J. Aust. Math. Soc. **85** (2008), 419–430
doi:10.1017/S1446788708000700

TANGENTIAL CONVERGENCE OF BOUNDED HARMONIC FUNCTIONS ON GENERALIZED SIEGEL DOMAINS

M. SUNDARI

(Received 13 December 2006; accepted 12 December 2007)

Communicated by G. A. Willis

To my parents

Abstract

Suppose that u is a bounded harmonic function on the upper half-plane such that $\lim_{x\to\infty} u(x, y_0) = a$ for some $y_0 > 0$. Then one can prove that $\lim_{x\to\infty} u(x, y) = a$ for any other positive y. In this paper, we shall consider the algebra of radial integrable functions on H-type groups and obtain a similar result for bounded harmonic functions on generalized Siegel domains.

2000 *Mathematics subject classification*: primary 22E; secondary 43A20.
Keywords and phrases: Poisson kernel, harmonic function, generalized Siegel domain.

1. Introduction

Suppose that u is a bounded function on the upper half-plane satisfying $\Delta u = 0$ and $\lim_{x\to\infty} u(x, y_0) = a$ for some positive number y_0. Then using classical methods, we can prove that $\lim_{x\to\infty} u(x, y) = a$ for any other positive y. Here Δ denotes the Laplacian in the two variables x and y.

Suppose that H_r denotes the Heisenberg group of homogeneous dimension $2r + 2$. It is known that H_r acts on the Siegel domain $D_r = \{(z, z_0) \in \mathbb{C}^r \times \mathbb{C} \mid \operatorname{Im} z_0 > |z|^2\}$ by translations. Under this action, H_r is identified with the boundary of D_r. By abuse of notation, we shall denote the Laplace–Beltrami operator for the Bergman metric on D_r by Δ. Then we have the following result.

THEOREM 1.1. *Let u be a bounded function on D_r such that $\Delta u = 0$. If, for an $\epsilon_0 > 0$, $\lim_{(z,t)\to\infty} u(z, t, \epsilon_0) = a$, then $\lim_{(z,t)\to\infty} u(z, t, \epsilon) = a$ for any $\epsilon > 0$.*

For any unexplained notation and terminology, the reader can refer to [12]. The proof in the case of the Heisenberg group depends on the explicit form of the Poisson kernel and the Gelfand spectrum of the commutative Banach algebra $L^1(H_r)^\natural$, of integrable radial functions on H_r. A similar result for rank-one symmetric spaces has been proved by Cygan [5]. Here we shall prove an analogue of Theorem 1.1 for H-type groups.

The paper is organized as follows. In Section 2, we shall define an H-type group and collect all the facts we require about H-type groups. In Section 3, we shall define the generalized Siegel domains and describe the action of H-type groups on them. In Section 4, we shall prove our main result as a consequence of a Tauberian theorem for H-type groups. The method of proof is that of Hulanicki and Ricci [12].

In the coming sections, we shall use the 'variable constant convention' according to which our constants are denoted by C, C' and so on and these are not necessarily equal at different occurrences.

2. H-type groups

In this section, we shall collect all the necessary information about the H-type groups, N, and describe the Gelfand transform for biradial functions on N. We shall denote the sets $\mathbb{R} \setminus \{0\}$ and $\{0, 1, 2, \ldots\}$ by $\mathbb{R}^*$ and $\mathbb{N}$, respectively, and the semi-infinite interval $(0, \infty)$ by $\mathbb{R}^+$. For more detailed information on the material covered in this section, the reader may refer to [2] and the references therein.

Let $\mathfrak{n}$ be a real two-step nilpotent Lie algebra endowed with an inner product $\langle \cdot\,, \cdot \rangle_{\mathfrak{n}}$, and let $\mathfrak{z}$ be the centre of $\mathfrak{n}$. Write $\mathfrak{n}$ as an orthogonal direct sum of two subspaces $\mathfrak{v}$ and $\mathfrak{z}$, that is, $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$. For each $Z \in \mathfrak{z}$, define the map $J_Z : \mathfrak{v} \to \mathfrak{v}$ by the formula

$$\langle J_Z X, Y \rangle_{\mathfrak{n}} = \langle [X, Y], Z \rangle_{\mathfrak{n}} \quad \forall X, Y \in \mathfrak{v}. \tag{2.1}$$

The Lie algebra $\mathfrak{n}$ is said to be H-type if, for every $Z \in \mathfrak{z}$,

$$J_Z^2 = -|Z|^2 I_{\mathfrak{v}}, \tag{2.2}$$

where $I_{\mathfrak{v}}$ denotes the identity transformation on $\mathfrak{v}$. A connected, simply connected Lie group N whose Lie algebra is H-type is said to be an H-type group. By (2.2), we can see that every unit element Z in $\mathfrak{z}$ induces a complex structure on $\mathfrak{v}$ via the map J_Z. Therefore, $\mathfrak{v}$ has even dimension, say $2m$. If k denotes the dimension of the centre $\mathfrak{z}$ of N, then $Q = m + k$ is the homogeneous dimension of N.

