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On Sha's Secondary Chern–Euler Class

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Abstract. For a manifold with boundary, the restriction of Chern's transgression form of the Euler curvature form over the boundary is closed. Its cohomology class is called the secondary Chern–Euler class and was used by Sha to formulate a relative Poincaré–Hopf theorem under the condition that the metric on the manifold is locally product near the boundary. We show that the secondary Chern–Euler form is exact away from the outward and inward unit normal vectors of the boundary by explicitly constructing a transgression form. Using Stokes' theorem, this evaluates the boundary term in Sha's relative Poincaré–Hopf theorem in terms of more classical indices of the tangential projection of a vector field. This evaluation in particular shows that Sha's relative Poincaré–Hopf theorem is equivalent to the more classical law of vector fields.

1 Introduction

Let X be a smooth oriented compact Riemannian manifold with boundary M. Throughout the paper we fix $\dim X = n \geq 2$ and hence $\dim M = n - 1$. On M, one has a canonical decomposition

$$TX|_M \cong \nu \oplus TM, \tag{1.1}$$

where ν is the rank 1 trivial normal bundle of M.

In his famous proof of the Gauss–Bonnet theorem, Chern [1, 2] constructed a differential form Φ (see (2.6)) of degree $n-1$ on the tangent sphere bundle STX, consisting of unit vectors in TX satisfying the following two conditions:

$$d\Phi = -\Omega, \tag{1.2}$$

where Ω is the Euler curvature form of X (pulled back to STX) when $\dim X$ is even and 0 otherwise, and

$$\widetilde{\Phi_0} = \widetilde{d\sigma_{n-1}},$$

i.e., the 0-th term $\widetilde{\Phi_0}$ of Φ is the relative unit volume form for the fibration $S^{n-1} \to STX \to X$ (see (2.7)).

By (1.2), one has $d\Phi = 0$ on $STX|_M$, since even if $\dim X$ is even, $\Omega|_M = 0$ by dimensional reason. Following [6], Φ on $STX|_M$ is called the *secondary Chern–Euler form*, whose cohomology class is called the secondary Chern–Euler class.

Secondary Chern–Euler classes are useful in studying the relative Poincaré–Hopf theorem. Let V be a smooth vector field on X. We assume that V has only isolated

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singularities, *i.e.*, the set $\operatorname{Sing} V := \{x \in X \mid V(x) = 0\}$ is finite, and that the restriction $V|_M$ is nowhere zero. Define the index $\operatorname{Ind}_x V$ of V at an isolated singularity x as usual (see [3, p. 136]), and let $\operatorname{Ind} V = \sum_{x \in \operatorname{Sing} V} \operatorname{Ind}_x V$ denote the sum of the local indices. Also define

$$\alpha_V : M \to STX|_M;\ x \mapsto \frac{V(x)}{|V(x)|}.$$

by rescaling V.

Following [6], we assume throughout the paper the following condition:

$$\text{the metric on } X \text{ is } \textit{locally product} \text{ near the boundary } M, \tag{1.3}$$

which in particular implies that M is a totally geodesic submanifold of X. The general case is addressed in [5].

Under condition (1.3), Sha [6] proved his version of the relative Poincaré–Hopf theorem

$$\operatorname{Ind} V - \int_{\alpha_V(M)} \Phi = \begin{cases} \chi(X) & \text{if } \dim X \text{ is even,} \\ 0 & \text{if } \dim X \text{ is odd.} \end{cases} \tag{1.4}$$

The starting point of this paper is to study Φ, or rather its certain restriction defined as follows. Let $\vec{n}$ denote the outward unit normal vector field of M. The images $\vec{n}(M)$ and $(-\vec{n})(M)$ in $STX|_M$ are the spaces of outward and inward unit normal vectors of M. Define

$$CSTM := STX|_M \backslash (\vec{n}(M) \cup (-\vec{n})(M)) \tag{1.5}$$

(C for cylinder) to be the complement.

