BULL. AUSTRAL. MATH. SOC. 08B99, 08B15, 05C99, 03C05
VOL. 39 (1989) [265-276]

GRAPH VARIETIES CONTAINING MURSKII'S GROUPOID

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In this paper varieties are investigated which are generated by graph algebras of undirected graphs and—in most cases—contain Murskii's groupoid (that is the graph algebra of the graph with two adjacent vertices and one loop). Though these varieties are inherently nonfinitely based, they can be finitely based as graph varieties (finitely graph based) like, for example, the varitey generated by Murskii's groupoid. Many examples of nonfinitely based graph varities containing Murskii's groupoid are given, too. Moreover, the coatoms in the subvariety lattice of the graph variety of all undirected graphs are described. There are two coatoms and they are finitely graph based.

1. INTRODUCTION

Graph algebras were invented by C. Shallon [15] to provide examples of non-finitely based finite algebras (see [5] for an account of these results, and [1] for the newer developments); further reference and other graph theoretic and algebraic applications can be found in [2-4] and [7-13].

DEFINITION 1.1: To recall this concept, let $G = (V, E)$ be a (directed) graph with vertex set V and edges $E \subseteq V \times V$ (in general we shall use the notation $V(G)$ and $E(G)$). Define the *graph algebra* $G^{\#}$ corresponding to G to have underlying set $V \cup \{\infty\}$ where ∞ is a symbol outside V, and two basic operations, a nullary operation pointing to ∞ and a binary operation denoted by juxtaposition, given by

$$uv = \begin{cases} u \text{ if } (u,v) \in E \\ \infty \text{ otherwise} \end{cases}$$

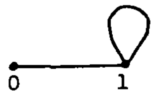
$u,\ v \in V \cup \{\infty\}$.

One of the first examples of a non-finitely based finite algebra was the three-element algebra called *Murskii's groupoid* (after its discoverer, see [6]). This algebra is the graph algebra $P_{01}^{\#}$ of the undirected graph P_{01} (denoted by G_0 in [2-4]) with two adjacent vertices and one loop. Figure 1 presents this graph and the multiplication table for the corresponding graph algebra.

It has been observed by S. Oates-Macdonald and M. Vaughan-Lee [7] that the lattice of subvarieties of the variety $\mathrm{Var}\{P_{01}^{\#}\}$ generated by Murskii's groupoid is also

Received 21 June 1988

https://doi.org/10.1017/S0004972700002756 Published online by Cambridge University Press



P_{01}

xy	$y =$ ∞	0	1
$x = \infty$	∞	∞	∞
0	∞	∞	0
1	∞	1	1

Figure 1.

very interesting. According to the main result of S. Oates-Williams[**10**], this lattice is uncountable and satisfies neither the minimum nor the maximum condition.

In this paper we go in another direction and shall investigate varieties containing Murskii's groupoid.

In [**3**] it is shown that the investigation of subvarieties of varities generated by graph algebras can be reduced to the investigation of so-called graph varieties.

DEFINITION 1.2: For a class $\mathcal{K}$ of graphs, let the *graph variety* $\mathrm{Var}_g\,\mathcal{K}$ *generated* by $\mathcal{K}$ be the class of all graphs G for which $G^\# \in \mathrm{Var}\,\mathcal{K}^\# = \mathrm{HSP}\{A^\# \mid A \in \mathcal{K}\}$. We call $\mathcal{K}$ a *graph variety* if $\mathrm{Var}_g\,\mathcal{K} = \mathcal{K}$.

By Birkhoff's Theorem, $\mathcal{K}$ is a graph variety if and only if there is some set Σ of identities such that

$$\mathcal{K} = \{G \mid G \text{ is a graph } \& G^\# \models \Sigma\}.$$

We have:

THEOREM 1.3. *([**3**, Theorem 1.5, 1.6]) Every subvariety of a variety generated by graph algebras is also generated by its graph algebras. Thus the lattice of subvarities of* $\mathrm{Var}\,\mathcal{K}^\#$ *(for some class* $\mathcal{K}$ *of graphs) is isomorphic to the lattice of graph subvarities of* $\mathrm{Var}_g\,\mathcal{K}^\#$.

Graph varities can be described by some closure properties. We mention here only the result for undirected graphs (for the directed case, see [**11**]).