As N is a connected, simply connected nilpotent group, we know that the exponential map $\exp : \mathfrak{n} \to N$ is surjective. Therefore, we shall identify N with $\mathfrak{v} \oplus \mathfrak{z}$ and denote a typical element n of N by (X, Z) where $X \in \mathfrak{v}$, $Z \in \mathfrak{z}$. Using the Campbell–Baker–Hausdorff formula, we get the product rule in N as

$$(X, Z)(X_1, Z_1) = (X + X_1, Z + Z_1 + \tfrac{1}{2}[X, X_1]) \quad \forall X, X_1 \in \mathfrak{v} \quad \forall Z, Z_1 \in \mathfrak{z}.$$

If dX and dZ denote the Lebesgue measures on $\mathfrak{v}$ and $\mathfrak{z}$, respectively, then $dn = dX\, dZ$ denotes a Haar measure on N.

There are two classes of irreducible unitary representations of an H-type group. Some are trivial on the centre and factor into characters on $\mathfrak{v}$. The others are parametrized by $\mathbb{R}^+ \times S_{\mathfrak{z}}$ (see [4] and [2]), where $S_{\mathfrak{z}}$ denotes the unit sphere in $\mathfrak{z}$. For w in $S_{\mathfrak{z}}$, we consider $\mathfrak{v}$ endowed with complex structure J_w. Denote by $I_w : \mathfrak{v} \to \mathbb{C}^m$ the corresponding isomorphism. Then the corresponding Hermitian inner product is given by

$$\{X, X_1\}_w = \langle X, X_1\rangle_{\mathfrak{n}} + i\langle J_w X, X_1\rangle_{\mathfrak{n}} \quad \forall X, X_1 \in \mathfrak{v}.$$

Define

$$\mathscr{H}_{\nu,w} = \left\{ \xi : \mathfrak{v} \to \mathbb{C} \,\middle|\, \xi \circ I_w^{-1} : \mathbb{C}^m \to \mathbb{C} \text{ is entire}, \right. \\ \left. \|\xi\|_\nu^2 = \int_{\mathfrak{v}} |\xi(X)|^2 e^{-\nu|X|^2/2}\, dX < \infty \right\}.$$

Thus $\mathscr{H}_{\nu,w}$ is a Hilbert space with respect to the inner product associated with the norm $\|\cdot\|_\nu$. For any multi-index j in $\mathbb{N}^m$, we define the following normalized polynomial:

$$\mathcal{P}_{\nu,j}(X) = \pi^{-m/2}\left(\frac{\nu}{2}\right)^{(m+|j|)/2} (j!)^{-1/2}(I_w(X))^j \quad \forall X \in \mathfrak{v},$$

where $|j| = j_1 + \cdots + j_m$, $j! = j_1! \cdots j_m!$ and $\zeta^j = \zeta_1^{j_1} \cdots \zeta_m^{j_m}$ for ζ in $\mathbb{C}^m$. One can check that the family $\{\mathcal{P}_{\nu,j}\}_{j\in\mathbb{N}^m}$ is an orthonormal basis of $\mathscr{H}_{\nu,w}$.

For any ν in $\mathbb{R}^+$ and any w in $S_{\mathfrak{z}}$, let $\pi_{\nu,w}$ be the unitary representation of N on $\mathscr{H}_{\nu,w}$ defined by

$$[\pi_{\nu,w}(X, Z)\xi](X_1) = \exp[-\nu(\tfrac{1}{4}|X|^2 + \tfrac{1}{2}\{X_1, X\}_w + i\langle Z, w\rangle_{\mathfrak{n}})]\xi(X + X_1) \\ \forall X_1 \in \mathfrak{v}, \forall \xi \in \mathscr{H}_{\nu,w}.$$

Given $f \in L^1(N)$, we shall define the group Fourier transform of f as an operator-valued function on $\mathscr{H}_{\nu,w}$ by

$$\pi_{\nu,w}(f) = \int_N \pi_{\nu,w}(n) f(n)\, dn.$$

A function f on N is said to be biradial if f is radial in both the variables X and Z. In other words, there exists a function f_0 on $\mathbb{R}^2$ such that

$$f(X, Z) = f_0(|X|, |Z|) \quad \forall (X, Z) \in N.$$

Let $L^1(N)^\natural$ be the space of all biradial integrable functions on the group N. We know from [9] that $L^1(N)^\natural$ is a commutative Banach algebra. The Gelfand spectrum of this commutative algebra is well known (see [1, 9, 13]) and can be described as follows.