Theorem 1.1 *Under condition* (1.3), Φ *is exact on* $CSTM$ (1.5). *More precisely, there is a differential form* Γ *of degree* $n-2$ *on* $CSTM$ *such that* $\Phi = d\Gamma$.

The definition of Γ is in Definition 2.4, and the above theorem is proved right after that.

Theorem 1.1 and Stokes' theorem then allow the following concrete evaluation of Sha's term $\int_{\alpha_V(M)} \Phi$ in (1.4) in terms of more classical local indices. For a generic vector field V, let ∂V be the projection of $V|_M$ to TM according to (1.1), and let $\partial_- V$ (resp. $\partial_+ V$) be the restriction of ∂V to the subspace of M, where V points inward (resp. outward) to X. Generically $\partial_\pm V$ have isolated singularities. (A non-generic V can always be modified by adding an extension to X of a normal vector field or a tangent vector field to M.)

Theorem 1.2 *Under condition* (1.3) *and for a generic vector field* V, *one has*

$$\int_{\alpha_V(M)} \Phi = \begin{cases} -\operatorname{Ind} \partial_- V & \textit{if } \dim X \textit{ is even,} \\ \frac{1}{2}(\operatorname{Ind} \partial_+ V - \operatorname{Ind} \partial_- V) & \textit{if } \dim X \textit{ is odd.} \end{cases} \tag{1.6}$$

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Remark 1.3 Generically $\operatorname{Ind}\partial_+ V + \operatorname{Ind}\partial_- V = \operatorname{Ind}\partial V = \chi(M)$ by the Poincaré–Hopf theorem. When $\dim X$ is even and hence $\dim M$ is odd, since $\chi(M) = 0$, one has equality between the two formulas in (1.6).

When $\dim X$ is odd, since $\chi(M) = 2\chi(X)$ by basic topological knowledge, one has the following reformulation of the odd case in (1.6)

$$\int_{\alpha_V(M)} \Phi = \frac{1}{2}\chi(M) - \operatorname{Ind}\partial_- V = \chi(X) - \operatorname{Ind}\partial_- V, \quad \text{if } \dim X \text{ is odd.} \tag{1.7}$$

We finish this introduction by explaining the relation of our result with the law of vector fields. For a generic vector field V, using the flow along $-V$ and counting fixed points with multiplicities, one has the following *law of vector fields*:

$$\operatorname{Ind} V + \operatorname{Ind}\partial_- V = \chi(X). \tag{1.8}$$

This was first proved by Morse [4] and later on publicized by Gottlieb, who also coined the term.

Our result (1.6) and the reformulation (1.7) of the odd case then directly show that the two relative Poincaré–Hopf theorems, (1.4) and (1.8), are equivalent. Therefore, following the route of the relative Poincaré–Hopf theorem of Sha [6] under condition (1.3), our result (1.6) gives a purely differential-geometric proof of the law of vector fields. Other differential-geometric proofs are given in [5].

2 Differential Forms

Throughout the paper, c_{j-1} denotes the volume of the unit $(j-1)$-sphere S^{j-1}.

Chern's transgression form Φ is defined as follows. Choose oriented local orthonormal frames $\{e_1, e_2, \dots, e_n\}$ for the tangent bundle TX. Let (ω_{ij}) and (Ω_{ij}) be the $\mathfrak{so}(n)$-valued connection forms and curvature forms for the Levi–Civita connection ∇ of the Riemannian metric on X defined by

$$\nabla e_i = \sum_{k=1}^{n} \omega_{ij} e_j, \tag{2.1}$$

$$\Omega_{ij} = d\omega_{ij} - \sum_{k=1}^{n} \omega_{ik}\omega_{kj}. \tag{2.2}$$

(In this paper, we closely follow Chern's notation and convention in [1, 2]. In particular we follow his convention in choosing the row and column indices in (2.1), which may not be the most standard. Also, products of differential forms always mean "exterior products", although we omit the notation $\wedge$ for simplicity.)