THEOREM 1.4. *([**2**]) A class of undirected graphs is a graph variety if and only if it is closed under isomorphic copies, direct products, induced subgraphs disjoint, and directed unions.*

Though most of the varieties generated by graph algebras are inherently non-finitely based (see [**15**, **5**, **1**]) they can be finitely based as graph varieties. Therefore we introduce the following notion.

DEFINITION 1.5: A graph variety $\mathcal{K}$ is called *finitely graph based* if there is a finite

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set Σ of identities, such that

$$\mathcal{K} = \operatorname{Mod}_g \Sigma := \{G \mid G \text{ is a graph } \& G^\# \models \Sigma\}.$$

A corresponding equational logic for graph algebras was developed in [**12**].

Example 1.6. Trivial examples of non-finitely based graph varieties which are finitely graph based are the class $\mathcal{G}_u$ of all undirected graphs and the class $\mathcal{G}_u^0$ of all loopless undirected graphs:

$$\begin{aligned}\mathcal{G}_u &= \{G \mid G \text{ graph } \& G^\# \models x_0x_1 \approx x_0(x_1x_0)\},\\ \mathcal{G}_u^0 &= \{G \mid G \text{ graph } \& G^\# \models \{x_0x_0 \approx \infty, x_0x_1 \approx x_0(x_1x_0)\}\}.\end{aligned}$$

Further examples can found also in [**13**, Remark 3.4]. This paper is motivated by the question as to whether the variety generated by Murskii's groupoid is finitely graph based. We have:

THEOREM 1.7. *The graph variety $\mathcal{V}_{01} = \operatorname{Var}_g\{P_{01}\}$ is finitely graph based.*

This result follows from the fact (Corollary 3.4) that the interval $[\operatorname{Var}_g\{P_{01}\}, \mathcal{G}_u]$ in subvariety lattice of $\mathcal{G}_u$ is atomic with two atoms. In Section 4 several parts of this interval are described in more detail; moreover it contains exactly one coatom $\mathcal{M}_1$ (which is finitely graph based (Theorem 5.2)). All coatoms of the subvariety lattice of $\mathcal{G}_u$ are described in Section 5.

Because there is a great difference between graph varieties and varieties with respect to equational bases, there arises the question as to what is the connection between equational bases for graph varieties and for the (usual) varieties generated by them. The answer was given in [**12**].

THEOREM 1.8. ([**12**, Proposition 1.9b]) *Let $\operatorname{Var}_g \mathcal{K} = \mathcal{K} = \operatorname{Mod}_g \Sigma$ be a graph variety determined by a set Σ of identities. Then the variety $\operatorname{Var} \mathcal{K}^\# = \operatorname{HSP} \mathcal{K}^\#$ generated by $\mathcal{K}^\# = \{G^\# \mid G \in \mathcal{K}\}$ is determined by $\Sigma \cup \Sigma_0$; that is*

$$\operatorname{Var} \mathcal{K}^\# = \{\mathcal{A} \mid \mathcal{A} \text{ algebra of type } \langle 0, 2\rangle,\ \mathcal{A} \models \Sigma \cup \Sigma_0\},$$

where Σ_0 is the set of identities which hold in all graph algebras.

Remark. The set Σ_0 was explicitly described in [**11**, Proposition 2.2], [**3**, Lemma 2.2]; see also [**12**, Proposition 1.4].

2. Notions, notation and preliminary results

We shall not go into details here concerning terms and identities in the language of graph algebras; we refer to [**3**] and [**12**] (see also [**5**, **1**]). Our most important tool for dealing with graph varieties is a criterion to decide whether or not a given graph belongs to a graph variety generated by a given set of graphs. In view of Theorem 1.4 the following Lemma provides such a criterion, at least for finite connected undirected graphs. In fact, a finite connected undirected graph G belongs to a graph variety generated by a set $\mathcal{K}$ of undirected graphs if and only if G is isomorphic to an induced subgraph of a direct product of members of $\mathcal{K}$ (see Theorem 1.4, [**12**, **11**]).