Let $\mathscr{J}_z$ be the generalized Bessel function defined for every x in $\mathbb{R}$ by the rule

$$\mathscr{J}_z(x) = \begin{cases} \dfrac{\Gamma(z+1)}{\Gamma((2z+1)/2)\Gamma(1/2)} \displaystyle\int_{-1}^{1} e^{ixs}(1-s^2)^{(2z-1)/2}\, ds & \text{if } z > -1/2, \\ \cos x & \text{if } z = -1/2, \end{cases}$$

https://doi.org/10.1017/S1446788708000700 Published online by Cambridge University Press

and let L_l^α be the lth Laguerre polynomial of order α, that is,

$$L_l^\alpha(x) = \sum_{j=0}^{l} \binom{l+\alpha}{l-j} \frac{(-x)^j}{j!} \quad \forall x \in \mathbb{R}.$$

The bounded spherical functions of the commutative algebra $L^1(N)^\natural$ are given by

$$\phi_{\nu,l}(X, Z) = e^{-\nu|X|^2/4} \frac{L_l^{m-1}(\nu|X|^2/2)}{\binom{l+m-1}{l}} \mathscr{J}_{(k-2)/2}(\nu|Z|) \quad \forall (X, Z) \in N,$$
$$\phi_\mu(X, Z) = \mathscr{J}_{m-1}(\mu|X|) \quad \forall (X, Z) \in N,$$

where $\nu > 0$, $\mu \geq 0$ and $l \in \mathbb{N}$. If f is a biradial integrable function on N, we have the Gelfand transform $\hat{f}$ of f as a function on $\mathbb{R}^+ \times \mathbb{N}$ defined by the rule

$$\hat{f}(\nu, l) = \int_N f(n)\phi_{\nu,l}(n)\, dn \quad \forall \nu > 0, \forall l \in \mathbb{N}. \tag{2.3}$$

3. Harmonic *NA* spaces and Poisson kernel

In this section, we shall describe the harmonic NA spaces, define harmonic functions on the generalized Siegel domain and give the explicit form of the Poisson kernel. For any unexplained terminology and notation in this section, the reader may refer to [9] and [3]. Let $S = NA$ be the semidirect product of the groups N and $A = \mathbb{R}^+$ with respect to the action of A on N given by the dilations

$$\delta_a : (X, Z) \mapsto (a^{1/2}X, aZ).$$

We shall denote the Lie algebras of A and S by $\mathfrak{a}$ and $\mathfrak{s}$ respectively. Any typical element $na = \exp(X + Z)a$ of NA is denoted by (X, Z, a) and the product law in NA is given by

$$(X, Z, a)(X', Z', a') = (X + a^{1/2}X', Z + aZ' + \tfrac{1}{2}a^{1/2}[X, X'], aa').$$

One can endow NA with a suitable left-invariant Riemannian metric that makes it a harmonic manifold [8]. Via the map

$$h(X, Z, a) = (X, Z, a + \tfrac{1}{4}|X|^2)$$

we can identify S with the generalized Siegel domain

$$\mathcal{D} = \{(X, Z, a) \in \mathfrak{s} : a > \tfrac{1}{4}|X|^2\}.$$

Under this identification, N gets identified with the boundary $\partial\mathcal{D}$ of $\mathcal{D}$.

Let $\mathcal{L}$ be the Laplace–Beltrami operator on S with respect to the Riemannian structure on S. Then, by [7, Theorem 2.1],

$$\mathcal{L} = \sum_{i=1}^{2m+k} E_i^2 + E_0^2 - QE_0,$$

where $E_1, \ldots, E_{2m}$ in $\mathfrak{v}$, $E_{2m+1}, \ldots, E_{2m+k}$ in $\mathfrak{z}$, E_0 in $\mathfrak{a}$ form an orthonormal basis of $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$. Bounded harmonic functions u on $\mathcal{D}$ are those functions that satisfy $\mathcal{L}u = 0$ and have boundary values almost everywhere (a.e.) on $\partial\mathcal{D}$, that is,

$$\lim_{a\to 0} u(X, Z, a) = \phi(X, Z) \quad \text{a.e.}, \tag{3.1}$$

where $\phi \in L^\infty(N)$ (see [6, Theorem 3.7]). Moreover

$$u(X, Z, a) = (\phi * P_a)(X, Z) \quad \forall (X, Z, a) \in S,$$

where P_a is the Poisson kernel on the nilpotent group N given by

$$P_a(X, Z) = \frac{Ca^Q}{((a + |X|^2/4)^2 + |Z|^2)^Q} = \frac{Ca^Q}{((a + |X|^2/4)^2 + |Z|^2)^{m+k}}$$

and the convolution is on N. Here the constant C is chosen in such a way that $\|P_a\|_1 = 1$. Note that $P_a \in L^1(N)^\natural$ and

$$P_a(X, Z) = \frac{Ca^Q}{(a + |X|^2/4 + i|Z|)^{m+k}(a + |X|^2/4 - i|Z|)^{m+k}}.$$

For each $a \in A$, we know that δ_a is an automorphism of N, hence δ_a defines an automorphism of $L^1(N)^\natural$ by

$$(\delta_a f)(X, Z) = a^{m+k} f(\delta_a(X, Z)).$$

Let $m(L^1(N)^\natural)$ be the set of nonzero multiplicative linear functionals on $L^1(N)^\natural$. Then δ_a induces a map δ_a^* on $m(L^1(N)^\natural)$ by

$$\langle f, \delta_a^* \psi\rangle = \langle \delta_a f, \psi\rangle \quad \forall \psi \in m(L^1(N)^\natural), \forall f \in L^1(N)^\natural.$$

It is easy to see that δ_a^* maps $m(L^1(N)^\natural)$ homeomorphically onto itself. If $f \in L^1(N)$ and $\|f\|_1 = 1$ then we can see that $\{\delta_a f\}$ is an approximate identity in $L^1(N)$ as $a \to 0$.