Let the u_i be the coordinate functions on STX in terms of the frames defined by

$$v = \sum_{i=1}^{n} u_i(v) e_i, \quad \forall v \in STX. \tag{2.3}$$

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Let the θ_i be the 1-forms on STX defined by

$$\theta_i = du_i + \sum_{k=1}^{n} u_k \omega_{ki}. \tag{2.4}$$

For $k = 0, 1, \dots, [\frac{n-1}{2}]$ (with $[\,\cdot\,]$ standing for the integral part), define the degree $n-1$ forms on STX

$$\Phi_k = \sum_{\tau} \epsilon(\tau) u_{\tau_1} \theta_{\tau_2} \cdots \theta_{\tau_{n-2k}} \Omega_{\tau_{n-2k+1}\tau_{n-2k+2}} \cdots \Omega_{\tau_{n-1}\tau_n}, \tag{2.5}$$

where the summation runs over all permutations τ of $\{1, 2, \dots, n\}$, and $\epsilon(\tau)$ is the sign of τ. (The index k stands for the number of curvature forms involved. Hence the restriction $0 \leq k \leq [\frac{n-1}{2}]$. This convention applies throughout the paper.) Define Chern's transgression form as

$$\begin{aligned}\Phi &= \frac{1}{(n-2)!!c_{n-1}} \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k \frac{1}{2^k k!(n-2k-1)!!} \Phi_k \\ &=: \frac{1}{(n-2)!!c_{n-1}} \sum_{k=0}^{[\frac{n-1}{2}]} \overline{\Phi_k} =: \sum_{k=0}^{[\frac{n-1}{2}]} \widetilde{\Phi_k}.\end{aligned} \tag{2.6}$$

(See (2.22) for an explanation, in the case of M with dimension $n-1$, for the coefficients involved.) The Φ_k and hence Φ are invariant under $\mathrm{SO}(n)$-transformations of the local frames and hence are intrinsically defined. Note that the 0-th term

$$\widetilde{\Phi_0} = \frac{1}{(n-2)!!c_{n-1}} \frac{1}{(n-1)!!} \Phi_0 = \frac{1}{c_{n-1}} d\sigma_{n-1} = \widetilde{d\sigma}_{n-1} \tag{2.7}$$

is the relative unit volume form of the fibration $S^{n-1} \to STX \to X$, since by (2.5)

$$\Phi_0 = \sum_{\tau} \epsilon(\tau) u_{\tau_1} \theta_{\tau_2} \cdots \theta_{\tau_n} = (n-1)! d\sigma_{n-1} \tag{2.8}$$

(see [1, (26)]).

Now we start to transgress Φ (2.6) on $CSTM$ (1.5). At $TX|_M$, we choose oriented local orthonormal frames $\{e_1, e_2, \dots, e_n\}$ such that $e_1 = \vec{n}$ is the outward unit normal vector of M. Therefore $\{e_2, \dots, e_n\}$ are oriented local orthonormal frames for TM. Let ϕ be the angle coordinate on $STX|_M$ defined by

$$\phi(v) = \angle(v, e_1) = \angle(v, \vec{n}), \ \forall v \in STX|_M.$$

One has from (2.3)

$$u_1 = \cos\phi. \tag{2.9}$$

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Let

$$p\colon CSTM = STX|_M \backslash (\vec{n}(M) \cup (-\vec{n})(M)) \to STM;\ v \mapsto \frac{\partial v}{|\partial v|}, \tag{2.10}$$

$$\text{(in coordinates)} \qquad (\cos\phi, u_2, \dots, u_n) \mapsto \frac{1}{\sin\phi}(u_2, \dots, u_n)$$

be the projection to the equator STM. By definition,

$$\text{for } x \in M,\ \partial V(x) = 0 \Leftrightarrow \alpha_V(x) = \pm\vec{n}(x), \tag{2.11}$$

$$p \circ \alpha_V = \alpha_{\partial V} \text{ when } \partial V \neq 0. \tag{2.12}$$

The locally product metric (1.3) near M means that $\nabla e_1 = \nabla \vec{n} = 0$. Hence from (2.1) one has

$$\omega_{1*} = -\omega_{*1} = 0. \tag{2.13}$$

From (2.4), (2.9), and (2.13), one has

$$\theta_1 = -\sin\phi\, d\phi. \tag{2.14}$$

From (2.2) and (2.13), one also has

$$\Omega_{1*} = -\Omega_{*1} = 0. \tag{2.15}$$

We use the convention that τ is a permutation of $(1, 2, \dots, n)$ and ρ is a permutation of $(2, \dots, n)$.