Lemma 2.1. *(See* [**3**, Lemma 4.6], [**14**]*). I) Let G be a graph and $\mathcal{K}$ a family of graphs. Then the following are equivalent:*

(1) *G is (isomorphic to) an induced subgraph of a direct product of members of $\mathcal{K}$.*

(2) *There exists a nonempty set ϕ of graph homomorphisms from G to members of $\mathcal{K}$ satisfying the following two conditons:*

  (a) *for every two different vertices u and v of G there is an $f \in \Phi$ such that $f(u)$ and $f(v)$ are different;*

  (b) *for every non-edge (u,v) of G there is an $f \in \Phi$ such that $(f(u), f(v))$ is also a non-edge.*

II) In particular, if $G \in \mathcal{G}_u$ is connected and finite, and if $\mathcal{K} \subseteqq \mathcal{G}_u$, then $G \in \mathrm{Var}_g \mathcal{K}$ if and only if condition (2) is satisfied.

We recall, a mapping $f\colon V(G) \to V(H)$ from a graph G into a graph H is a *(graph) homomorphism* if for all edges $(u,v) \in E(G)$ (loops $(u,u) \in E(G)$ included) we have $(f(u), f(v)) \in E(H)$. A *non-edge* is a pair (u,v) of vertices with $(u,v) \notin E(G)$.

The proof of 2.1 is elementary and is left to the reader (see [**14**] for the general case of arbitary relational systems).

Let us consider now finitely graph based graph varieties.

Proposition 2.2. *(i) A graph variety $\mathcal{W}$ is finitely graph based if and only if there is no infinite descending chain*

$$\mathcal{W}_0 \supset \mathcal{W}_1 \supset \ldots \supset \mathcal{W}_n \supset \ldots \supset \mathcal{W}$$

of graph varieties $\mathcal{W}_i$ with $\bigcap_{i=0}^{\infty} \mathcal{W}_i = \mathcal{W}$.

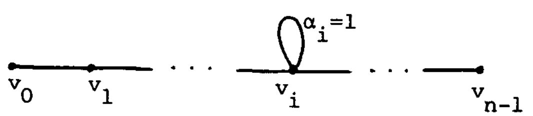
(ii) If $\mathcal{G}$ is a finitely graph based graph variety containing the graph variety $\mathcal{W}$, then (i) remains true by adding the assumption $\mathcal{W}_0 = \mathcal{G}$.

Proof: The proof is totally analogous to the corresponding fact for ordinary varieties. In fact, let $\mathcal{W}_i = \mathrm{Mod}_g\, \Sigma_i$, $\mathcal{W} = \mathrm{Mod}_g\, \Sigma$ and let $\langle \Sigma_i \rangle$, $\langle \Sigma \rangle$ denote the set of all identities which hold in $\mathcal{W}_i$, $\mathcal{W}$, respectively. Then we have $\Sigma_i \subseteqq \langle\Sigma_i\rangle \subseteqq \langle\Sigma\rangle$, and $\bigcap_{i=0}^{\infty} \mathcal{W}_i = \mathcal{W}$ is equivalent to $\langle\Sigma\rangle = \bigcup_{i=0}^{\infty} \langle\Sigma_i\rangle$ or $\Sigma \subseteqq \bigcup_{i=0}^{\infty} \langle\Sigma_i\rangle$. Thus, for finite Σ there is an $n \in \mathbb{N} = \{0,1,\ldots\}$ with $\Sigma \subseteqq \bigcup_{i=0}^{n} \langle\Sigma_i\rangle$; consequently $\bigcap_{i=0}^{n} \mathcal{W}_i = \mathcal{W}$ and $\mathcal{W}_i = \mathcal{W}$ for $i \geqq n$. Conversely, for a non-finitely graph based graph variety $\mathcal{W} = \mathrm{Mod}_g\,\Sigma$ let $\Sigma = \{g_0, g_1, g_2, \ldots\}$ and $\Sigma_i = \{g_1, \ldots, g_i\}$. Then $\mathcal{W}_i = \mathrm{Mod}_g\,\Sigma_i$ gives an infinite descending chain with $\bigcap_{i=0}^{\infty} \mathcal{W}_i = \mathcal{W}$. Thus (i) if proved.