We shall now make a small computation that we need in the next section (see [4, Lemma 3.4]). For $w \in S_{\mathfrak{z}}$, we shall denote by $w^\perp$ the orthogonal complement of w in $\mathfrak{z}$. Then

$$\begin{aligned}
&\int_{\mathfrak{z}} \exp\,(-i\nu\langle Z, w\rangle_{\mathfrak{n}}) P_a(X, Z)\, dZ \\
&\quad = \int_{\exp(w^\perp)} \int_{\mathbb{R}} \exp\,(-i\nu\langle tw + Z', w\rangle_{\mathfrak{n}}) P_a(X, tw + Z')\, dt\, dZ' \\
&\quad = \int_{\mathbb{R}} e^{-i\nu t} \int_{\exp(w^\perp)} P_a(X, tw + Z')\, dZ'\, dt \\
&\quad = \int_{\mathbb{R}} e^{-i\nu t} \int_{\exp(w^\perp)} Ca^Q\left(\left(a + \frac{|X|^2}{4}\right)^2 + t^2 + |Z'|^2\right)^{-m-k} dZ'\, dt \\
&\quad = \int_{\mathbb{R}} e^{-i\nu t} Ca^Q \int_{\exp(w^\perp)} (u^2)^{-m-k}\left(1 + \left(\frac{|Z'|}{u}\right)^2\right)^{-m-k} dZ'\, dt
\end{aligned} \tag{3.2}$$

https://doi.org/10.1017/S1446788708000700 Published online by Cambridge University Press

where

$$u^2 = \left(\left(a + \frac{|X|^2}{4}\right)^2 + t^2\right).$$

Now by a change of variable argument, we can show that the above integral is equal to

$$\int_{\mathbb{R}} e^{-i\nu t} C' a^Q \left(\left(a + \frac{|X|^2}{4}\right)^2 + t^2\right)^{-((m+k+1)/2)} dt. \tag{3.3}$$

In the next section, we shall prove our main result as a consequence of a Tauberian theorem on H-type groups.

4. Main result

In this section, we shall show that $R = L^1(N)^\natural$ is a regular $*$-algebra, and state and prove a Wiener–Tauberian theorem for $L^1(N)$. Further, we show that the Gelfand transform of the Poisson kernel never vanishes. We shall conclude our main result as a consequence of the Wiener–Tauberian theorem. Our proof of the Wiener–Tauberian theorem is based on that of Hulanicki and Ricci [12].

PROPOSITION 4.1. *The commutative Banach algebra R is regular.*

PROOF. Given $f \in R$, define f^* by $f^*(n) = \overline{f(n^{-1})}$. It is easy to see that $f^* \in R$, * defines an involution on R and R is symmetric. Let $\tilde{R}$ be the commutative *-Banach algebra obtained from R by adjoining the unit element 1. As in [11], we can check that the set of multiplicative linear functionals $m(\tilde{R})$ on $\tilde{R}$ is actually equal to $m(R) \cup \{\infty\}$. Note that $m(\tilde{R})$ is compact and $\hat{R}$ separates points. Further, if $f \in R \subset \tilde{R}$, then $\hat{f}(\infty) = 0$. Since R is *-closed, $\hat{R}$ is self-adjoint.

Let $C \subset m(R)$ be closed and $\xi \in m(R) \setminus C$. To show that R is regular, we need to show that there exists $f \in R$ such that $\hat{f}(C) = 0$ but $\hat{f}(\xi) = 1$. Since C is closed in $m(R)$, $C \cup \{\infty\}$ is compact in $m(\tilde{R})$ and $\xi \notin C \cup \{\infty\}$. As $m(\tilde{R})$ is compact and Hausdorff, by Urysohn's lemma, we can obtain a continuous function ϕ on $m(\tilde{R})$ such that $\phi(C) = 0$, $\phi(\infty) = 0$, but $\phi(\xi) = 1$. By [14, Theorem 2, p. 217], closure of $\tilde{R} = C_\infty(m(\tilde{R})) = C_0(m(R))$. So there exists $f \in R$ such that $\sup_{\eta \in m(\tilde{R})} |\phi(\eta) - \hat{f}(\eta)| < 1/4$. Let F be a smooth real-valued function defined on $\mathbb{R}$ as follows:

$$F(\alpha) = \begin{cases} 0 & \text{if } \alpha \le \frac{1}{3}, \\ 1 & \text{if } \frac{2}{3} \le \alpha \le \frac{4}{3}. \end{cases}$$

Then by Dixmier [10], $F \circ \hat{f} \in \hat{R}$. But if $\eta \in C$, then $|\hat{f}(\eta)| < 1/4 < 1/3$ and so $(F \circ \hat{f})(\eta) = 0$ and $|\hat{f}(\xi) - 1| < 1/4$. This implies that $3/4 < \hat{f}(\xi) < 5/4$, which in turn implies that $(F \circ \hat{f})(\xi) = 1$. This proves the proposition. □

For $f \in R$, let C_f be the support of $\hat{f}$ in $m(R)$. Let

$$\mathscr{B} = \{f \in R \mid C_f \text{ is compact in } m(R)\}.$$

Now we shall state and prove a Wiener–Tauberian theorem for R.