In view of (2.15) on $STX|_M$, the index 1 in the formula (2.5) for Φ_k appears in either u_{τ_1} or one of the θ_{τ_i} for $2 \leq i \leq n-2k$. There are totally $n-2k-1$ possibilities for the second case.

Therefore, on $STX|_M$, one has the following more concrete

$$\Phi_k = u_1 \Xi_k - (n-2k-1)\theta_1 \Upsilon_k, \quad k = 0, \dots, \left[\frac{n-1}{2}\right], \tag{2.16}$$

where

$$\Upsilon_k = \sum_\rho \epsilon(\rho) u_{\rho_2} \theta_{\rho_3} \cdots \theta_{\rho_{n-2k}} \Omega_{\rho_{n-2k+1}\rho_{n-2k+2}} \cdots \Omega_{\rho_{n-1}\rho_n}, \qquad k = 0, \dots, \left[\frac{n-2}{2}\right], \tag{2.17}$$

$$\Xi_k = \sum_\rho \epsilon(\rho) \theta_{\rho_2} \theta_{\rho_3} \cdots \theta_{\rho_{n-2k}} \Omega_{\rho_{n-2k+1}\rho_{n-2k+2}} \cdots \Omega_{\rho_{n-1}\rho_n}, \qquad k = 0, \dots, \left[\frac{n-1}{2}\right]. \tag{2.18}$$

The negative sign in (2.16) is from $\epsilon(\tau)$ in (2.5) when one moves θ_1 in front of u_{τ_1}. (When $[\frac{n-1}{2}] = [\frac{n-2}{2}] + 1$, one can either define $\Upsilon_{[\frac{n-1}{2}]} = 0$ by dimensional reason or observe its coefficient in (2.16) $n - 2k - 1 = 0$ for $k = [\frac{n-1}{2}]$. This observation applies throughout the section.)

We use the convention to write superscript ^{e} for functions and forms defined on the equator STM of $STX|_M$. Using the $\{e_2, \dots, e_n\}$ as oriented local orthonormal frames for TM, we define u_i^e, θ_i^e ,and Φ_k^e as functions and forms on STM in the same way as in (2.3), (2.4), and (2.5). Note that $\Omega_{ij}^e = \Omega_{ij}$ for $2 \le i, j \le n$ by (2.2) and (2.13). Therefore one has the degree $n-2$ forms on STM

$$\Phi_k^e = \sum_\rho \epsilon(\rho) u_{\rho_2}^e \theta_{\rho_3}^e \cdots \theta_{\rho_{n-2k}}^e \Omega_{\rho_{n-2k+1}\rho_{n-2k+2}} \cdots \Omega_{\rho_{n-1}\rho_n}, \quad k = 0, \dots, \left[\frac{n-2}{2}\right]. \tag{2.19}$$

Following [1], also define the degree $n-1$ forms on STM

$$\Psi_k^e = \sum_\rho \epsilon(\rho) \theta_{\rho_2}^e \theta_{\rho_3}^e \cdots \theta_{\rho_{n-2k}}^e \Omega_{\rho_{n-2k+1}\rho_{n-2k+2}} \cdots \Omega_{\rho_{n-1}\rho_n}, \ k = 0, \dots, \left[\frac{n-1}{2}\right]. \tag{2.20}$$

Note that the Φ_k^e and the Ψ_k^e are just the Υ_k in (2.17) and the Ξ_k in (2.18) with the superscript ^{e}. By dimensional reasoning one has

$$\Psi_0^e = 0. \tag{2.21}$$

On STM, Chern's basic formulas [1] are

$$d\Phi_k^e = \Psi_k^e + \frac{n-2k-2}{2(k+1)} \Psi_{k+1}^e, \ k = 0, \dots, \left[\frac{n-2}{2}\right]. \tag{2.22}$$

(This also explains, over M with dimension $n-1$, the construction of Φ in (2.6) for the purpose of consecutive cancellations.)

