0, n-p)$-form that can be integrated against an (n, p)-form t_b.

Definition 5.6 The pairing $\langle a \cdot b \rangle$ of polar cycles is called the *polar intersection index*.

Remark 5.7 If Conjecture (4.4) is true, this pairing is non-degenerate.

5.8 Let us consider now the case when the cycles a and b are smooth and transverse. That is $a = (A, \alpha)$ and $b = (B, \beta)$, where A is a smooth p-dimensional variety and α a holomorphic p-form on it (and similarly for (B, β) in dimension $n - p$) and it is assumed that A and B intersect transversely. Then, we have the following formula for the polar intersection index.

Theorem 5.9 *The polar intersection index of two smooth transverse cycles* (A, α) *and* (B, β) *is given by the following sum over the set of points in* $A \cap B$*:*

$$\langle (A, \alpha) \cdot (B, \beta) \rangle = \sum_{P \in A \cap B} \frac{\alpha(P) \wedge \beta(P)}{\mu(P)}$$

Here $\alpha(P)$ *and* $\beta(P)$ *are understood as exterior forms on* $T_P M = T_P A \times T_P B$ *obtained by the pull-back from the corresponding factors.*

The ratio in the right-hand-side can be understood as the comparison of the polar orientations brought to the intersection point P by the two cycles with the polar orientation $\mu(P)$ of the ambient manifold at that point.

Proof As we have already mentioned, the homomorphism ρ of Proposition 4.3 can be conveniently described in terms of the following natural map of polar chains:

$$\varphi \colon \mathcal{C}_k(M) \to \mathcal{D}^{n,n-k}(M),$$

where $\mathcal{D}^{p,q}(M)$ is the space of currents of degree (p, q) (*i.e.*, a space of linear functionals on smooth $(n-p, n-q)$-forms, see [GH]). As a matter of fact, this map is described by the integral (4.1). For a p-dimensional submanifold $A \subset M$, let the current $\delta_A \in \mathcal{D}^{n-p,n-p}(M)$ denote the linear functional on (p, p)-forms corresponding

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to the integration over A. The current δ_A is supported on A. Therefore, for a p-form α defined on A, the product $\delta_A \wedge \alpha$ makes sense and defines a current in $\mathcal{D}^{n,n-p}(M)$. Recalling the isomorphism of the cohomology of currents with the cohomology of smooth forms,

$$H^j\big(\mathcal{D}^{i,\bullet}(M), \bar\partial\big) \xrightarrow{\sim} H^{i,j}_{\bar\partial}(M),$$

we can use $\delta_A \wedge \alpha$ and $\delta_B \wedge \beta$ in place of t_a and t_b in the integral (5.5). Thus, for a transverse intersection of smooth polar cycles we derive that[7]

$$\langle (A,\alpha)\cdot(B,\beta)\rangle = \int_M \delta_B \wedge \beta \wedge \frac{\delta_A \wedge \alpha}{\mu} = \int_M \Big(\frac{\alpha\wedge\beta}{\mu}\Big)\cdot \delta_A \wedge \delta_B.$$

The second equality can be checked in local coordinates. This proves the theorem, since $\delta_A \wedge \delta_B$ is supported on $A \cap B$. ■

(B) Polar Intersection Product

5.10 Now consider the case when on a polar oriented closed manifold (M, μ) we have two polar cycles of arbitrary dimensions p and q (not necessarily complimentary ones). Similarly to the pairing (5.4), we may consider the following chain of homomorphisms:

$$\begin{aligned}\mathrm{HP}_p(M) \otimes \mathrm{HP}_q(M) &\xrightarrow{s_{12}\circ(\rho\otimes\rho)} H^{n,n-q}_{\bar\partial}(M) \otimes H^{n,n-p}_{\bar\partial}(M)\\ &\xrightarrow{\mathrm{id}\otimes\mu^{-1}} H^{n,n-q}_{\bar\partial}(M)\otimes H^{0,n-p}_{\bar\partial}(M) \longrightarrow H^{n,2n-p-q}_{\bar\partial}(M),\end{aligned}$$

where s_{12} is the transposition of tensor factors and the last term is understood as zero unless $p + q \geqslant n$. Let Λ denote the resulting composition:

$$\Lambda\colon\ \mathrm{HP}_p(M) \otimes \mathrm{HP}_q(M) \to H^{n,2n-p-q}_{\bar\partial}(M).$$

5.11 If Conjecture (4.4) holds, the above homomorphism and the inverse of ρ in (4.3) define an intersection product on polar homology,

$$\mathrm{HP}_p(M) \otimes \mathrm{HP}_q(M) \to \mathrm{HP}_{p+q-n}(M).$$

However, even without this hypothesis we will show that if a and b are two smooth transverse cycles, $[a] \in \mathrm{HP}_p(M)$, $[b] \in \mathrm{HP}_q(M)$, their polar intersection product can be represented in $\mathrm{HP}_{p+q-n}(M)$ by a smooth cycle c.

[7] Although the (exterior) product of currents is not in general defined, the product of the cohomology classes of $\bar\partial$-closed currents is always well defined. In some cases it is easy to find a representative of the product of cohomology classes. For instance, for two submanifolds V and W in a generic position the product of cohomology classes of the corresponding currents δ_V and δ_W is represented by $\delta_{V\cap W}$. In this sense we can write $\delta_V \wedge \delta_W = \delta_{V\cap W}$.