To see (ii), let $\mathcal{W}_i \supseteqq \mathcal{W}$ as above, $\mathcal{G} = \mathrm{Mod}_g\,\Sigma'$, Σ' finite. Since $\mathcal{W} \subseteqq \mathcal{G}$ and therefore $\Sigma' \subseteqq \langle\Sigma\rangle = \bigcup_{i=0}^{\infty} \langle\Sigma_i\rangle$, there is some $n \in \mathbb{N}$ with $\Sigma' \subseteqq \bigcup_{i=0}^{n} \langle\Sigma_i\rangle = \langle\Sigma_n\rangle$; that is $\mathcal{G} \supseteqq \mathcal{W}_i$ for $i \geqq n$. Consequently, if $(\mathcal{W}_i)_{i\in\mathbb{N}}$ is an infinite chain, then $(\mathcal{G} \cap \mathcal{W}_i)_{i\in\mathbb{N}}$ is infinite, too. This implies (ii). ∎

In the next sections we are concerned with special graphs and graph varieties for which we introduce the following notation:

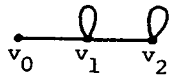
DEFINITIONS 2.3: Let $P_{\alpha_0\alpha_1\ldots\alpha_{n-1}}(\alpha_0, \ldots, \alpha_{n-1} \in \{0,1\}, n \geqq 1)$ denote the graph which is the undirected path of n consecutive vertices $v_0, v_1, \ldots, v_{n-1}$ with a loop at vertex v_i if and only if $\alpha_i = 1$:



Let $\mathcal{V}_{\alpha_0\alpha_1\ldots\alpha_{n-1}} = \mathrm{Var}_g\{P_{\alpha_0\alpha_1\ldots\alpha_{n-1}}\}$ be the corresponding graph variety. For $P_{\underbrace{10\ldots01}_{k}}$ we write P_{10^k1} (analogously $\mathcal{V}_{10^k1}$).

The *join* of two graph varieties is denoted by $\mathcal{V} \vee \mathcal{W}$ (it is the least graph variety containing both $\mathcal{V}$ and $\mathcal{W}$).

For example we have: P_{01} is the graph given in Figure 1, P_{011} is the graph



and $\mathcal{V}_{01} \vee \mathcal{V}_{11} = \mathrm{Var}_g\{P_{01}, P_{11}\}$. We note that many of the varieties $\mathcal{V}_{\alpha_0\alpha_1\ldots\alpha_{n-1}}$ coincide, for example $\mathcal{V}_{001} = \mathcal{V}_{0001}$ (proof by 2.1).

LEMMA 2.4. *A connected graph G belongs to $\mathcal{V}_{01}$ if and only if it is either loopless or contains exactly one loop at a vertex adjacent to every other vertex.*

PROOF: $\mathcal{G}_u^0 \subsetneqq \mathcal{V}_{01}$ (see 1.6) has been mentioned or proved in many papers; we refer to [**15**, Theorem 7.4], [**8**, Theorem 1], [**5**, p.211], [**13**, 4.5] and [**4**, Proposition 1.1]. The remaining parts of the proof easily follow from 1.4. ∎

Remark 2.5. In the next sections we shall consider undirected graphs only. Because of 2.2(ii) this does not affect the property of being finitely graph based. Moreover, without loss of generality we assume that all graphs under consideration are finite, undirected and connected (since, by 1.4, a graph G belongs to a subvariety $\mathcal{V}$ of $\mathcal{G}_u$ if and only if it is undirected and every finite connected induced subgraph of G belongs to $\mathcal{V}$).

3. Murskii's groupoid is finitely graph based

LEMMA 3.1. $P_{001} \in \mathcal{V}_{101}$.

PROOF: The proof follows easily from criterion 2.1(II) since all the required homomorphisms $P_{001} \to P_{101}$ exist. The Lemma will follow also from 4.1. ∎

LEMMA 3.2. *Let $\mathcal{W}$ be a graph variety such that $\mathcal{V}_{01} \subsetneqq \mathcal{W} \subseteqq \mathcal{G}_u$. Then $P_{001} \in \mathcal{W}$ or $P_{11} \in \mathcal{W}$.*

PROOF: Let $G \in \mathcal{W} \backslash \mathcal{V}_{01}$ be a finite connected graph. By 2.4, G has at least two vertices and one loop. If G has two loops, consider a shortest path connecting these loops. Obviously, G must have an induced subgraph isomorphic to P_{11}, P_{001} or P_{101}. If G has exactly one loop at some vertex, say v, then v cannot be adjacant to every other vertex (otherwise $g \in \mathcal{V}_{01}$ by 2.4). Consequently, G has the subgraph P_{001}. Thus $P_{001} \in \mathrm{Var}_g\{G\}$ or $P_{11} \in \mathrm{Var}_g\{G\}$ (since $P_{001} \in \mathrm{Var}\{P_{101}\}$ by 3.1). ∎