Lemma 2.1 *One has on* $CSTM$ (1.5), *for* $k = 0, \dots, [\frac{n-1}{2}]$,

$$\Phi_k = \sin^{n-2k-1}\phi \cos\phi \, p^* \Psi_k^e + (n-2k-1) \sin^{n-2k-2}\phi \, d\phi \, p^* \Phi_k^e. \tag{2.23}$$

Proof For $2 \le i \le n$ and from (2.10), one has

$$p^* u_i^e = \frac{1}{\sin\phi} u_i. \tag{2.24}$$

Differentiating the above and using (2.4) and (2.13), one has

$$p^* \theta_i^e = \frac{1}{\sin\phi} \theta_i - \frac{\cos\phi}{\sin^2\phi} d\phi \, u_i \tag{2.25}$$

Because of the presence of $d\phi$ and in view of (2.19), (2.24), and (2.25), one has

$$d\phi\, p^*\Phi_k^e = \frac{1}{\sin^{n-2k-1}\phi}\, d\phi\, \Upsilon_k \Rightarrow d\phi\, \Upsilon_k = \sin^{n-2k-1}\phi\, d\phi\, p^*\Phi_k^e, \tag{2.26}$$

where $n-2k-1$ is the number of u and θ's in (2.19). Hence by (2.14), one has

$$\theta_1 \Upsilon_k = -\sin^{n-2k}\phi\, d\phi\, p^*\Phi_k^e. \tag{2.27}$$

Now the pullback of Ψ_k^e in (2.20) is slightly harder, since $d\phi$ may come up as in (2.25), but only once among the $(n-2k-1)$ θ^e's. Therefore

$$p^*\Psi_k^e = \frac{1}{\sin^{n-2k-1}\phi}\Xi_k - (n-2k-1)\frac{\cos\phi}{\sin^{n-2k}\phi} d\phi\, \Upsilon_k.$$

Using (2.9) and (2.26), one then has

$$\begin{aligned} u_1\Xi_k &= \sin^{n-2k-1}\phi \cos\phi\, p^*\Psi_k^e + (n-2k-1)\frac{\cos^2\phi}{\sin\phi} d\phi\, \Upsilon_k \\ &= \sin^{n-2k-1}\phi\cos\phi\, p^*\Psi_k^e + (n-2k-1)\sin^{n-2k-2}\phi\cos^2\phi\, d\phi\, p^*\Phi_k^e. \end{aligned} \tag{2.28}$$

Combining (2.16), (2.27), and (2.28), one has

$$\begin{aligned} \Phi_k &= u_1\Xi_k - (n-2k-1)\theta_1\Upsilon_k \\ &= \sin^{n-2k-1}\phi\cos\phi\, p^*\Psi_k^e + (n-2k-1)\sin^{n-2k-2}\phi\cos^2\phi\, d\phi\, p^*\Phi_k^e \\ &\qquad + (n-2k-1)\sin^{n-2k}\phi\, d\phi\, p^*\Phi_k^e \\ &= \sin^{n-2k-1}\phi\cos\phi\, p^*\Psi_k^e + (n-2k-1)\sin^{n-2k-2}\phi\, d\phi\, p^*\Phi_k^e, \\ &= \text{the right-hand side of (2.23)} \end{aligned}$$