5.12 $\mathbb{C}$-Orientations of Vector Spaces

Let W be an n-dimensional complex vector space, and μ be a non-zero complex volume form on W ($\mu \in \bigwedge^n W^*$). Let $V_A, V_B \subset W$ be vector subspaces of dimensions $\dim V_A = p$, $\dim V_B = q$, $p+q \geqslant n$ that intersect transversely, *i.e.* $V_A + V_B = W$ (or, $r := \dim V_A \cap V_B = p+q-n$). Suppose we are also given complex volume forms on each of V_A and V_B, that is $\alpha \in \bigwedge^p V_A^*$ and $\beta \in \bigwedge^q V_B^*$. (We may say that all three spaces W, V_A, and V_B are $\mathbb{C}$-oriented.) Then, one can define a complex volume form γ on the intersection $V_A \cap V_B$ (*i.e.*, one can $\mathbb{C}$-orient it) as follows.

Let $\lambda_A \in \bigwedge^{n-p} W^*$ be a non-zero exterior form conormal to V_A (*i.e.*, λ_A vanishes on any vector from V_A and is non-zero as an element of $\bigwedge^{n-p}(W/V_A)^*$). Similarly, let $\lambda_B \in \bigwedge^{n-q} W^*$ be a non-zero exterior form conormal to V_B. Note that in this case $\lambda_A \wedge \lambda_B$ is a form conormal to $V_A \cap V_B$.

Definition-Lemma 5.13 *Given complex orientations (i.e., complex volume forms) α, β, and μ of the vector subspaces V_A, V_B and the space W respectively, the following complex orientation γ of the intersection $V_A \cap V_B$ is defined by the following relation:*

$$\lambda_A \wedge \lambda_B \wedge \gamma = \Big(\frac{\lambda_A \wedge \alpha}{\mu}\Big) \cdot \Big(\frac{\lambda_B \wedge \beta}{\mu}\Big) \cdot \mu.$$

Here α, β, and γ are understood as arbitrary extensions of these forms to the whole space W. The r-form γ on $V_A \cap V_B$ depends neither on these extensions, nor on the choice of the auxiliary forms λ_A and λ_B.

Proof A straightforward verification. ■

Corollary 5.14 *For a transverse intersection of subspaces of complimentary dimensions ($p+q = n$ and $W = V_A \oplus V_B$), the 0-form γ is just the following complex number:*

$$\gamma = \frac{\alpha \wedge \beta}{\mu},$$

where α and β are now understood as exterior forms on $W = V_A \times V_B$ obtained by pull-backs from the corresponding factors.

5.15 Let $a = (A, \alpha)$ and $b = (B, \beta)$ be two smooth polar cycles of dimension p and q respectively in a polar oriented closed manifold (M, μ), $p+q \geqslant n = \dim M$. Suppose they intersect transversely. Then we can define a $(p+q-n)$-cycle $c = (C, \gamma)$, where $C = A \cap B$ is a smooth subvariety in M and γ is a holomorphic $(p+q-n)$-form on it defined by (5.13). Let us denote this as $[c] = [a] \cdot [b]$ and call it the intersection product. (Of course, the product $[a] \cdot [b]$ equals zero if $p+q < n$.)

Theorem 5.16 *The polar intersection product $[a] \cdot [b]$ of two smooth transverse cycles $a = (A, \alpha)$ and $b = (B, \beta)$ defined above agrees with the homomorphism (5.10), i.e.,*

$$\Lambda([a] \otimes [b]) = \rho([a] \cdot [b]),$$

where $\rho\colon \operatorname{HP}_k(X) \to H^{n,n-k}_{\bar\partial}(X)$ was defined in (4.1)–(4.3).

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Proof This will be similar to the proof of Theorem 5.9 and will use the same notations. We first represent the polar cycles a and b by the currents $\delta_A \wedge \alpha$ and $\delta_B \wedge \beta$ respectively, then

$$\Lambda([a] \otimes [b]) = \left[\delta_B \wedge \beta \wedge \frac{\delta_A \wedge \alpha}{\mu}\right],$$

where [ ] on the right is understood as taking the $\bar{\partial}$-cohomology class. On the other hand, for $c = (C, \gamma)$ introduced in (5.15), the current representing c is $\delta_C \wedge \gamma$ and it is easy to show that

$$\delta_C \wedge \gamma = \delta_B \wedge \beta \wedge \frac{\delta_A \wedge \alpha}{\mu},$$

which implies the statement of the theorem. The last equality is easily checked by noticing that δ_A is an $(n-p, n-p)$-form (in fact, a current) conormal to A and similarly for δ_B, while $\delta_C = \delta_A \wedge \delta_B$. This is to be compared to λ_A and λ_B in (5.13). One has to note only that, *e.g.*, δ_A is conormal to A over $\mathbb{R}$ (that is in the sense of $(n-p, n-p)$-forms) while λ_A is conormal to it over $\mathbb{C}$ (that is in the sense of $(n-p, 0)$-forms). ■

Remark 5.17 We have defined the polar intersection on any complex manifold M that can be equipped with a *holomorphic* non-vanishing volume form μ. This is analogous to the topological intersection theory on a compact smooth oriented manifold without boundary. (Note that the Poincaré duality in this context should correspond to the Serre duality.) Furthermore, the consideration above easily extends to the case of a complex manifold possessing a *meromorphic* non-vanishing form μ (in particular, to a complex projective space), *i.e.*, to the case of a polar oriented manifold (M, μ) with boundary $(N, \operatorname{res}\mu)$ (*cf.* 5.2). The latter setting is similar to the topological intersection theory on manifolds with boundary. In this case the above formulas can be used to define the pairing between polar homology $\mathrm{HP}_k(M)$ and polar homology relative to the boundary $\mathrm{HP}_{n-k}(M, N)$.

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