LEMMA 3.3. *The graph varieties $\mathcal{V}_{001}$ and $\mathcal{V}_{11} \vee \mathcal{V}_{01}$ are different and properly contain $\mathcal{V}_{01}$.*

PROOF: The proof is a typical application of 2.1. We have $P_{001} \notin \mathcal{V}_{11} \vee \mathcal{V}_{01}$ since there is no homomorphism from P_{001} into P_{11} or P_{01} which maps the non-edge $(v_0, v_2) \notin E(P_{001})$ to a non-edge. Moreover, $P_{11} \notin \mathcal{V}_{001}$ since there is no homomorphism from P_{11} into P_{001} which maps the different looped vertices of P_{11} into different vertices. Finally, $P_{11} \notin \mathcal{V}_{01}$ and $P_{001} \notin \mathcal{V}_{01}$ for the same reasons (or by 2.4). ∎

COROLLARY 3.4. *The interval $[\mathcal{V}_{01}, \mathcal{G}_u]$ of graph varieties containing P_{01} is an atomic subvariety lattice of $\mathcal{G}_u$ with the two atoms $\mathcal{V}_{001}$ and $\mathcal{V}_{11} \vee \mathcal{V}_{01}$. (3.2, 3.2).* ∎

By 2.2, this Corollary implies Theorem 1.7: $\mathcal{V}_{01}$ is finitely graph based.

A direct proof of Theorem 1.7 may be given by presenting a concrete finite equational basis, as follows:

THEOREM 3.5. *Let Σ be the set consisting of the following three identities:*

(i) $x_0x_1 \approx x_0(x_1x_0)$;

(ii) $(x_0x_0)(x_1x_2) \approx (x_0x_0)(x_2x_1)$;

(iii) $(x_0x_0)(x_1x_1) \approx (x_1x_1)(x_0x_0)$.

Then $\mathcal{V}_{01} = \mathrm{Mod}_g\,\Sigma$.

PROOF: Let $\mathcal{W} = \mathrm{Mod}_g\,\Sigma$. It is straightforward to check that P_{01} satisfies (i)–(iii); that is $\mathcal{V}_{01} \subseteqq \mathcal{W}$. To see the converse, note that (i) expresses the property of being undirected (see 1.6), (ii) forces every looped vertex x_0 to be adjacent to every other vertex which is connected (by a path) with x_0 and (iii) expresses the property that adjacent looped vertices must coincide. Thus, by 2.4, every $G \in \mathcal{W}$ belongs to $\mathcal{V}_{01}$. ∎

4. GRAPH VARIETIES CONTAINING P_{01}

We start with a characterisation by closure properties of graph varieties containing P_{01}.

THEOREM 4.1. *Let $\mathcal{W} \subseteqq \mathcal{G}_u$ be a graph variety contianing P_{01}. Then $\mathcal{W}$ is closed with respect to deleting loops and edges between non-looped vertices. Conversely, if graph variety $\mathcal{W} \subseteqq \mathcal{G}_u$ is closed with respect to these operations and if $\mathcal{W}$ contains a connected graph with at least one loop and two vertices then $P_{01} \in \mathcal{W}$.*

PROOF: Let $(u_1, u_2) \in E(G)$ and $G \in \mathcal{W}$, where u_1, $u_2 \in V(G)$ are non-looped when $u_1 \neq u_2$. Let G' be the graph obtained from G by deleting the edge (or loop if $u_1 = u_2$) (u_1, u_2). We have to show $G' \in \mathcal{W}$. Obviously, the identity mapping $\iota\colon V(G') \to V(G)$ and the mapping $f\colon V(G') \to V(P_{01}) = \{v_0, v_1\}$ with $f(u_1) = f(u_2) = v_0$ and $f(x) = v_1$ otherwise, are homomorphisms from G' into G and P_{01}, respectively. These two homomorphisms fulfil conditions 2.1(a) and (b); thus $G' \in \mathrm{Var}_g\{G, P_{01}\} \subseteqq \mathcal{W}$.