by $\cos^2\phi + \sin^2\phi = 1$. ∎

Since $\Psi_0^e = 0$ (2.21), one has, from (2.23),

$$\Phi_0 = (n-1)\sin^{n-2}\phi\, d\phi\, p^*\Phi_0^e. \tag{2.29}$$

Remark 2.2 In view of (2.8), (2.29) is just the relation (due to condition (1.3)) between the relative volume forms $d\sigma_{n-1}$ of $S^{n-1} \to STX|_M \to M$ and $d\sigma_{n-2}$ of $S^{n-2} \to STM \to M$,

$$d\sigma_{n-1} = \sin^{n-2}\phi\, d\phi\, p^* d\sigma_{n-2}.$$

On one fixed sphere and its equator, this is an easy fact and follows from using spherical coordinates, which also accounts for the basic formula

$$c_{n-1} = c_{n-2}\int_0^\pi \sin^{n-2}\phi\, d\phi. \tag{2.30}$$

Our goal is to find a differential form Γ such that $d\Gamma = \Phi$. We do this inductively starting from the above Φ_0 in (2.29). Therefore we need to use an antiderivative of $\sin^{n-2}\phi$.

Definition 2.3 For a non-negative integer b, define functions of ϕ,

$$I_b(\phi) = \int \sin^b \phi \, d\phi,$$

where we require the arbitrary constants to be 0. More precisely,

$$I_b(\phi) = \begin{cases} \displaystyle\int_0^\phi \sin^b t \, dt & \text{if } b \text{ is even,} \\ \displaystyle\int_{\frac{\pi}{2}}^\phi \sin^b t \, dt & \text{if } b \text{ is odd.} \end{cases} \tag{2.31}$$

Integration by parts gives

$$bI_b(\phi) + \sin^{b-1}\phi \cos\phi = (b-1)I_{b-2}(\phi), \quad b \geq 2. \tag{2.32}$$

Clearly $I_0(\phi) = \phi$ and $I_1(\phi) = -\cos\phi$. These also inductively determine $I_b(\phi)$.

Definition 2.4 We define the following differential forms of degree $n-2$ on $CSTM$:

$$\Gamma_k = I_{n-2k-2}(\phi) p^* \Phi_k^e, \; k = 0, \dots, \left[\frac{n-2}{2}\right], \tag{2.33}$$

$$\begin{aligned} \overline{\Gamma_k} &= (-1)^k \frac{1}{2^k k!(n-2k-3)!!} \Gamma_k \\ &= (-1)^k \frac{1}{2^k k!(n-2k-1)!!} (n-2k-1)\Gamma_k \end{aligned} \tag{2.34}$$

(with the convention $(-1)!! = 1$), and

$$\begin{aligned} \Gamma &= \frac{1}{(n-2)!!c_{n-1}} \sum_{k=0}^{[\frac{n-2}{2}]} \overline{\Gamma_k} \\ &= \frac{1}{(n-2)!!c_{n-1}} \sum_{k=0}^{[\frac{n-2}{2}]} (-1)^k \frac{1}{2^k k!(n-2k-3)!!} I_{n-2k-2}(\phi) p^* \Phi_k^e. \end{aligned} \tag{2.35}$$

With this definition of Γ, now we prove Theorem 1.1.

Proof of Theorem 1.1 First by Chern's basic formula (2.22), for $k = 0, \dots, [\frac{n-2}{2}]$,

$$\begin{aligned} d\Gamma_k = {} & \sin^{n-2k-2}\phi \, d\phi \, p^* \Phi_k^e + I_{n-2k-2}(\phi) p^* \Psi_k^e \\ & + \frac{n-2k-2}{2(k+1)} I_{n-2k-2}(\phi) p^* \Psi_{k+1}^e. \end{aligned} \tag{2.36}$$

Define

$$\overline{L_k} = (-1)^k \frac{n-2k}{2^k k!(n-2k-1)!!} I_{n-2k}(\phi) p^* \Psi_k^e,\ k = 0, \dots, \left[\frac{n}{2}\right] \tag{2.37}$$

(L for leftover).