PROPOSITION 4.2. *The set $\mathscr{B}$ is dense in R.*

PROOF. Let f be a radial function in $L^1(N)$ having compact support. Choose a function F in $C^k(\mathbb{R})$ such that $F(1) = 1$ and $F(x) = 0$ for $|x| \leq 1/2$. By [10], we know that $\hat{f}_1 = F \circ \hat{f} \in \hat{R}$ for some $f_1 \in R$. Note that

$$\int_N f_1(X, Z)\, dX\, dZ = (F \circ \hat{f})(0, 0) = 1$$

and C_{f_1} is compact. Therefore $(\widehat{\delta_a f_1})$ has compact support in $m(R)$ for every $a > 0$. We know that $\{\delta_a f_1\}$ is an approximate identity in $L^1(N)$ as $a \to 0$. Therefore $f * \delta_a f_1 \to f$ as $a \to 0$. But $(\widehat{f * \delta_a f_1}) = \hat{f}(\widehat{\delta_a f_1})$ has compact support in $m(R)$. This proves that $\mathscr{B}$ is dense in $C_c(N)$. But $C_c(N)$ is dense in $L^1(N)$, hence $\mathscr{B}$ is dense in $L^1(N)$. □

As a consequence of the above we have the following result.

PROPOSITION 4.3. *Let I be a proper closed right ideal in $L^1(N)$. Then there exists a $\psi \in m(R)$ such that, for all $f \in I \cap R$, $\hat{f}(\psi) = \psi(f) = 0$.*

PROOF. Since I is a proper closed right ideal in $L^1(N)$, an approximate identity argument shows that $I \cap R$ is a proper closed ideal in R.

To prove the proposition, we need the following local Wiener–Tauberian theorem (see [16]). Suppose that $G \in R$ with C_G compact. Let $f \in R$ be such that $\hat{f}(\psi) \neq 0$ for all $\psi \in C_G$. Then there exists $g \in R$ such that $g * f * G = G$. In order to prove this claim, note that $\hat{f}$ is continuous on the compact set C_G. Hence there exists $\beta > 0$ such that $\hat{f}(\phi) > \beta$ for all $\phi \in C_G$. Now choose F in $C^k(\mathbb{R})$ such that

$$F(\alpha) = \begin{cases} \dfrac{1}{\alpha} & \text{for } \alpha \geq \beta, \\ 0 & \text{for } \alpha \leq 0. \end{cases}$$

Then $F \circ \hat{f}|_{C_G} = 1/\hat{f}$. But by [10], we have $g \in R$ such that $\hat{g} = F \circ \hat{f} \in \hat{R}$. Therefore

$$\widehat{g * f * G}(\psi) = (\hat{g}\hat{f}\hat{G})(\psi) = \begin{cases} 0 & \text{if } \psi \notin C_G, \\ \hat{G} & \text{if } \psi \in C_G, \end{cases}$$

that is, $\hat{g}\hat{f}\hat{G} = \hat{G}$. By the uniqueness of the Gelfand transform, $g * f * G = G$.

We shall now prove the proposition. Assume on the contrary that, for every $\psi \in m(R)$, there exists f in $I \cap R$ such that $\hat{f}(\psi) \neq 0$. Let $G \in R$ be such that C_G is compact. By assumption, given any $\psi \in C_G$ we can find $f_\psi \in I \cap R$ such that $\hat{f}_\psi(\psi) \neq 0$. In fact $\hat{f}_\psi(\psi) > 0$. By continuity, $\hat{f}$ does not vanish in a neighbourhood U_ψ of ψ. The collection of open sets $\{U_\psi\}_{\psi \in C_G}$ forms an open cover for C_G. Hence we can find $\psi_1, \psi_2, \ldots, \psi_n$ in C_G such that $\{U_{\psi_1}, U_{\psi_2}, \ldots, U_{\psi_n}\}$ forms a finite subcover of C_G. Consider the function $f = f_{\psi_1} + f_{\psi_2} + \cdots + f_{\psi_n}$. Then $f \in I \cap R$

as I is an ideal and $\hat{f}(\psi) \neq 0$ for all $\psi \in C_G$. By our claim above, then there exists $g \in R$ such that $g * f * G = G$. But $I \cap R$ is an ideal in R and $f \in I \cap R$. Hence $g * f * G \in I \cap R$. This implies that $G \in I \cap R$. This in turn implies that $\mathscr{B} \subset I \cap R$. Therefore, $R = \overline{\mathscr{B}} \subset \overline{I \cap R} = I \cap R$ as $I \cap R$ is closed. This contradicts the fact that $I \cap R$ is a proper closed ideal in R. This completes the proof of the proposition. □