The second part of 4.1 is trivial. ∎

There is no full description of the undirected graph varieties containing P_{01}. Figure 2 describes a part of the interval $[\mathcal{V}_{01}, \mathcal{G}_u]$ together with the varieties $\mathcal{G}_u^0$, $\mathcal{V}_1$, $\mathcal{V}_{11}$ and $\mathcal{V}_{111}$. The coatoms $\mathcal{M}_1$, $\mathcal{M}_2$ will be described in Section 5. One can prove that:

The lattice given in Figure 2 is a sublattice of the subvariety lattice of $\mathcal{G}_u$.

We shall not give the proof of this fact (the proof is more or less straightforward via Lemma 2.1; see for example the proof of 3.3). As an example we prove the following:

PROPOSITION 4.2. $\mathcal{G}_u = \mathcal{V}_{01} \vee \mathcal{V}_{111}$, *in particular* $\mathcal{G}_u = \mathcal{V}_{0111}$.

PROOF: Let $G = (V, E) \in \mathcal{G}_u$. We show the existence of graph homomorphisms satisfying 2.1(a) and (b) for $\mathcal{K} = \{P_{01}, P_{111}\}$ (then 2.1 applies). In fact, for vertices

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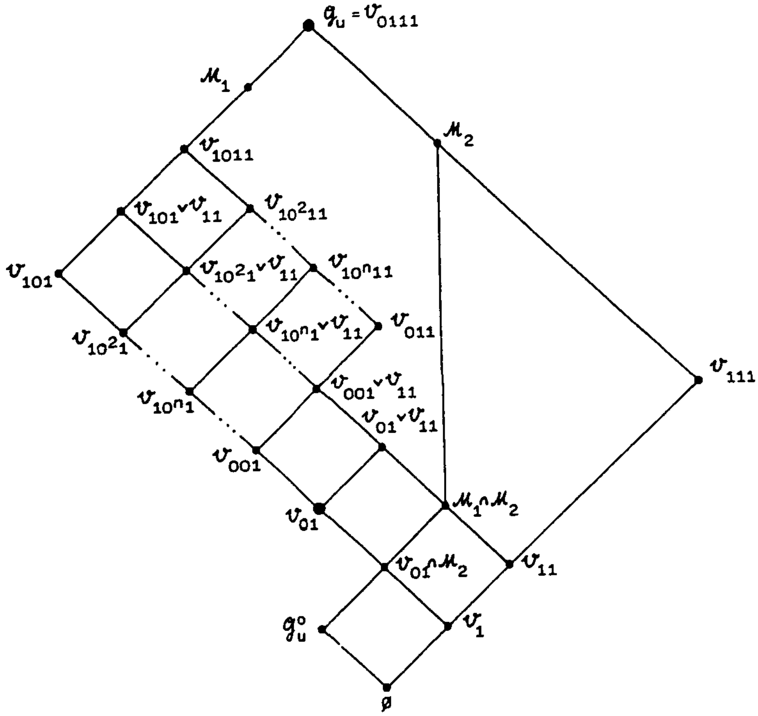

Figure 2

$u \neq v$ we choose the mapping $f\colon V \to V(P_{111})$, $f(u) = v_0$ and $f(x) = v_1$ for $x \neq v$. Given a non-edge $(u,u) \notin E$ with $u \in V$, we choose the mapping $g\colon V \to V(P_{01})$ with $g(u) = v_0$ and $g(x) = v_1$ for $x \neq u$. Given a non-edge $(u,v) \notin E$ with $u \neq v$ we take the mapping $h\colon V \to V(P_{111})$ with $h(u) = v_0$, $h(v) = v_2$ and $h(x) = v_1$ for $x \notin \{u,v\}$. All these mappings are graph homomorphisms. ∎

However there are many more graph varieties between $\mathcal{V}_{01}$ and $\mathcal{G}_u$. We mention, in particular, that almost all graph varieties in Figure 2 are non-finitely graph based (thus there exist infinite descending chains; see 4.4).

The exceptions, that is *the finitely graph based graph varieties among the varieties in* Figure 2 are

$$\mathcal{G}_u^0, \quad \mathcal{G}_u, \quad \mathcal{M}_1, \quad \mathcal{M}_2, \quad \mathcal{V}_{111}, \quad \mathcal{V}_{01},$$

their intersections

$$\mathcal{M}_1 \cap \mathcal{M}_2 = \mathcal{G}_u^0 \vee \mathcal{V}_{11}, \qquad \mathcal{V}_{01} \cap \mathcal{M}_2 = \mathcal{G}_u^0 \vee \mathcal{V}_1,$$
$$\mathcal{M}_1 \cap \mathcal{V}_{111} = \mathcal{V}_{11}, \qquad \mathcal{V}_{01} \cap \mathcal{V}_{111} = \mathcal{V}_1,$$

and the empty graph variety $\emptyset = \{G \in \mathcal{G}_u \mid G^{\#} \models x = y\}$.