Claim 2.5 *For* $k = 0, \dots, [\frac{n-2}{2}]$, *one has*

$$\sum_{i=0}^{k} \overline{\Phi_i} - d(\sum_{i=0}^{k} \overline{\Gamma_i}) = \overline{L_{k+1}}. \tag{2.38}$$

Proof of the Claim We proceed by induction. Actually (2.38) clearly holds for $k = -1$, since both sides are zero by natural reasons, (2.37) and (2.21). (One can also check the $k = 0$ case using the same reasoning as in the following induction step.)

Now assume (2.38) holds for $k-1$. Then using this induction hypothesis, plugging in all the formulas (2.37), (2.36), (2.34), (2.23), and by (2.32), one has

$$\begin{aligned}
\sum_{i=0}^{k} \overline{\Phi_i} - d(\sum_{i=0}^{k} \overline{\Gamma_i}) &= \overline{L_k} + \overline{\Phi_k} - d\overline{\Gamma_k} \\
&= (-1)^k \frac{1}{2^k k!(n-2k-1)!!} \Big[((n-2k) I_{n-2k}(\phi) + \sin^{n-2k-1}\phi \cos\phi) p^* \Psi_k^e \\
&\quad + (n-2k-1) \sin^{n-2k-2}\phi\, d\phi\, p^* \Phi_k^e - (n-2k-1) \sin^{n-2k-2}\phi\, d\phi\, p^* \Phi_k^e \\
&\quad - (n-2k-1) I_{n-2k-2}(\phi) p^* \Psi_k^e \\
&\quad - \frac{(n-2k-2)}{2(k+1)} (n-2k-1) I_{n-2k-2}(\phi) p^* \Psi_{k+1}^e \Big] \\
&= (-1)^{k+1} \frac{n-2k-2}{2^{k+1}(k+1)!(n-2k-3)!!} I_{n-2k-2}(\phi) p^* \Psi_{k+1}^e = \overline{L_{k+1}}. \qquad \blacksquare
\end{aligned}$$

When $n = 2m$ for $m \geq 1$, $[\frac{n-2}{2}] = [\frac{n-1}{2}] = m-1$. Therefore to prove $\Phi = d\Gamma$, in view of (2.6) and (2.35), it suffices by (2.38) to proceed as follows:

$$\sum_{i=0}^{m-1} \overline{\Phi_i} - d\Big(\sum_{i=0}^{m-1} \overline{\Gamma_i}\Big) = \overline{L_m} = 0,$$

since $\overline{L_m} = 0$ from (2.37) due to the coefficient $n-2k$ on the top.

When $n = 2m+1$ for $m \geq 1$, $[\frac{n-2}{2}] = m-1$ and $[\frac{n-1}{2}] = m$. In view of (2.6), (2.35), (2.38), (2.37), and (2.23), one has

$$\sum_{i=0}^{m} \overline{\Phi_i} - d(\sum_{i=0}^{m-1} \overline{\Gamma_i}) = \overline{L_m} + \overline{\Phi_m} = (-1)^m \frac{1}{2^m m!} (I_1(\phi) + \cos\phi) p^* \Psi_m^e = 0,$$

since $I_1(\phi) = -\cos\phi$ by Definition 2.3. The proof is now complete. ■

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3 Indices

Now we are ready for the proof of Theorem 1.2 using Stokes' theorem.

Proof of Theorem 1.2 Let $B_r^M(\operatorname{Sing}\partial V)$ (resp. $S_r^M(\operatorname{Sing}\partial V)$) denote the union of small open balls (resp. spheres) of radii r in M around the finite set of points $\operatorname{Sing}\partial V$. Then by (2.11), $\alpha_V(M\backslash B_r^M(\operatorname{Sing}\partial V)) \subset CSTM$. By Theorem 1.1 and Stokes' theorem,

$$
\begin{aligned}
\int_{\alpha_V(M)} \Phi &= \lim_{r\to 0}\int_{\alpha_V(M\backslash B_r^M(\operatorname{Sing}\partial V))} \Phi = \lim_{r\to 0}\int_{\alpha_V(M\backslash B_r^M(\operatorname{Sing}\partial V))} d\Gamma \\
&= -\lim_{r\to 0}\int_{\alpha_V(S_r^M(\operatorname{Sing}\partial V))} \Gamma \\
&= -\lim_{r\to 0}\int_{\alpha_V(S_r^M(\operatorname{Sing}\partial V))} \frac{1}{(n-2)!!c_{n-1}}\overline{\Gamma_0},
\end{aligned}
\tag{3.1}
$$

since all the other $\overline{\Gamma_k}$ for $k \geq 1$ in (2.35) involve curvature forms and hence do not contribute in the limit when integrated over small spheres (see [2, §2]).