Put $s = (m + k - 1)/2$. Note that $s \geq 0$. For $a \in A$, recall that the Poisson kernel P_a is given by

$$\begin{aligned} P_a(X, Z) &= \frac{Ca^Q}{((a + |X|^2/4)^2 + |Z|^2)^{2s+1}} \\ &= \frac{Ca^Q}{(2s!)^2} \frac{(2s!)^2}{(a + |X|^2/4 - i|Z|)^{2s+1}(a + |X|^2/4 + i|Z|)^{2s+1}} \end{aligned}$$

for all $(X, Z) \in N$.
Consider, for any $r > 0$,

$$F_a(X, Z) = \frac{1}{r!} \frac{r!}{(a + |X|^2/4 + i|Z|)^{r+1}}.$$

Using the Laplace transform techniques, one can easily show that

$$F_a(X, Z) = \frac{1}{r!} \int_0^\infty \exp(i\alpha|Z|) \exp(-\alpha(a + |X|^2/4 + i|Z|))\alpha^r \, d\alpha.$$

Therefore,

$$\begin{aligned} P_a(X, Z) &= \frac{Ca^Q}{(2s!)^2} \int_0^\infty \int_0^\infty \exp(-\alpha(a + |X|^2/4 - i|Z|)) \\ &\quad \times \exp(-\beta(a + |X|^2/4 + i|Z|))(\alpha\beta)^{2s} \, d\alpha \, d\beta \\ &= \frac{1}{(2s!)^2} \int_0^\infty \int_0^\infty \exp(i(\alpha - \beta)|Z|) \\ &\quad \times \exp(-(\alpha + \beta)(a + |X|^2/4))(\alpha\beta)^{2s} \, d\alpha \, d\beta. \end{aligned} \tag{4.1}$$

As the last step in our proof of the main result, we have the following result.

PROPOSITION 4.4. *For every $a > 0$, the Gelfand transform $\hat{P}_a$ of P_a is never zero on $m(R)$.*

PROOF. We need to check that $\hat{P}_a(\nu, l)$ and $\hat{P}_a(0, \mu)$ do not vanish for $\nu > 0$, $\mu \geq 0$ and $l \in \mathbb{N}$.

Consider the integral

$$\begin{aligned} &\int_{\mathfrak{z}} P_a(X, Z) \exp(-i\nu\langle Z, w\rangle_{\mathfrak{n}}) \, dZ \\ &\quad = \int_{\mathbb{R}} \int_{\exp(w^\perp)} P_a(X, tw + Z') \exp(-i\langle tw + Z', \nu w\rangle_{\mathfrak{n}}) \, dZ' \, dt \end{aligned}$$

https://doi.org/10.1017/S1446788708000700 Published online by Cambridge University Press

$$= \int_{\mathbb{R}} e^{-i\nu t} \int_{\exp(w^\perp)} P_a(X, tw + Z') \exp(-i\langle Z', \nu w\rangle_{\mathfrak{n}})\, dZ'\, dt$$
$$= \int_{\mathbb{R}} e^{-i\nu t} \int_{\exp(w^\perp)} P_a(X, tw + Z')\, dZ'\, dt$$
$$= Ca^Q \int_{\mathbb{R}} e^{-i\nu t} \left(\left(a + \frac{|X|^2}{4}\right)^2 + t^2\right)^{-(s+1)} dt \tag{4.2}$$

by (3.2). Consider the expression

$$\left(\left(a + \frac{|X|^2}{4}\right)^2 + t^2\right)^{-(s+1)} = \left(a + \frac{|X|^2}{4} + it\right)^{-(s+1)} \left(a + \frac{|X|^2}{4} - it\right)^{-(s+1)}$$
$$= \frac{1}{(s!)^2} \int_0^\infty \int_0^\infty \exp(i(\alpha - \beta)t)$$
$$\times \exp(-(\alpha + \beta)(a + |X|^2/4))\alpha^s \beta^s\, d\alpha\, d\beta \tag{4.3}$$

by (4.1). Now using Fourier transform techniques together with (4.2) and (4.3), we get

$$\int_{\mathfrak{z}} P_a(X, Z) e^{-i\nu\langle Z, w\rangle_{\mathfrak{n}}}\, dZ$$
$$= \frac{Ca^Q}{(s!)^2} \int_0^\infty \exp(-(2\beta + \nu)(a + |X|^2/4))(\beta + \nu)^s \beta^s\, d\beta. \tag{4.4}$$

Now if we take $\nu = 0$ and evaluate the Fourier transform in the variable X, we obtain $\hat{P}_a(0, \mu)$.

Therefore

$$\hat{P}_a(0, \mu) = \hat{P}_a(0, |Y|)$$
$$= \frac{Ca^Q}{(s!)^2} \int_{\mathfrak{v}} \int_0^\infty \exp(-2\beta(a + |X|^2/4))\beta^{2s} \exp(-i\langle Y, X\rangle)\, d\beta\, dX$$
$$= \frac{Ca^Q}{(s!)^2} \int_0^\infty \exp(-2\beta a)\beta^{2s} \int_{\mathfrak{v}} \exp(-\beta(|X|^2/2)) \exp(i\langle Y, X\rangle)\, dX\, d\beta$$
$$= \frac{Ca^Q}{(s!)^2} \int_0^\infty \exp(-\mu^2/(2\beta)) \exp(-2\beta a)\beta^{2s}\, d\beta \tag{4.5}$$

where $Y \in \mathfrak{v}$.