Finite bases for the identities of these exceptional cases are given in 1.6, 3.5, 5.1 and in the next Proposition. Of course, the intersection (like $\mathcal{V}_1$, $\mathcal{V}_{11}$) of graph varieties is characterised by the union of the corresponding identities. Nevertheless we mention here a simpler characterisation for $\mathcal{V}_1$ and $\mathcal{V}_{11}$.

PROPOSITION 4.3.

$$\mathcal{V}_{111} = \mathrm{Mod}_g\{x_0x_0 \approx x_0,\ x_0x_1 \approx x_0(x_1x_0)\},$$
$$\mathcal{V}_{11} = \mathrm{Mod}_g\{x_0x_0 \approx x_0,\ x_0(x_1x_2) \approx (x_0x_1)x_2\},$$
$$\mathcal{V}_1 = \mathrm{Mod}_g\{x_0x_0 \approx x_0,\ x_0x_1 \approx x_1x_0\}.$$

PROOF: This follows from the definitions without difficulties; see also [**1**]. ∎

Note that the indicated identities imply undirectedness, for instance in the case of $\mathcal{V}_{11}$, the second identity implies $x_0(x_1x_0) \approx (x_0x_1)x_0$ which, together with $x_0x_0 \approx x_0$ and the always valid identity $(x_0x_1)x_0 \approx (x_0x_0)x_1$ ($\in \Sigma_0$; see for example [**12**, 1.4]) gives $(x_0x_1)x_0 \approx x_0x_1$, and this characterises undirectedness (see 1.6).

The following Theorem will serve as an example that the other varieties in Figure 2 are not only non-finitely based (as shown in [**5**, Theorem 6]) but also non-finitely graph based.

THEOREM 4.4. *$\mathcal{V}_{101}$ is non-finitely graph based.*

PROOF: Let D_n be the graph given in Figure 3.

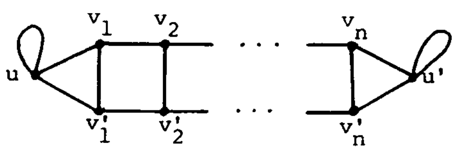

Figure 3

Via Lemma 2.1 one shows that $D_{n+1} \in \mathrm{Var}_g\{D_n\}$ but $D_n \notin \mathrm{Var}_g\{D_{n+1}, P_{101}\}$ (since there is no homomorphism from D_n into D_{n+1} or P_{101} for which u and u' have different images). Consequently we get an infinite descending chain $\mathcal{W}_1 \supset \mathcal{W}_2 \supset$

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$\ldots \supset \mathcal{W}_n \supset \ldots \supset \mathcal{V}_{101}$ where $\mathcal{W}_n = \mathrm{Var}_g\{D_n, P_{101}\}$. By 2.2 it suffices to prove $\bigcap_{n=1}^{\infty} \mathcal{W}_n = \mathcal{V}_{101}$. Let $\mathcal{W} = \bigcap_{n=1}^{\infty} \mathcal{W}_n$ and let $G \in \mathcal{W}$ be finite and connected. Let m be the diameter of G, that is the greatest distance of vertices of G, and let $n > m$. Since the vertices u, u' have distance $n+1$ in the graph D_n, the images of the homomorphisms which have to exist by 2.1 for $G \in \mathcal{W}_n = \mathrm{Var}_g\{D_n, P_{101}\}$ do not contain both u and u'. Thus, again by 2.1, $G \in \mathrm{Var}\{\tilde{D}_n, P_{101}\}$ where $\tilde{D}_n$ is the graph arising form D_n by deleting the vertex u'. Now we are going to show $\tilde{D}_n \in \mathcal{V}_{101}$, which implies $G \in \mathcal{V}_{101}$ and therefore $\mathcal{W} = \mathcal{V}_{101}$.