One has by Definition 2.4

$$
\begin{aligned}
\frac{1}{(n-2)!!c_{n-1}}\overline{\Gamma_0} &= \frac{1}{(n-2)!!c_{n-1}}\frac{1}{(n-3)!!}I_{n-2}(\phi)p^*\Phi_0^e \\
&= \frac{1}{c_{n-1}}I_{n-2}(\phi)p^*d\sigma_{n-2}
\end{aligned}
\tag{3.2}
$$

with $d\sigma_{n-2}$ being the relative volume form of $S^{n-2} \to STM \to M$, since $\Phi_0^e = (n-2)!d\sigma_{n-2}$ (see (2.8)).

Continuing (3.1) and using (3.2), one has

$$
\begin{aligned}
\int_{\alpha_V(M)} \Phi &= -\frac{1}{c_{n-1}}\lim_{r\to 0}\int_{\alpha_V(S_r^M(\operatorname{Sing}\partial_+ V)\cup S_r^M(\operatorname{Sing}\partial_- V))} I_{n-2}(\phi)p^*d\sigma_{n-2} \\
&= -\frac{1}{c_{n-1}}\Big[I_{n-2}(0)\lim_{r\to 0}\int_{\alpha_{\partial V}(S_r^M(\operatorname{Sing}\partial_+ V))} d\sigma_{n-2} \\
&\qquad + I_{n-2}(\pi)\lim_{r\to 0}\int_{\alpha_{\partial V}(S_r^M(\operatorname{Sing}\partial_- V))} d\sigma_{n-2}\Big] && (3.3)\\
&= -\frac{c_{n-2}}{c_{n-1}}(I_{n-2}(0)\operatorname{Ind}\partial_+ V + I_{n-2}(\pi)\operatorname{Ind}\partial_- V) && (3.4)\\
&= \begin{cases} -\operatorname{Ind}\partial_- V & \text{if } n=\dim X \text{ is even,}\\ \frac{1}{2}(\operatorname{Ind}\partial_+ V - \operatorname{Ind}\partial_- V) & \text{if } n=\dim X \text{ is odd.}\end{cases} && (3.5)
\end{aligned}
$$

Here equality (3.3) uses (2.12) and

$$
\phi(\alpha_V(x)) \to \pi \text{ for } x\in S_r^M(\operatorname{Sing}\partial_- V), \text{ as } r\to 0,
$$

$$
\phi(\alpha_V(x)) \to 0 \text{ for } x\in S_r^M(\operatorname{Sing}\partial_+ V), \text{ as } r\to 0.
$$

Equality (3.4) is by the definition of index. In view of (2.31), one has

$$I_{n-2}(0) = 0,\ I_{n-2}(\pi) = \int_0^\pi \sin^{n-2}\phi\, d\phi,\ \text{if } n \text{ is even},$$

$$(3.6)\ I_{n-2}(0) = -\frac{1}{2}\int_0^\pi \sin^{n-2}\phi\, d\phi,\ I_{n-2}(\pi) = \frac{1}{2}\int_0^\pi \sin^{n-2}\phi\, d\phi,\ \text{if } n \text{ is odd},$$

where (3.6) uses symmetry of integrals. Then equality (3.5) follows from (2.30). ■

Remark 3.1 If, instead of (2.31), one also defines $I_b(\phi) = \int_0^\phi \sin^b t\, dt$ for the odd case, it can be checked that one gets formulas different from, but equivalent to, ours.

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