Let

$$I = \int_0^\infty \exp(-2\beta a)\beta^{2s} \exp(-\mu^2/(2\beta))\, d\beta.$$

Choose $0 < \epsilon_1 < \epsilon_2 < \infty$. Then

$$I = \int_0^{\epsilon_1} + \int_{\epsilon_1}^{\epsilon_2} + \int_{\epsilon_2}^\infty \exp(-2\beta a)\beta^{2s} \exp(-\mu^2/(2\beta))\, d\beta.$$

Note that for β in $[\epsilon_1, \epsilon_2]$ the integrand $\exp(-2\beta a)\beta^{2s}\exp(-\mu^2/(2\beta)) > 0$, and the integrand is nonnegative for all other values in the interval $[0, \infty)$. Hence

$$\int_{\epsilon_1}^{\epsilon_2} \exp(-2\beta a)\beta^{2s}\exp(-\mu^2/(2\beta))\, d\beta > 0$$

and the integral on the intervals $[0, \epsilon_1)$, $[\epsilon_2, \infty)$ are nonnegative. Therefore $I > 0$, which in turn implies that $\hat{P}_a(0, \mu) > 0$. For $\nu \neq 0$,

$$\begin{aligned}
\hat{P}_a(\nu, l) &= \frac{Ca^Q}{(s!)^2} \int_{\mathfrak{v}} \int_0^\infty \exp(-(2\beta+\nu)(a+|X|^2/4))(\beta+\nu)^s \beta^s \\
&\quad \times \exp(-\nu|X|^2/4) \frac{L_l^{m-1}(\nu|X|^2/2)}{\binom{l+m-1}{l}}\, d\beta\, dX \\
&= \frac{Ca^Q}{(s!)^2} \int_0^\infty \exp(-(2\beta+\nu)a)(\beta+\nu)^s \beta^s \int_{\mathfrak{v}} \exp(-\nu|X|^2/4) \\
&\quad \times \frac{L_l^{m-1}(\nu|X|^2/2)}{\binom{l+m-1}{l}}\, dX\, d\beta \\
&= \frac{Ca^Q}{(s!)^2} \frac{|S_{\mathfrak{v}}|}{\binom{l+m-1}{l}} \int_0^\infty \exp(-(2\beta+\nu)a)(\beta+\nu)^s \beta^s \\
&\quad \times \left(\int_0^\infty \exp(-\nu k^2/4) L_l^{m-1}\left(\frac{\nu k^2}{2}\right) k^{2m-1}\, dk \right) d\beta \\
&= \frac{Ca^Q}{(s!)^2} \frac{|S_{\mathfrak{v}}|}{\binom{l+m-1}{l}} \frac{2^{m-1}}{\nu^m} \int_0^\infty \exp(-(2\beta+\nu)a)(\beta+\nu)^s \beta^s \\
&\quad \times \int_0^\infty e^{-y/2} L_l^{m-1}(y) y^{m-1}\, dy\, d\beta
\end{aligned} \tag{4.6}$$

by a change of variable. But we know from [15] that

$$L_l^{m-1}(x) = \frac{1}{l!} e^x x^{-(m-1)} \frac{d^l}{dx^l}(e^{-x} x^{l+m-1}),$$

and so

$$\begin{aligned}
\hat{P}_a(\nu, l) &= \frac{Ca^Q}{(s!)^2} \frac{|S_{\mathfrak{v}}|}{\binom{l+m-1}{l}} \frac{2^{m-1}}{\nu^m} \int_0^\infty \exp(-(2\beta+\nu)a)(\beta+\nu)^s \beta^s \\
&\quad \times \int_0^\infty e^{-y/2} e^y y^{-(m-1)} \frac{d^l}{dy^l}(e^{-y} y^{l+m-1}) y^{m-1}\, dy\, d\beta
\end{aligned}$$

Integrating by parts (4.6) implies that

$$\begin{aligned}\hat{P}_a(\hat{\nu}, l) &= \frac{Ca^Q}{(s!)^2} \frac{|S_{\mathfrak{v}}|}{\binom{l+m-1}{l}} \frac{2^{m-1}}{\nu^m} \int_0^\infty \exp\,(-(2\beta+\nu)a)(\beta+\nu)^s \beta^s \\ &\quad \times \int_0^\infty e^{y/2} \frac{d^l}{dy^l}(e^{-y} y^{l+m-1})\, dy\, d\beta \\ &= \frac{Ca^Q}{(s!)^2} \frac{|S_{\mathfrak{v}}|}{\binom{l+m-1}{l}} \frac{2^{m-1}}{\nu^m} \left(\frac{-1}{2}\right)^l \int_0^\infty \exp\,(-(2\beta+\nu)a)(\beta+\nu)^s \beta^s \\ &\quad \times \int_0^\infty e^{y/2} e^{-y} y^{l+m-1}\, dy\, d\beta \\ &= \frac{Ca^Q}{(s!)^2} \frac{|S_{\mathfrak{v}}|}{\binom{l+m-1}{l}} \frac{2^{m-1}}{\nu^m} \left(\frac{-1}{2}\right)^l \int_0^\infty \exp\,(-(2\beta+\nu)a)(\beta+\nu)^s \beta^s \\ &\quad \times \int_0^\infty e^{-y/2} y^{l+m-1}\, dy\, d\beta \\ &= \frac{Ca^Q}{(s!)^2} \frac{|S_{\mathfrak{v}}|}{\binom{l+m-1}{l}} \frac{2^{2m-1}(-1)^l}{\nu^m (l!)} (l+m-1)! \\ &\quad \times \int_0^\infty \exp\,(-(2\beta+\nu)a)(\beta+\nu)^s \beta^s\, d\beta. \end{aligned} \tag{4.7}$$