In fact, the following mappings are graph homomorphisms and show (via 2.1) that $\tilde{D}_n \in \mathrm{Var}_g\{P_{101}\}$. For different vertices w_1, $w_2 \in V\left(\tilde{D}_n\right)$ at least one, say w_1, has no loop (since $\tilde{D}_n$ has only one loop); we choose the mapping $f\colon V\left(\tilde{D}_n\right) \to V(P_{101})$ with $f(w_1) = v_1$, $f(x) = v_0$ for $x \neq w_1$ (notation see 2.3). The same mapping serves for the non-loops $(w_1, w_1) \notin E\left(\tilde{D}_n\right)$. For the non-edges $(w_1, w_2) \notin E\left(\tilde{D}_n\right)$, take the mapping $g\colon V\left(\tilde{D}_n\right) \to V(P_{101})$ with $g(w_1) = v_0$, $g(w_2) = v_2$ and $g(x) = v_1$ for $x \notin \{w_1, w_2\}$ ∎

Remark. Similar proofs exist for the other varieties in Figure 2, other than those already shown to be finitely graph based. Moreover, one can show that: *every connected graph G which contains exactly one loop and P_{001} as induced subgraph, generates a non-finitely graph based graph variety $\mathcal{W}$ ($\mathcal{W}_n = \mathrm{Var}_g\{G, P_{10^n1}\}$, $n = 1, 2, \ldots$ give an infinite descending chain with $\bigcap \mathcal{W}_n = \mathcal{W}$).*

5. Maximal graph varieties in $\mathcal{G}_u$

Since the graph variety $\mathcal{G}_u$ is finitely generated, $\mathcal{G}_u = \mathrm{Var}_g\{P_{01}, P_{111}\}$ (see 4.2), the subvariety lattice of $\mathcal{G}_u$ is dually atomic. Therefore there arises the question of describing the maximal subvarieties, that is the dual atoms in this lattice. By 2.2 every such maximal graph variety is finitely graph based.

Definition 5.1: Let

$$\begin{aligned} \mathcal{M}_1 = \mathrm{Mod}_g\, \{x_0x_1 \approx x_0(x_1x_0), ((x_0x_0)(x_1x_1))(x_2x_2) \\ \approx (x_0x_0)((x_1x_1)(x_2x_2))\}, \\ \mathcal{M}_2 = \mathrm{Mod}_g\{x_0x_1 \approx x_0(x_1x_0), (x_0x_0)x_1 \approx (x_0x_0)(x_1x_1)\}. \end{aligned}$$

Theorem 5.2. *(i) $\mathcal{M}_1$ consists of all undirected graphs which do not contain P_{111} as induced subgraph.*

(ii) $\mathcal{M}_2$ consists of all undirected graphs which do not contain P_{01} as induced subgraph (that is, every connected $G \in \mathcal{M}_2$ is totally looped or totally non-looped).

(iii) Both $\mathcal{M}_1$ and $\mathcal{M}_2$ are maximal subvarieties (coatoms) of $\mathcal{G}_u$.

(iv) There are no maximal subvarieties other than $\mathcal{M}_1$ and $\mathcal{M}_2$.

PROOF: The second identity defining $\mathcal{M}_1$ (see 5.1) expresses precisely the property that if a looped vertex has two looped neighbours then these neighbours must be adjacent, too. This is equivalent to (i). Analogously, the second identity defining $\mathcal{M}_2$ expresses the property (which is equivalent to (ii)), that every neighbour of a looped vertex must be looped, too. (iii) immediately follows from (i), (ii) and 4.2. To see (iv), let $\mathcal{M}$ be a subvariety of $\mathcal{G}_u$. If $P_{111} \notin \mathcal{M}$ then $\mathcal{M} \subseteq \mathcal{M}_1$, if $P_{01} \notin \mathcal{M}$ then $\mathcal{M} \subseteq \mathcal{M}_2$ (we always use the fact that graph varieties are closed with respect to induced subgraphs; see 1.4). Otherwise $\{P_{111}, P_{01}\} \subseteq \mathcal{M}$, thus $\mathcal{M} = \mathcal{G}_u$ by 4.2. ∎

There are several open problems for further research, for instance: describe and classify all graph varieties in the interval $[\mathcal{V}_{01}, \mathcal{G}_u]$, find full sublattices of this interval, describe all finitely graph based graph varieties of undirected graphs. The results of this paper are a contribution to these problems. Many more results can be obtained by using Lemma 2.1.

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