By repeating a similar argument as in the case of $\hat{P}_a(0, \mu)$ we can show that $\hat{P}_a(\nu, l) > 0$. This completes the proof of the assertion. □

We shall prove the tangential convergence of the bounded harmonic functions on the generalized Siegel domain $\mathcal{D}$.

THEOREM 4.5. *Suppose that u is a bounded harmonic function on $\mathcal{D}$ and*

$$\lim_{(X,Z)\to\infty} u(X, Z, a_0) = \alpha$$

for some $a_0 > 0$. Then for all $a > 0$, the limit $\lim_{(X,Z)\to\infty} u(X, Z, a)$ exists and is equal to α.

PROOF. Let ϕ be a function in $L^\infty(N)$ satisfying (3.1). Consider the right ideal in $L^1(N)$ given by

$$I = \left\{ g \in L^1(N) \,\middle|\, \phi * g = \alpha \int_N g(X, Z)\, dX\, dZ \right\}.$$

By our assumption, $P_{a_0} \in I \cap R$. But by Proposition 4.4, $\hat{P_{a_0}}$ does not vanish anywhere on the Gelfand spectrum $m(R)$ of R. By Proposition 4.3, this would imply that $I = L^1(N)$. This completes the proof of the theorem. □

https://doi.org/10.1017/S1446788708000700 Published online by Cambridge University Press

Acknowledgements

The author would like to thank Michael Cowling for his encouragement and many useful discussions on the subject of this paper. This work was done during her visit to The University of New South Wales in June 2005.

References

[1] F. Astengo and B. Di Blasio, 'A Paley–Wiener theorem on NA harmonic spaces', *Colloq. Math.* **80** (1999), 211–233.
[2] ———, 'The Schwartz space and homogeneous distributions on H-type groups', *Monatsh. Math.* **132** (2001), 197–214.
[3] M. Cowling, A. Dooley, A. Korányi and F. Ricci, 'An approach to symmetric spaces of rank one via groups of Heisenberg type', *J. Geom. Anal.* **8** (1998), 199–237.
[4] M. Cowling and U. Haagerup, 'Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one', *Invent. Math.* **96** (1989), 507–549.
[5] J. Cygan, 'A tangential convergence for bounded harmonic functions on a rank one symmetric space', *Trans. Amer. Math. Soc.* **265** (1981), 405–418.
[6] E. Damek, 'Harmonic functions on semidirect extensions of type H nilpotent groups', *Trans. Amer. Math. Soc.* **290** (1985), 375–384.
[7] ———, 'A Poisson kernel on Heisenberg type nilpotent groups', *Colloq. Math.* **53** (1987), 239–247.
[8] E. Damek and F. Ricci, 'A class of nonsymmetric harmonic Riemannian spaces', *Bull. Amer. Math. Soc.* **27** (1992), 139–142.
[9] ———, 'Harmonic analysis on solvable extensions of H-type groups', *J. Geom. Anal.* **2** (1992), 213–248.
[10] J. Dixmier, 'Opérateurs de rang fini dans les représentations unitaires', *Publ. Math. Inst. Hautes Études Sci.* **6** (1960), 13–25.
[11] G. B. Folland, *A Course in Abstract Harmonic Analysis*, Studies in Advanced Mathematics (CRC Press, Boca Raton, FL, 1995).
[12] A. Hulanicki and F. Ricci, 'A Tauberian theorem and tangential convergence for bounded harmonic functions on balls in $\mathbb{C}^n$', *Invent. Math.* **62** (1980), 325–331.
[13] A. Korányi, 'Some applications of Gelfand pairs in classical analysis', in: *Harmonic Analysis and Group Representations* (CIME, Napoli, 1982).
[14] M. A. Naimark, *Normed Rings*(ed. L. F. Boron) (Wolters-Noordhoff, Groningen, 1970), Translated from first Russian edition.
[15] S. Thangavelu, *Lectures on Hermite and Laguerre Expansions*, Mathematical Notes, 42 (Princeton University Press, Princeton, NJ, 1993).
[16] M. K. Vemuri, 'Realizations of the canonical representation', *Proc. Indian Acad. Sci. (Math. Sci.)* **118** (2008), 115–131.

M. SUNDARI, Chennai Mathematical Institute, Plot H-1, SIPCOT IT Park, Padur P O, Siruseri, 603 103, India
e-mail: sundari@cmi.ac.in